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Quality-Adjusted Life Years (QALY) Utility Models under Expected Utility and Rank Dependent Utility Assumptions

John M. Miyamoto

University of Washington

Quality-adjusted life years (QALY) utility models are multiattribute utility models of survival duration and health quality. This paper formulates six classes of QALY utility models and axiomatizes these models under expected utility (EU) and rank-dependent utility (RDU) assumptions. The QALY models investigated in this paper include the standard linear QALY model, the power and exponential multiplicative models, and the general multiplicative model. Emphasis is placed on a preference assumption, the zero condition, that greatly simplifies the axiomatizations under EU and RDU assumptions. The RDU axiomatizations of QALY models are generally similar to their EU counterparts, but in some cases, they require modification because linearity in probability is no longer assumed, and rank dependence introduces asymmetries between the domains of better-than-death health states and worse-than-death health states.

This paper concerns the foundations of quality-adjusted life years (QALY) utility models. QALY utility models are widely used in the expected utility analysis of health decisions because they provide an outcome measure that integrates the duration and quality of survival. Before discussing the specifics of these models, it will be helpful to motivate the discussion by describing the role played by QALY utility models in health decision analysis (Weinstein *et al*., 1980; Sox, Blatt, Higgins, & Marton, 1988; Gold, Siegel, Russell, & Weinstein,1996, Drummond, O'Brien, Stoddart, & Torrance, 1997).

A typical application of the QALY model would involve the utility analysis of a decision in which a patient must choose between two or more therapies. One component of the analysis involves the construction of probability models for each therapeutic choice. Each such model describes the possible sequences of health states that could occur given a therapeutic choice and assigns probabilities to these

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Reprint requests should be addressed to John Miyamoto, Department of Psychology, Box 351525, University of Washington, Seattle, WA 98195-1525.

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sequences. In a simple case, the probability model might be described as a lottery for specific health outcomes. In more complex cases, the model might be described as a decision tree or Markov model. The probabilities in the probability model are determined by means of actuarial data and the clinical judgments of health professionals.

A second component of the decision analysis involves the construction of a utility model of the possible health outcomes. Within expected utility theory, utility is conceptualized as a quantitative representation of value that is inferred from preferences among real or hypothetical lotteries. In health decision analyses, the actual source of the preferences varies in different studies. In some studies, a utility model is proposed by health professionals based on their own clinical judgment of the relative values of different health states. In other studies, attempts may be made to assess the preferences of patients or of people sampled from the general public. There are many considerations affecting the choice of a population from which preferences are sampled (see Gold, Patrick, *et al.*, 1996, for a discussion of this issue). In this paper, I will assume that patients who face a given health decision are the source of the preference judgments to be used in a decision analysis. This is the most straightforward motivation for the analysis of QALY models, and alternative assumptions would not affect the mathematical analysis.

Assuming, then, that probability models have been constructed for each therapeutic choice, and a utility model has been constructed for the possible health outcomes, one can calculate the expected utility of each therapeutic choice. The goal of an applied medical decision analysis is usually to determine which therapeutic choice has the highest expected utility. It should be recognized that many simplifications enter into any decision analysis. The advantage of a formal decision analysis is that even if it is flawed, the analysis makes explicit the assumptions and relationships that are posited in the analysis, so that other researchers can criticize and improve the analysis. Moreover, any decision analysis should be evaluated not only in terms of how well it captures the full structure of the decision problem, but also whether it represents an improvement over current theory and practice.

One point should be clarified from the outset of this discussion. Whereas the normative validity of expected utility (EU) theory is assumed throughout this paper, this paper takes the position that the descriptive validity of EU theory should not be assumed in the assessment of the utilities for the decision analysis. As is well known, the preferences of individuals violate EU theory in systematic ways (Kahneman & Tversky, 1979, 1984; Luce, 1992; Slovic, Lichtenstein, & Fischhoff, 1988). In this paper it is assumed that health utilities should be based on a descriptive model of patient preferences. Therefore, this paper will explore the axiomatic foundations of QALY utility models in a non-EU framework, specifically under rank dependent utility theory. The goal is to construct QALY utility models that are descriptively valid, or are at least good descriptive approximations. If such models can be discovered, they can be used to construct utility scales for individuals sampled from relevant populations. These utility scales can then be combined with probability models in the normative analysis of health decisions. The normative analysis assumes the validity of EU theory, not as a theory of the preferences of the individuals whose utilities have been assessed, but as a guide for the reasoning of the decision theorist.

The remainder of this paper will focus on axiomatic foundations of several classes of QALY models that are useful in health utility analyses. The first section develops a notation for discussing QALY utility models and presents several classes of QALY models. The second section describes axiomatizations of QALY utility models for constant (chronic) health states under EU theory assumptions. Several of these axiomatizations extend previous formalizations to include worse-than-death health states in the representation. Although the assumptions of EU theory are not descriptively valid, it is of interest to see how QALY models are axiomatized in the EU framework because these axiomatizations are relevant to normative decision theory and also because the comparison of EU and non-EU axiomatizations of QALY models is instructive. The third section of this paper presents axiomatizations of QALY models in the rank-dependent utility (RDU) theory framework. Both the EU and RDU axiomatizations emphasize the role of a useful, simplifying assumption, the zero condition. A final section of the paper discusses similarities and differences between the EU and RDU axiomatizations.

QUALITY-ADJUSTED LIFE YEARS (QALY) UTILITY MODELS

Notation

Let $\mathscr{Y}$ stand for a set of possible survival durations. Unless otherwise stated, I will assume that $\mathscr{Y} = [0, M)$; in other words, death immediately is a possible survival duration and utility modeling applies to survivals within an interval. Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}^+$ the strictly positive real numbers. Let $\mathscr{Q}$ stand for a non empty set of health states. A few proofs require that $\mathscr{Q}$ be infinite, and this requirement will be explicitly mentioned in these cases, but the majority of proofs allow $\mathscr{Q}$ to be finite or infinite. When testing QALY models, it is useful if axiomatizations allow for finite $\mathscr{Q}$ because some applications may contain only be a small finite set of relevant health states, e.g., as in McNeil, Weichselbaum, and Pauker (1981).

The set $\mathscr{H} = \mathscr{Q} \times \mathscr{Y}$ stands for the set of possible state/duration combinations. Members of $\mathscr{Q}$ will be denoted by lower case letters from the beginning of the alphabet and members of $\mathscr{Y}$ will be denoted by lower case letters from the end of the alphabet. Elements of $\mathscr{H}$ will be denoted by juxtaposed symbols, e.g., $bx \in \mathscr{H}$ represents a survival of length x in a constant health state b followed by death. Sequences will be represented as ordered n-tuples, e.g., (aw, bx, cy) represents a survival of w years in state a, followed by x years in state b, followed by y years in state c, followed by death. Of course, depending on the application, it may be more natural to interpret the unit of duration as months or days rather than years.

Let $\mathscr{T}$ stand for the set of all finite sequences of members of $\mathscr{H}$. Let $\mathscr{L}^*$ denote the set of all probability distributions over $\mathscr{T}$. The choice of a therapy for a given patient can be interpreted as a choice of a specific element of $\mathscr{L}^*$. Let $\mathscr{L}$ denote the set of all finite probability distributions (lotteries) over $\mathscr{H}$. Elements of $\mathscr{L}$ are represented in the notation, $(a_1x_1, p_1; \ldots; a_nx_n, p_n) \in \mathscr{L}$, indicating a p_1 chance of $a_1x_1, \ldots,$ and a p_n chance of a_nx_n. Let $\succcurlyeq$ represent a binary relation on $\mathscr{L}$ to be interpreted as a preference relation on lotteries. It will assume that $\succcurlyeq$ is a weak order, i.e., it is transitive (if $f \succcurlyeq g$ and $g \succcurlyeq h$, then $f \succcurlyeq h$) and connected (either $f \succcurlyeq g$ or $g \succcurlyeq f$ for any $f, g \in \mathscr{L}$). Define relations $\succ$ and $\sim$ by the usual conditions: $r \succ s$ iff $r \succcurlyeq s$ and not $s \succcurlyeq r$, and $r \sim s$ iff $r \succcurlyeq s$ and $s \succcurlyeq r$.

It may be helpful to mention that elements of $\mathscr{L}$ (finite probability distributions over survivals in constant health states) do not generally represent real possibilities for real individuals. The real health choices are choices between elements of $\mathscr{L}^*$ (possibly infinite probability distributions over sequences of health states). The elements of $\mathscr{L}^*$ are generally too complex, however, to serve as stimuli in preference studies. Therefore one uses elements of $\mathscr{L}$ as stimuli in preference studies whose goal is to determine the utility structure in the outcome domain. Once theoretical and empirical studies have identified a utility model for the given domain, a fit of the model can be used in the EU analysis of therapy selection, i.e., in determining a rational choice between elements of $\mathscr{L}^*$.

Basic Classes of QALY Utility Models

It is assumed throughout this section that there exists a utility function $U\colon \mathscr{H} \to \mathbb{R}$. This section presents various QALY models that postulate specific forms for U. The axiomatic analysis of these models will be undertaken in a subsequent section.

The linear QALY model is the simplest and most widely used QALY utility model. This model postulates the existence of a function $H\colon \mathscr{Q} \to \mathbb{R}$ such that

$$U(bx) = k \cdot H(b) \cdot x \tag{1}$$

for every $b \in \mathscr{Q}$ and $x \in \mathscr{Y}$. The scaling constant k is chosen so that the utilities fall on a convenient range of numbers. The most common practice is to define H such that $H(\text{full health}) = 1$ where "full health" represents whatever is regarded as the best possible health state; and to choose k such that $U(\text{full health}, M) = 100$. In all of the QALY models discussed in this paper, there will be similar constants that play no substantive role other than to force scale values to fall on a convenient range of numbers. It should be mentioned that on logical grounds, one could also have a nonzero additive constant in Eq. (1), i.e., $U(bx) = k \cdot H(b) \cdot x + s$, but the additive constant is usually assigned the value of zero in order that we have $U(b, 0) = 0$.

To accommodate the possibility of changes in health state over time, it is usually assumed that the utility of survival is additive over time periods, i.e.,

$$U(b_1 x_1, ..., b_n x_n) = \sum H(b_i) \cdot x_i. \tag{2}$$

Equations (1) and (2) characterize the *linear QALY model.* Because the linear QALY model is, with very few exceptions, the only QALY model that is ever employed in applied health decision analysis, it is usually called *the* QALY utility model in the health decision and policy literature (see, for example, Gold, Siegel, *et al.*, 1996 or Weinstein *et al.*, 1980). This paper uses a nonstandard terminology, calling (1) and (2) the linear QALY model, because it also investigates QALY models in which linearity is not assumed. The history of the linear QALY model is recounted in Fryback (1999) and Drumond *et al.* (1997).

The assumption that utility is linear with respect to duration is a serious limitation, because it implies that the preference order is equivalent to the ordering of expected values. In the EU framework, an individual is said to be *risk averse* if she always prefers the expected value of a gamble to the gamble itself, *risk seeking* if she always prefers a gamble to its expected value, and *risk neutral* if she is always indifferent between a gamble and its expected value. Under EU assumptions, risk averse, risk neutral, and risk seeking preferences correspond to utility functions that are, respectively, concave, linear, and convex (Keeny & Raiffa, 1976). The curvature of a utility function is sometimes said to indicate the *risk attitude* of the individual. In this paper, I will use the more neutral expression, *utility curvature*, to refer to the shape of the utility function for survival duration. The standard terminology of risk attitude carries theoretical connotations that could be misleading in the context of recent research in which classically risk averse behavior is attributed in part to nonlinear weighting of probability, and not exclusively to the shape of the utility function (Wakker, 1994; Wakker & Stiggelbout, 1995), and where furthermore, a distinction is drawn between a component of utility curvature that is due to a riskless process of decreasing marginal value, and a component that is specific to preferences under risk (Keller, 1985; Sarin, 1982). Utility curvature with respect to survival duration is important in medical decision analysis because therapies can differ in their tradeoffs for short- and long-term survival (McNeil & Pauker, 1982; McNeil, Weichselbaum, & Pauker, 1978).

The class of *power multiplicative QALY models* is a class of utility models that contains a parametric representation of utility curvature: Models in this class postulate the existence of a function $H: \mathscr{Q} \to \mathbb{R}$, and parameter $\theta \in \mathbb{R}^+$, such that

$$U(bx) = k \cdot H(b) \cdot x^{\theta} \tag{3}$$

for every $bx \in \mathscr{H}$. The scaling constant $k \in \mathbb{R}^+$ is not of substantive importance. The parameter θ represents utility curvature. The power multiplicative models were proposed as a QALY representation by Pliskin, Shepard, and Weinstein (1980), and were further developed in Miyamoto and Eraker (1985).

The class of *exponential multiplicative QALY models* is another class of models that contains a parametric representation of utility curvature: Models in this class postulate the existence of a function $H: \mathscr{Q} \to \mathbb{R}$, and parameter $\lambda \in \mathbb{R}^+$, such that

$$U(bx) = \begin{cases} k \cdot H(b) \cdot [\exp(\lambda \cdot x) - 1], & \text{if} \quad \lambda > 0 \\ k \cdot H(b) \cdot x, & \text{if} \quad \lambda = 0 \\ (-k) \cdot H(b) \cdot [\exp(\lambda \cdot x) - 1], & \text{if} \quad \lambda < 0 \end{cases} \tag{4}$$

for every $bx \in \mathscr{H}$. Once again, $k \in \mathbb{R}^+$ is a scaling constant. The linear utility function is substituted for the case where $\lambda = 0$ because the preference order over lotteries that is implied by an exponential utility with $\lambda \neq 0$ becomes arbitrarily similar to the expected value ordering as λ approaches 0. Exponential utility functions for survival duration were used in Pauker's (1976) decision analysis of coronary artery surgery, and the exponential multiplicative QALY model was employed in a medical decision analysis by Cher, Miyamoto, and Lenert (1997).

The power multiplicative and exponential utility models are special cases of a general multiplicative QALY model that was investigated by Maas and Wakker (1994), Miyamoto and Eraker (1988), Miyamoto (1992), and Miyamoto, Wakker, Bleichrodt, and Peters (1998). The *multiplicative QALY model* postulates the existence of functions $H\colon \mathcal{Q} \to \mathbb{R}$ and $F\colon \mathcal{Y} \to \mathbb{R}$ such that

$$U(bx) = H(b) \cdot F(x), \tag{5}$$

for every $bx \in \mathcal{H}$. For reasons to be explained later, it makes sense in the context of QALY measurement to assume that $F(0) = 0$. This constraint will be assumed to be part of the definition of the multiplicative QALY model.

The power multiplicative, exponential multiplicative, and general multiplicative models all assume that health state in constant during the period of survival. To extend these representations to sequences of health states, we postulate additivity over time periods. Under this assumption, the multiplicative model (2) implies that

$$U(a_1x_1, ..., a_nx_n) = \sum_{i=1}^{n} H(a_i) \cdot \left[F\left(\sum_{k=0}^{i} x_i \right) - F\left(\sum_{k=0}^{i-1} x_i \right) \right], \tag{6}$$

where $x_0 = 0$. Substituting model (3) or (4) in (6) yields extensions of these models to the utility of sequences of health states. Equation (6) asserts that the utility of a sequence is the sum of the utility increments during the separate periods.

All of the QALY models presented up to this point are versions of the multiplicative model, Eq. (5). Later, I will present arguments in favor of the multiplicative QALY model, but it will be useful to consider some nonmultiplicative models. The general family of power QALY models is defined by

$$U(bx) = k \cdot H(b) \cdot x^{G(b)}, \tag{7}$$

where $H\colon \mathcal{Q} \to \mathbb{R}$ and $G\colon \mathcal{Q} \to \mathbb{R}^+$ are functions, and k is a scaling constant. The general family of exponential QALY models is defined by

$$U(bx) = \begin{cases} k \cdot H(b) \cdot [\exp(G(b) \cdot x) - 1], & \text{if} \quad G(b) > 0 \\ k \cdot H(b) \cdot x, & \text{if} \quad G(b) = 0 \\ (-k) \cdot H(b) \cdot [\exp(G(b) \cdot x) - 1], & \text{if} \quad G(b) < 0 \end{cases} \tag{8}$$

where $H\colon \mathcal{Q} \to \mathbb{R}$ and $G\colon \mathcal{Q} \to \mathbb{R}$ are any functions, and k is a scaling constant. Both the general power and general exponential models allow utility curvature to vary as a function of health state. Models (7) and (8) are nonmultiplicative because the joint utility function cannot be decomposed into a product of factors that depend, respectively, on health state and duration. As I will show in the next section, the general power and general exponential models violate a critical axiom for a multiplicative model.

If one retains the assumption that utility is additive across time periods, then Eq. (6) can be extended to sequences of health states as follows: For any sequence, $[b_1 x_1, ..., b_n x_n]$, let $x_0 = 0$. Then,

$$U[b_1 x_1, ..., b_n x_n] = \sum_{i=1}^{n} \left[U\left(b_i, \sum_{k=0}^{i} x_i\right) - U\left(b_i, \sum_{k=0}^{i-1} x_i\right)\right]. \tag{9}$$

Substituting model (7) or (8) in (9) yields extensions of these models to the utility of sequences of health states.

EU AXIOMATIZATIONS OF QALY MODELS

The discussion of EU axiomatizations is divided into two parts. The first part axiomatizes the most general QALY models, the general power, the general exponential, and the general multiplicative models. Combining ideas from these axiomatizations yields axiomatizations of the power multiplicative model and exponential multiplicative model. A further specialization yields an axiomatization of the linear QALY model. This line of development passes from more general to more specific models by adding assumptions to the axiomatizations. All of the axiomatizations emphasize an assumption, the zero condition, that greatly simplifies QALY axiomatizations. The second part of this section compares the initial axiomatizations to alternative axiomatizations from the literature.

All of the axiomatizations in this section assume that the relational structure $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ satisfies a sufficient set of axioms for the EU representation. A number of different axiomatizations have been published (Fishburn, 1982; von Neumann & Morgenstern, 1944). The specific choice of axioms for EU theory does not affect the present axiomatizations. What matters is that these axiomatizations imply the existence of a utility function $U: \mathscr{L} \to \mathbb{R}$ such that:

(i) If $f, g \in \mathscr{L}$, then $f \succcurlyeq g$ iff $U(f) \geqslant U(g)$;

(ii) If $g \in \mathscr{L}$ and $g = [a_1 x_1, p_1; ...; a_n x_n, p_n]$, then $U(g) = \sum_{i=1}^{n} p_i \cdot U(a_i x_i)$.

The following uniqueness condition for the EU representation is frequently invoked in proofs of QALY representation theorems.

(iii) If $U^*: \mathscr{L} \to \mathbb{R}$ is any other function that satisfies (i) and (ii), then there exists constants $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that $U^* = \alpha \cdot U + \beta$.

Condition (iii) asserts that U is an interval scale.

It will simplify the axiomatic analysis if we assume that utility is a continuous function of survival duration for any fixed choice of health state. The following restricted solvability assumption combines with a sign dependence assumption (Definition 3 below) to imply continuity of the utility of survival duration.

DEFINITION 1. Preferences satisfy *restricted solvability* with respect to survival duration iff for every $b \in \mathscr{Q}$, every $x, z \in \mathscr{Y}$, and every $g \in \mathscr{L}$, if $bx \succcurlyeq g \succcurlyeq bz$, then there exists $y \in \mathscr{Y}$ such that $by \sim g$.

Restricted solvability is reasonable under the intended interpretation in health preferences. Continuity of the utility of survival duration is needed to deduce solutions to functional equations that arise in QALY representation theorems.

EU Axiomatizations of QALY Models Based on the Zero Condition

The *zero condition* is the assumption that preferences between health qualities disappear when survival duration is 0 (Bleichrodt, Wakker, & Johannesson, 1997; Miyamoto & Eraker, 1988; Miyamoto *et al.*, in press). This assumption is defined as follows:

DEFINITION 2. Preferences for survival durations and health states satisfy *the zero condition* iff $a0 \sim b0$ for every $a, b \in \mathcal{Q}$, where $0 \in \mathcal{Y}$ stands for death immediately.

In combination with EU assumptions, the zero condition implies that all utility functions intersect at a single point when duration equals zero. The zero condition is a natural assumption because all combinations of health states with zero duration are the same physical object; hence, preference should not distinguish between them (Miyamoto *et al.*, in press). It might be questioned whether this claim is meaningful—can we really associate a health quality with a survival duration of zero? The following heuristic argument offers another approach to interpreting the zero condition (Miyamoto & Eraker, 1988). Consider the subjective difference between living x years in excellent health versus x years in mediocre health. For example, consider the subjective difference between 10 years in excellent health and 10 years in mediocre health, or between 1 year in excellent health and 1 year in mediocre health. As the common duration gets shorter, 5 years, 5 days, 5 seconds, etc., the subjective difference between survival in the two health states gets arbitrarily close to indifference. The zero condition is the limiting case where duration is zero. This argument is not cogent from a purely EU standpoint because it refers to subjective differences which do not have an operational interpretation in terms of preferences among lotteries, but it has heuristic value in the context of psychological investigations.

Preferences for duration and quality of survival exhibit several qualitative properties that are diagnostic of a multiplicative model. The first is the zero condition itself. Model (5) predicts that the zero condition must obtain because $H(a) \cdot F(0) = H(b) \cdot F(0)$ for all $a, b \in \mathcal{Q}$. Second, longer survival is typically preferred to shorter survival. Nevertheless some health states are so undesirable that shorter survival is actually preferred to longer survival in these states (Patrick, Cain, Pearlman, Starks, & Uhlmann, 1993). In other words, some health states are *better than death* in the sense that $bx \succ by$ whenever $x > y$, and other health states are *worse than death* in the sense that $bx \prec by$ whenever $x > y$. The reversal in preference can be represented by the assumption that F is an increasing function of duration, and H takes on positive values at better-than-death health states and negative values at worse-than-death health states. It will also be useful to recognize the possibility that some health states are perceives to be *equal to death* in the sense that there may exist $b \in \mathcal{Q}$ such that $by \sim b0$ for all $y \in \mathcal{Y}$.

The existence of better-than-death and worse-than-death health states are examples of a measurement property called sign dependence that is diagnostic of a multiplicative relationship (Krantz *et al.*, 1971, Chapter 9).

DEFINITION 3. Survival duration is *sign dependent* on health state iff $\mathscr{Q}$ can be partitioned into three sets, $\mathscr{Q}^+$, $\mathscr{Q}^0$ and $\mathscr{Q}^-$, such that for all $x, y \in \mathscr{Y}$, if $a, b \in \mathscr{Q}^+$ or $a, b \in \mathscr{Q}^-$, then $ax \succcurlyeq ay$ iff $bx \succcurlyeq by$, if $a \in \mathscr{Q}^+$ and $b \in \mathscr{Q}^-$, $ax \succcurlyeq ay$ iff $bx \preccurlyeq by$; and if $a \in \mathscr{Q}^0$, $ax \sim ay$. Health state is *sign dependant* on survival duration iff $\mathscr{Y}$ can be partitioned into three sets, $\mathscr{Y}^+$, $\mathscr{Y}^0$, and $\mathscr{Y}^-$, such that for all $a, b \in \mathscr{Q}$ if $x, y \in \mathscr{Y}^+$ or $x, y \in \mathscr{Y}^-$, then $ax \succcurlyeq bx$ iff $ay \succcurlyeq by$; if $x \in \mathscr{Y}^+$ and $y \in \mathscr{Y}^-$, then $ax \succcurlyeq bx$ iff $ay \preccurlyeq by$; and if $x \in \mathscr{Y}^0$, then $ax \sim bx$.

In the case of QALYs, it is natural to interpret $\mathscr{Q}^+$ as the set of better-than-death health states, $\mathscr{Q}^0$ as the equal-to-death health states, and $\mathscr{Q}^-$ as the worse-than-death health states. For duration, it is natural to interpret $\mathscr{Y}^+ = \mathscr{Y} - \{0\}$, $\mathscr{Y}^0 = \{0\}$, and $\mathscr{Y}^- = \varnothing$. These interpretations will be assumed in this paper; thus, to say that $a \in \mathscr{Q}^+$ will be taken to mean that a is better than death, etc.

Sign dependence between survival duration and health state is suggestive of a multiplicative model (Miyamoto & Eraker, 1988; Miyamoto *et al.*, in press), but it is not sufficient prove its existence. To arrive at sufficient conditions, we must define some preference relations that can be used to characterize utility curvature.

DEFINITION 4. For any $b \in \mathscr{Q}$ and x, y, z $\mathscr{Y}$, if $(bx, p; bz, 1-p) \sim by$, then p is said to be the *probability equivalent* of by with respect to the endpoints bx and bz, and y will be said to be the *certainty equivalent* of the lottery with respect to the health state b. If $p = 1/2$, then y will be called the 50/50 *certainty equivalent* of the lottery with respect to the health state b.

Under EU assumptions, two utility functions have identical utility curvature with respect to survival duration if they imply precisely the same probability equivalents for every binary lottery. Continuous utility functions have identical utility curvature if they imply the same certainty equivalents for every 50/50 lottery. This motivates the following definition.

DEFINITION 5. Probability equivalents will be said to be *invariant under changes in health state* iff for any health states $b, c \in \mathscr{Q}$ that are not equal to death, the following holds:

$$(bx, p; bz, 1-p) \sim by \qquad \text{iff} \quad (cx, p; cz, 1-p) \sim cy \tag{10}$$

for every $x, y, z \in \mathscr{Y}$. 50/50 certainty equivalents will be said to be *invariant under changes in health state* iff (10) holds for $p = 1/2$.

The following lemma states that these invariances are equivalent to a functional equation on the utility function. The hypotheses of the lemma include the assumptions that survival duration satisfies restricted solvability and is sign dependent on health state because these properties imply that utility is continuous with respect to survival duration.

LEMMA 1. *Let* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfy the EU axioms, and let* U *be a utility function. Suppose that survival duration satisfies restricted solvability and is sign dependent on health state. Then the following three conditions are equivalent.*

(i) *Probability equivalents are invariant under changes in health state.*

(ii) 50/50 *certainty equivalents are invariant under changes in health state.*

(iii) *There exist functions* $\alpha\colon \mathscr{Q} \to \mathbb{R}$, $\beta\colon \mathscr{Q} \to \mathbb{R}$, *and* $F\colon \mathscr{Y} \to \mathbb{R}$ *such that* $U(b, y) = \alpha(b) \cdot F(y) + \beta(b)$ *for every* $b \in \mathscr{Q}$ *and* $y \in \mathscr{Y}$.

Lemma 1 is proved in the Appendix. Because (i) and (ii) are equivalent in the context of restricted solvability and sign dependence, they can play similar roles in the axiomatizations presented in this paper. This paper will employ the invariance of 50/50 certainty equivalents in the axiomatizations because it is usually the more easily tested assumption.

The following theorem states sufficient conditions for the multiplicative model.

THEOREM 1. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) 50/50 *certainty equivalents are invariant under changes in health state.*

Then the multiplicative model (5) *is satisfied.*

The zero condition and the invariance of 50/50 certainty equivalents are necessary conditions for the multiplicative model (5). Sign dependence and restricted solvability are not necessary, but are plausible under the intended interpretation.

To prove Theorem 1, one applies Lemma 1 to yield the functional equation, $U(b, y) = \alpha(b) \cdot F(y) + \beta(b)$ for every $b \in \mathscr{Q}$ and $y \in \mathscr{Y}$. Algebraic manipulation and the zero condition then yield that $\beta(b)$ is a constant function of b, thereby establishing the multiplicative model. Details of the argument can be found in Miyamoto (1992). Miyamoto and Eraker (1988) stated this result, referring the proof of Miyamoto's (1985) dissertation. Miyamoto *et al.* (1998) pointed out that the zero condition and the invariance of probability equivalents[1] were sufficient for the multiplicative QALY model. This result did not require the assumptions of sign dependence with respect to health state and restricted solvability because the invariance of probability equivalents is logically stronger than the invariance of 50/50 certainty equivalents.

Next we will axiomatize the general power model (7) and the general exponential models (8). To do this, we require some notation. For any $c \in \mathscr{Q}$ let $\mathscr{H}/c = c \times \mathscr{Y} = \{cx \text{ such that } x \in \mathscr{Y}\}$, and let $\mathscr{L}/c = \{(cx_1, p_1; \ldots; cx_n, p_n) \in \mathscr{L}\}$. For any $c \in \mathscr{Q}$ let $U \mid c\colon \mathscr{Y} \to \mathbb{R}$ be the function determined by the condition: for $x \in \mathscr{Y}$, $(U \mid c)(x) = U(cx)$. Finally, if $g = (cx_1, p_1; \ldots; cx_n, p_n) \in \mathscr{L}/c$, $s \in \mathbb{R}^+$, and $t \in \mathbb{R}$, define the opera-

[1] Invariance of probability equivalents is referred to as standard gamble invariance in Miyamoto *et al.* (1999).

tions $g \cdot s$ and $g+t$ by the conditions, $g \cdot s = [c(sx_1), p_1; ...; c(sx_n), p_n]$ and $g+t = [c(t+x_1), p_1; ...; c(t+x_n), p_n]$, respectively.

It is well known that power utility functions and exponential utility functions are characterized by invariants of the preference order. These invariants are defined here in the QALY framework.

Definition 6. Suppose that $c \in \mathscr{Q}$ is any health state. Then preferences for gambles in $\mathscr{L}/c$ are said to satisfy *constant risk posture* iff

$$g \succcurlyeq h \qquad \text{iff} \quad g+t \succcurlyeq h+t \tag{11}$$

for every $t \in \mathbb{R}$ and every $g, h \in \mathscr{L} \mid c$ such that $g+t, h+t \in \mathscr{L}/c$.

Definition 7. Suppose that $c \in \mathscr{Q}$ is any health state. Then preferences for gambles in $\mathscr{L}/c$ are said to satisfy *constant proportional risk posture* iff

$$g \succcurlyeq h \qquad \text{iff} \quad g \cdot s \succcurlyeq h \cdot s \tag{12}$$

for every $s \in \mathbb{R}^+$ and every $g, h \in \mathscr{L} \mid c$ such that $g \cdot s, h \cdot s \in \mathscr{L} \mid c$.

As is well known, constant risk posture and constant proportional risk posture characterize, respectively, the exponential and power utility functions.

Lemma 2. *Suppose that $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ satisfies the EU axioms. Let U be a utility function, and suppose that the following conditions are satisfied.*

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *Preferences over gambles in $\mathscr{L} \mid c$ satisfy constant risk posture for $c \in \mathscr{Q}$.*

Then either $[U \mid c](x) = \alpha(c) \cdot \exp(\lambda(c) \cdot x) + \beta(c)$ or $[U \mid c](x) = \alpha(c) \cdot x + \beta(c)$ for some constants $\alpha(c)$, $\beta(c)$, and $\lambda(c)$ that depend in general on the value of c.

Lemma 3. *Suppose that $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ satisfies the EU axioms. Let U be a utility function, and suppose that the following conditions are satisfied.*

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *Preferences over gambles in $\mathscr{L} \mid c$ satisfy constant proportional risk posture for $c \in \mathscr{Q}$.*

Then $[U \mid c](x) = \alpha(c) \cdot x^{\theta(c)} + \beta(c)$ for some constants, $\theta(c) > 0$, and $\alpha(c), \beta(c) \in \mathbb{R}$ that depend in general on the value of c.

The proofs of Lemmas 2 and 3 are well known and will not be given here. Nevertheless they deserve a few comments. Pratt (1964) proposed a risk aversion measure, $-U''(x)/U'(x)$, and used properties of this measure to prove Lemmas 2 and 3 (see, also, the proofs in Keeney & Raiffa, 1976). This approach requires that second derivatives of the utility function exist everywhere. There is a different approach by means of functional equations that does not require the assumption that U is twice differentiable (see Aczel, 1966; Luce, 1959; and Pfanzagl, 1959). Miyamoto (1988) and Miyamoto and Wakker (1996) give the details of the functional equations proofs of Lemmas 2 and 3 under more general assumptions than the EU assumptions. Sign dependence and restricted solvability are assumed in Lemmas 2 and 3 because the continuity of $U \mid c$ is needed to solve the functional equation. If 0 were not in the domain of the utility function, the logarithmic function and power functions would be possible solutions to the conditions of Lemma 3, but these functions are excluded because 0 is in the domain, and the logarithm of 0 and 0 raised to a negative power are not a real numbers.

Lemmas 3 and 2 combine with the zero condition to yield representation theorems for the general power and general exponential models.

THEOREM 2. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant risk posture holds with respect to every* $c \in \mathscr{Q}$.

Then the general exponential model (8) *holds.*

THEOREM 3. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant proportional risk posture holds with respect to every* $c \in \mathscr{Q}$.

Then the general power model (7) *holds.*

Conditions (i), (ii), and (iii) of Theorems 2 and 3 are all necessary for the respective representations.

Theorems 2 and 3 allow utility curvature to vary from one health state to another. It is easy to show that this is incompatible with a multiplicative model (Keeney & Raiffa, 1976). To axiomatize the exponential multiplicative and power multiplicative models, it suffices to augment the assumptions of Theorems 2 and 3 with an assumption that excludes changes in utility curvature, e.g., with the assumption that 50/50 certainty equivalents are invariant under changes in health state.

THEOREM 4. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant risk posture holds with respect to every* $c \in \mathscr{Q}$.

(iv) 50/50 *certainty equivalents are invariant under changes in health state.*

Then the exponential multiplicative model (4) *holds.*

Theorem 5. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant proportional risk posture holds with respect to every* $c \in \mathscr{Q}$.

(iv) 50/50 *certainty equivalents are invariant under changes in health state.*

Then the power multiplicative model (3) *holds.*

Conditions (i), (ii), (iii), and (iv) of Theorems 4 and 5 are all necessary for their respective representations. To prove these theorems, one applies Theorem 1 to infer that $U(by) = H(b) \cdot F(x)$. Constant risk posture implies that F is linear or exponential, and constant proportional risk posture implies that F is a power function. Cher *et al.* (1997) stated and proved Theorem 4. Theorem 5 has not appeared before to my knowledge.

We can now axiomatize the linear QALY model (1). The key new assumption is risk neutrality.

Definition 8. For any $c \in \mathscr{Q}$ and any gamble $g \in \mathscr{L}/c$, let $EV(g)$ stand for the expected value of survival duration given g. Preferences for survival duration are said to be *risk neutral* iff for every $c \in \mathscr{Q}$ and $g \in \mathscr{L}/c$, $g \sim c[EV(g)]$.

It preferences are risk neutral in the sense of Definition 8, then the utility functions for survival duration are straight line. However, they need not have a common point of intersection (Bleichrodt *et al.*, 1997). Combining risk neutrality with the zero condition forces all of the utility functions to have a common point of intersection at duration 0.

Theorem 6. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ *satisfied the EU axioms, and the following conditions hold*:

(i) *The zero condition is satisfies.*

(ii) *Risk neutrality is satisfied.*

Then the linear QALY model (1) *holds.*

The zero condition and risk neutrality are both necessary for the linear QALY model. A proof of Theorem 6 is given in Bleichrodt *et al.* (1997).

Alternative EU Axiomatizations of QALY Models

Pliskin, Shepard, and Weinstein (1980) published the first axiomatization of the linear QALY model for constant health states. They also identified conditions that were sufficient for the multiplicative model and power multiplicative model, although they did not organize these conditions into separate representation theorems. Maas and Wakker (1994) approached the multiplicative QALY model from a rather different standpoint from other axiomatizations. I will discuss these developments and compare them to the axiomatizations of the previous section.

Alternative Axiomatizations of the Multiplicative QALY Model (5)

Pliskin *et al.* (1980) proposed that duration and health state satisfy a property called mutual utility independence. To define this property precisely, recall that $\mathscr{H}/c$ is a set of outcomes that vary only on duration, and that $\mathscr{L}/c$ is set of lotteries with outcomes in $\mathscr{H}/c$. To define mutual utility independence, we need the analogous sets for constant duration: For any $y \in \mathscr{Y}$, let $\mathscr{H}/y = \mathscr{Q} \times y = \{cy$ such that $c \in \mathscr{Q}\}$, and $\mathscr{L}/y = \{[c_1 y, p_1; ...; c_n y, p_n] \in \mathscr{L}\}$.

Definition 9. Survival duration is said to be *utility independent of health state* iff for every $c, d \in \mathscr{Q}$ the preference order over $\mathscr{L}/c$ is the same as the preference order over $\mathscr{L}/d$. Health state is said to be *utility independent of survival duration* iff for every $x, y \in \mathscr{Y}$, the preference order over $\mathscr{L}/x$ is the same as the preference order over $\mathscr{L}/y$. Survival duration and health state are said to be *mutually utility independent* iff each is utility independent of the other.

Utility independence is more constraining than invariance of probability equivalents or 50/50 certainty equivalents. If duration is utility independent of health state, there cannot be any equal-to-death or worse-than-death health states[2] in $\mathscr{Q}$. Moreover, if health state is utility independent of duration, then either the zero condition is rejected, or else $0 \notin \mathscr{Y}$. Thus, mutual utility independence of duration and quality is a rather severe limitation on the domain of a health utility model.

Raiffa (1969) and Keeney (1971) showed that if preferences satisfy EU axioms, and if two attributes are mutually utility independent, the utility function over these attributes must be additive or multiplicative. Fishburn (1965) pointed out that a utility function over multiple attributes is additive iff gambles are equally preferred whenever they have identical marginal probability distributions over attribute levels. I will call this the *marginality property* (also called, additive independence or value independence in some studies). As pointed out by Pliskin *et al.* (1980), it is doubtful whether preferences for QALYs satisfy the marginality property. They give the following example:

Gamble A: [(pain-free, 10 years), 1/2; (pain, 1 year), 1/2]

Gamble B: [(pain, 10 years), 1/2; (pain-free, 1 year), 1/2]

The marginality property implies that Gambles A and B should be equally preferred because in both gambles, the marginal probability of pain is 1/2, of no pain is 1/2, of 10 years is 1/2, and of 1 year is 1/2. Many people, however, prefer Gamble A over Gamble B (Weinstein, Pliskin, & Stason, 1977). Under EU assumptions, this observation is sufficient to reject an additive utility model (not so, however, under rank dependent utility). Thus, under EU assumptions, the mutual utility independence of survival duration and health state and the violation of marginality are sufficient to imply a multiplicative utility model. This result is summarized in the following theorem.

Theorem 7. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

[2] Assuming that some health status in $\mathscr{Q}$ are better than death.

(i) *Survival duration and health state are mutually utility independent.*

(ii) *The marginality property is violated.*

Then there exists a utility function U: $\mathscr{H} \to \mathbb{R}$, *and functions* J: $\mathscr{Q} \to \mathbb{R}$ *and* K: $\mathscr{Y} \to \mathbb{R}$ *such that* $U(bx) = J(b) \cdot K(x)$ *for all* $bx \in \mathscr{H}$, *such that either* $J(\mathscr{Q}) \subseteq \mathbb{R}^+$ *or* $-J(\mathscr{Q}^+) \subseteq \mathbb{R}^+$, *and either* $K(\mathscr{Y}) \subseteq \mathbb{R}^+$ *or* $-K(\mathscr{Y}) \subseteq \mathbb{R}^+$.

The requirement that $J(\mathscr{Q}) \subseteq \mathbb{R}^+$ or $-J(\mathscr{Q}) \subseteq \mathbb{R}^+$ implies that $\mathscr{Q}$ cannot contain equal-to-death health states, nor can it contain both better-than-death and worse-than-death health states. The requirement that $K(\mathscr{Y}) \subseteq \mathbb{R}^+$ or $-K(\mathscr{Y}) \subseteq \mathbb{R}^+$ implies that either $0 \notin \mathscr{Q}$ or the zero condition is not satisfied. Mutual utility independence (Condition (i)) is a necessary condition for a multiplicative representation in which sign changes and zeroes are excluded. The violation of the marginality property is also necessary provided that there exist $a, b \in \mathscr{Q}$ such that $ax \succ bx$ for some $x \in \mathscr{Y}$. A proof of Theorem 7 will not be given here because it is a straightforward consequence of well-known results (see Keeney and Raiffa (1976), Theorems 5.1 and 5.2, and the discussion in their Section 5.4.3.)

Maas and Wakker (1994) proposed a rather different axiomatization of the multiplicative QALY model. Up to now, I have not distinguished notationally between the riskless preference order over outcomes and the preference order over gambles. To be more clear in the present context, let us define a riskless preference relation $\gtrapprox$ *by*

$$ax \gtrapprox by \qquad \text{iff} \quad (ax, 1) \succcurlyeq (by, 1). \tag{13}$$

Thus $\gtrapprox$ is unambiguously a binary relation between outcomes, whereas $\succcurlyeq$ is a binary relation on gambles or outcomes. Suppose, first, that the riskless preference structure $(\mathscr{H}, \gtrapprox)$ satisfies the additive conjoint measurement axioms of Krantz *et al.* (1971, Chapter 6, Definition 2); then there exist functions, V: $\mathscr{H} \to \mathbb{R}$, V_1: $\mathscr{Q} \to \mathbb{R}$, and V_2: $\mathscr{Y} \to \mathbb{R}$ such that V preserves $\gtrapprox$ and

$$V(ax) = V_1(a) + V_2(x) \tag{14}$$

for all $a \in \mathscr{Q}$ and $x \in \mathscr{Y}$. Suppose, in addition, that $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ satisfied the EU axioms; then there exists a utility function U: $\mathscr{H} \to \mathbb{R}$ that preserves the preference order, $\succcurlyeq$, over $\mathscr{L}$. Because the riskless and risky preference orders coincide on outcomes, we must have $U = \phi \circ V$ for some strictly monotonic ϕ: $\mathbb{R} \to \mathbb{R}$ (Maas & Wakker, 1994). Maas and Wakker proved a general theorem which, when applied to QALY models, implies that if survival duration is utility independent of health quality, then ϕ must be linear or exponential. As they noticed, a linear π is excluded because marginality is implausible in the QALY domain. Therefore ϕ must be exponential. Because the exponential of an additive function is multiplicative, this argument yields a representation theorem for a multiplicative model. This result is summarized in the following theorem.

THEOREM 8. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

(i) $(\mathscr{H}, \succsim)$ *satisfies the additive conjoint measurement axioms in Krantz et al.* (1971, *Definition* 2).

(ii) *Survival duration is utility independent of health state.*

(iii) *Marginality is violated.*

Then there exists a utility function $U: \mathscr{H} \to \mathbb{R}$, *and functions* $J: \mathscr{Q} \to \mathbb{R}$ *and* $K: \mathscr{Y} \to \mathbb{R}$ *such that* $U(bx) = J(b) \cdot K(x)$ *for all* $bx \in \mathscr{H}$, *such that either* $J(\mathscr{Q}) \subseteq \mathbb{R}^+$ *or* $-J(\mathscr{Q}^+) \subseteq \mathbb{R}^+$, *and either* $K(\mathscr{Y}) \subseteq \mathbb{R}^+$ *or* $-K(\mathscr{Y}) \subseteq \mathbb{R}^+$.

Theorem 8 will not be proved here because it is an immediate consequence of a more general theorem proved in Maas and Wakker (1994; see their Theorem 3.2, and their Section 4). The Maas/Wakker axiomatization of the multiplicative QALY model is interesting because it constructs the model from the relations in the riskless preference structure, $(\mathscr{H}, \succsim)$, and then extends the representation to gamble. Its main limitations are the restrictions on the ranges of J and K, and the requirement imposed by the additive conjoint axioms that $\mathscr{Q}$ be infinite.

The assumptions of Theorems 7 and 8 force the ranges of J and K to be strictly positive or strictly negative, and under the intended interpretation, this would imply that all survival durations are strictly positive and all health states are better than death. These restrictions on the ranges of J and K can be relaxed if one replaces the utility independence assumptions with a form of generalized utility independence that was developed in Fishburn and Kenney (1975). In the case of Theorem 8, it is also necessary to replace the additive conjoint independence assumption with the sign dependence assumption described in Definition 3. Essentially, these modifications take into account the fact that multiplication by zeroes or negative values creates degenerate or inverse preference orders over lotteries and outcomes.

Overall, it should be clear that Theorem 1 is the simplest and most general axiomatization of the multiplicative QALY model (5). The zero condition is extremely plausible, and the invariance of probability equivalents or of 50/50 certainty equivalents are easy to test (Miyamoto & Eraker, 1988). Theorem 1 applies to health state domains that contain worse-than-death and equal-to-death health states as well as better-than-death health states, it allows for the possibility that $\mathscr{Q}$ is finite, and it allows duration 0 to be in the domain.

Alternative Axiomatization of the Power Multiplicative Model (3)

Pliskin *et al.* (1980) proposed a set of properties from which the power multiplicative model (3) can be derived. The key new property, called constant proportional time trade-off, is extremely important in health utility analysis because it is the basis for one of the most ubiquitous methods of health utility assessment.

DEFINITION 10. Let $c, b \in \mathscr{Q}$ be any health states that satisfy $cy \succcurlyeq by \succ c0$ with respect to some $y > 0$. A duration y^* is said to be the *time trade-off* for b with respect to c at duration y iff $cy^* \sim by$. If y^* is the time trade-off between b and c at duration y, then the *proportional time trade-off* is the ratio y^*/y. Time trade-off satisfy *constant proportional time trade-off* iff for every $c, b \in \mathscr{Q}$ if $cy \succcurlyeq by \succ c0$ with respect to some $y > 0$, then there exists a constant $K_{cb} \in \mathbb{R}^+$ such that $c(K_{cb} \cdot x) \sim bx$ for every $x \in \mathcal{Y}$.

Time trade-offs may be assumed to exist provided that utility is continuous in survival duration. The property of constant proportional time trade-off asserts that the proportional time trade-off is independent of the base duration y. Definition 10 applies only to better-than-death health states. One could formulate a more general definition that would encompass worse-than-death health states, but time trade-offs with respect to worse-than-death health states will be omitted because they are difficult to elicit.

In most applied work, time trade-offs are elicited with respect to a best health state $c^* \in \mathcal{Q}$ that is referred to as full or normal health. It is not hard to see that if c^* exists, and if proportional time trade-offs are constant with respect to c^*, then they must be constant between all pairs of better-than-death health states as stated in Definition 10. The following theorem shows that constant proportional time trade-off constrains the form of the utility of survival duration.

THEOREM 9. *Suppose that* $(\mathcal{L}, \mathcal{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration restricted solvability* (*Definition* 1).

(ii) *For any* $b \in \mathcal{Q}$ *and* $x, y \in \mathcal{Y}$, *if* $x > y$, *then* $bx \succ by$.

(iii) *For every* $x \in \mathcal{Y}$, *every* $b, c \in \mathcal{Q}$ *and every* $g \in \mathcal{L}$, *if* $bx \succcurlyeq g \succcurlyeq cx$, *then there exists* $d \in \mathcal{Q}$ *such that* $dx \sim g$. (*Restricted solvability with respect to health state.*)

(iv) *Duration and health state are mutually utility independent.*

(v) *The marginality property is violated.*

(vi) *Constant proportional time trade-off is satisfied.*

There exists a utility function $U: \mathcal{H} \to \mathbb{R}$, *a function* $J: \mathcal{Q} \to \mathbb{R}$, *and* $\theta \in \mathbb{R} - \{0\}$ *such that* $U(bx) = J(b) \cdot x^\theta$ *for all* $bx \in \mathcal{H}$. *Furthermore, if* $\theta > 0$, *then* $J(\mathcal{Q}) \subseteq \mathbb{R}^+$, *and if* $\theta < 0$, *then* $-J(\mathcal{Q}^+) \subseteq \mathbb{R}^+$.

The proof of Theorem 9, which is given in the Appendix, is based on a functional equation argument. Pliskin *et al.* proved a theorem that is close to Theorem 9 except that they did not require that marginality be violated nor did they require the structural assumptions (i)–(ii) (see Pliskin *et al.*, 1980, pp. 212–213). Instead of the structural assumptions, Pliskin *et al.* assumed that K is twice differentiable, and that $\lim_{y \to 0}(-y \cdot K''(y)/K'(y))$ exists. The effect of allowing marginality is that it allows for the possibility that survival duration and health state combine additively. If they combine additively, constant proportional tradeoff implies that the utility of survival duration is logarithmic. As also noted by Pliskin *et al.*, there is reason to suspect that marginality fails and hence, additivity and the logarithmic utility function can be excluded.

Theorem 9 and the similar result proved in Pliskin *et al.* (1980) are limited in their domain of application because mutual utility independence excludes the possibility that $0 \in \mathcal{Y}$, and that $\mathcal{Q}$ contains worse than death health states as well as better than death health states. Theorem 9 is also limited by the fact it implies that the set of health states $\mathcal{Q}$ is infinite.

Alternative Axiomatization of the Linear QALY Model (1)

Evidently, if the assumptions of Theorem 9 are sufficient for the power multiplicative model, then one can axiomatize the linear QALY model by the addition of the assumption of risk neutrality. If we add risk neutrality, however, we can eliminate some of the assumptions because they are redundant.

THEOREM 10. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration and health state are mutually utility independent.*

(ii) *Constant proportional time trade-off is satisfied.*

(iii) *Risk neutrality is satisfied.*

Then there exists a utility function $U\colon \mathscr{H} \to \mathbb{R}$, *and a function* $J\colon \mathcal{Q} \to \mathbb{R}$ *such that* $U(bx) = J(b) \cdot x$ *for all* $bx \in \mathscr{H}$. *Furthermore, either* $J(\mathcal{Q}) \subseteq \mathbb{R}^+$, *or* $-J(\mathcal{Q}^+) \subseteq \mathbb{R}^+$.

A proof of Theorem 10 is given in the Appendix. Theorem 10 is the axiomatization of the linear QALY model that was proposed by Pliskin *et al.* (1980), except that they neglected to point out that if health state is utility independent of survival duration, then 0 cannot be included among the possible durations. As previously noted, this limitation could be lifted by replacing mutual utility independent with the assumption that duration and health state are generalized utility independent of each other

Summary of Axiomatizations under EU Assumptions

Multiplicative Model (5)

Theorem 1

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. 50/50 certainty equivalents are invariant under changes in health state (Def. 5).

Theorem 7[3]

1. Survival duration and health state are mutually utility independent.
2. Marginality is violated.

Theorem 8[3]

1. $(H, \gtrsim)$ satisfies the additive conjoint measurement axioms.

[3] These axioms are inconsistent with the zero condition. Therefore they imply that the zero condition is violated or $0 \notin \mathscr{Y}$.

2. Survival duration is utility independent of health state.
3. Marginality is violated.

General Exponential Model (8)

Theorem 2

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. Zero condition (Def. 2).
4. Constant risk posture (Def. 6).

General Power Model (7)

Theorem 3

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. Zero condition (Def. 2).
4. Constant proportional risk posture (Def. 7).

Exponential Multiplicative Model (3)

Theorem 4

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. Constant risk posture (Def. 6).
5. 50/50 certainty equivalents are invariant under changes in health state (Def. 5).

Power Multiplicative Model (3)

Theorem 5(i)

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. Zero condition (Def. 2).
4. Constant proportional risk posture (Def. 7).
5. 50/50 certainty equivalents are invariant under changes in health state (Def. 5).

Theorem 9[4]

1. Restricted solvability with respect to survival duration (Def. 1).

[4] These axioms are inconsistent with the zero condition. Therefore they imply that the zero condition is violated or $0 \notin \mathscr{Y}$.

2. For any $b \in \mathcal{Q}$ and $x, y \in \mathcal{Y}$, if $x > y$, then $bx \succ by$.
3. Restricted solvability with respect to health state.
4. Duration and health quality are mutually utility independent (Def. 9).
5. The marginality property is violated.
6. Constant proportional time trade-off (Def. 10).

Linear QALY Model (1)

Theorem 6

1. Zero condition (Def. 2).
2. Risk neutrality (Def. 4).

Theorem 10

1. Survival duration and health state are mutually utility independent (Def. 9).
2. Constant proportional time tradeoff (Def. 10).
3. Risk neutrality (Def. 8).

RDU AXIOMATIZATIONS OF QALY UTILITY MODELS

Rank-dependent utility (RDU) theory is a successful attempt to incorporate a nonlinear transformation of probabilities into a model of decision under risk. Whereas utility theories had been proposed in which the probabilities of lottery outcomes were subjected to a nonlinear transformation (Edwards, 1954; Handa, 1977), these theories were found to violate stochastic dominance (Fishburn, 1978; Quiggin, 1982). Quiggin's (1982) anticipated utility theory showed how to transform probabilities without violating stochastic dominance. RDU theory is essentially anticipated utility theory after removing a constraint that the transformed weight of a 0.5 probability be 0.5. Although RDU theory has been developed primarily as a model of preferences for monetary lotteries, it is straightforward to translate the RDU representation to the QALY utility framework.

Like EU theory, RDU theory proposes that there exists a utility function $U: \mathcal{L} \to \mathbb{R}$ that preserves the preference order over lotteries:

(i) If $f, g \in \mathcal{L}$, then $f \succcurlyeq g$ iff $U(f) \geqslant U(g)$;

To evaluate the utility of lotteries, RDU theory proposes that probabilities are transformed to decision weights by a process that takes into account the decumulative[5] probability distribution of a lottery. To define this process, let $\mathcal{L}\downarrow = \{(a_1x_1, p_1; ...; a_nx_n, p_n) \in \mathcal{L}$ such that $a_1x_1 \succcurlyeq \cdots \succcurlyeq a_nx_n\}$. RDU theory proposes that there exists a transformation $w: [0, 1] \to [0, 1]$ such that w is continuous, $w(0) = 0$, $w(1) = 1$, and:

(ii) If $g = (a_1x_1, p_1; ...; a_nx_n, p_n) \in \mathcal{L}\downarrow$, then

$$U(g) = \sum_{i=1}^{n} \left[w\left(\sum_{k=0}^{i} p_k \right) - w\left(\sum_{k=0}^{i-1} p_k \right) \right] \cdot U(a_ix_i), \tag{15}$$

[5] The *decumulative* probability distribution is one minus the cumulative probability distribution.

where $p_0 = 0$ by convention. If $g \in \mathscr{L} - \mathscr{L}\downarrow$, then $U(g)$ is determined by permuting g into a decreasing preference order, and applying (15) to the permutation of g.

The function w is called the *weighting* function, and the *decision weights* are the differences,

$$w\left(\sum_{k=0}^{i} p_k\right) - w\left(\sum_{k=0}^{i-1} p_k\right),$$

in the transformed decumulative distribution. For the special case of a binary lottery, $(bx, p; bz, 1-p)$, Eq. (15) reduces to

$$U(bx, p; bz, 1-p) = w(p)\, U(bx) + (1 - w(p))\, U(bz). \tag{16}$$

The uniqueness result for the RDU representation asserts that w is unique, and U is an interval scale.

(iii) If $U^*: \mathscr{L} \to \mathbb{R}$ and $w^*: [0, 1] \to [0, 1]$ are any other functions that satisfy (i) and (ii), then $w^* = w$, and there exist $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that $U^* = \alpha \cdot U + \beta$.

All of the representation theorems in this section assume that $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ satisfies an axiom system that is sufficient to guarantee (i), (ii), and (iii). The formal basis for the RDU representation is discussed in Luce (1988), Nakamura (1992), Quiggin (1982, 1993), Quiggin and Wakker (1994), and Wakker (1994). For the sake of explicitness, I will assume that the RDU representation is axiomatized in the manner of Wakker (1994, Theorem 12). I will also assume that w is continuous. Although Wakker's (1994) axiomatization allows for discontinuities in w, he states a simple condition that implies continuity of w in the context of his axiomatization. Psychological interpretations of this representation are given in Birnbaum and Sutton (1992), Wakker, Erev, and Weber (1994), Weber (1996), and Weber and Kirsner (1997).

RDU Axiomatizations of QALY Models Based on the Zero Condition

Most of the EU axiomatizations that were based on the zero condition carry the same implications under RDU assumptions because the proofs are based on functional equations arguments that depend primarily on the interval scale uniqueness of the utility scale. The axiomatizations of the multiplicative QALY model (5) and the linear QALY model (1) are exceptions to this assertion. The RDU axiomatization of the linear QALY model must modify the EU criterion for risk neutrality because the EU definition of risk neutrality is predicated on the assumption that utility is linear in probability. The RDU axiomatization of the multiplicative QALY model is made more complex by the fact that the rank order of preference over durations is inverted when health state is changed from a better-than-death state to a worse-than-death state. This latter point will be taken up first.

Under EU assumptions, the multiplicative QALY model implies the invariance of probability equivalents under changes in health state. The same implication does not hold under RDU assumptions because probability weights are dependent on the rank order of the outcomes. To see the difficulty, suppose that b is better-than-death, $x > y > z$, and that model (5) obtains. If $(bx, p; bz, 1-p) \sim by$, then by (5) and (16), we have $U(bx, p; bz, 1-p) = w(p)\, H(b)\, F(x) + (1 - w(p))\, H(b)\, F(z) = H(b)\, F(y) = U(by)$. If c is worse-than-death, then $cx \prec cz$, so $U(cx, p; cz, 1-p) = [1 - w(1-p)]\, H(c)\, F(x) + w(1-p)\, H(c)\, F(z) \neq H(c)\, F(y) = U(cy)$, where the last inequality obtains whenever $w(p) + w(1-p) \neq 1$. Therefore, probability equivalents are invariant under changes from better-than-death to worse-than-death health states only in the special case where $w(p) + w(1-p) = 1$ for all p.

Obviously there is no problem if we restrict attention to better-than-death health states only, or to worse-than-death health states only. Therefore, it is straightforward to axiomatize the multiplicative model for health states that are exclusively better than death, or exclusively worse than death. The problem will be to link these two separate multiplicative representations. The following definition states a restricted version of invariance of probability equivalents and of certainty equivalents that is consistent with the multiplicative QALY model under RDU assumptions.

DEFINITION 11. Probability equivalents will be said to be *invariant under same valence changes in health state* iff for any $b, c \in \mathcal{Q}^+$ or $b, c \in \mathcal{Q}^-$, the following holds:

$$(bx, p; bz, 1-p) \sim by \qquad \text{iff} \quad (cx, p; cz, 1-p) \sim cy \tag{17}$$

for every $x, y, z \in \mathcal{Y}$. 50/50 certainty equivalents will be said to be *invariant under same valence changes in health state* iff (17) holds for $p = 1/2$.

Next we state a lemma that derives a key functional equation separately for better-than-death and worse-than-death health states.

LEMMA 4. *Let $(\mathcal{L}, \mathcal{H}, \succcurlyeq)$ satisfy the RDU axioms, and let U be a utility function. Suppose that survival duration satisfies restricted solvability and is sign dependent on health state. Then, the following three conditions are equivalent.*

(i) *Probability equivalents are invariant under same valence changes in health state.*

(ii) 50/50 *certainty equivalents are invariant under same valence changes in health state.*

(iii) *For $i = +, -$, there exist functions $\alpha^i: \mathcal{Q}^i \to \mathbb{R}$, $\beta^i: \mathcal{Q}^i \to \mathbb{R}$, and $F^i: \mathcal{Y} \to \mathbb{R}$ such that $U(b, y) = \alpha^i(b) \cdot F^i(y) + \beta^i(b)$ for every $b \in \mathcal{Q}^i$ and $y \in \mathcal{Y}$.*

A proof of Lemma 4 is given in the Appendix. Miyamoto (1988) proved similar results for the utility independence of one attribute from another, but utility independence excludes the existence of zeroes or worse-than-death health states. As in the EU section, the remainder of this section will use condition (ii) in its axiomatizations, but (i) could replace it without altering the conclusions of theorems.

Lemma 4 provides the basis for separate multiplicative representations for better-than-death and worse-than-death health states. The following assumption is designed to force the utility scales for duration in better-than-death and worse-than-death health states to be linear with respect to each other.

DEFINITION 12. The structure $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ will be said to satisfy *the interlocking condition* iff for any $b \in \mathscr{Q}^+$ and $c \in \mathscr{Q}^-$ and for any $(bw, p; bs, 1-p)$, $(bx, p; bt, 1-p)$, $(by, p; bs, 1-p)$, $(bz, p; bt, 1-p)$, $(cu, q; cw, 1-q)$, $(cv, q; cx, 1-q) \in \mathscr{L}\downarrow$, if

$$(bw, p; bs, 1-p) \sim (bx, p; bt, 1-p), \tag{18}$$

$$(by, p; bs, 1-p) \sim (bz, p; bt, 1-p) \tag{19}$$

and

$$(cu, q; cw, 1-q) \sim (cv, q; cx, 1-q) \tag{20}$$

then $(cu, q; cy, 1-q)$, $(cv, q; cz, 1-q) \in \mathscr{L}\downarrow$, and

$$(cu, q; cy, 1-q) \sim (cv, q; cz, 1-q). \tag{21}$$

To understand the purpose of the interlocking condition, suppose that we have possibly different multiplicative representations for better-than-death and worse-than-death health states: $U(bx) = H(b)\, F(x)$ for $b \in \mathscr{Q}^+$, and $U(cx) = J(c)\, K(x)$ for $c \in \mathscr{Q}^-$. The interlocking condition implies that K is linear with respect to F. To see this, let $b \in \mathscr{Q}^+$ and $c \in \mathscr{Q}^-$. To simplify the notation, let $P = w(p)$, $Q = w(q)$, $h = H(b)$, and $j = J(c)$. In this notation, (18), (19), (20), and (21) imply that

$$P \cdot h \cdot F(w) + (1-P) \cdot h \cdot F(s) = P \cdot h \cdot F(x) + (1-P) \cdot h \cdot F(t) \tag{22}$$

$$P \cdot h \cdot F(y) + (1-P) \cdot h \cdot F(s) = P \cdot h \cdot F(z) + (1-P) \cdot h \cdot F(t) \tag{23}$$

$$Q \cdot j \cdot K(u) + (1-Q) \cdot j \cdot K(w) = Q \cdot j \cdot K(v) + (1-Q) \cdot j \cdot K(x) \tag{24}$$

$$Q \cdot j \cdot K(u) + (1-Q) \cdot j \cdot K(y) = Q \cdot j \cdot K(v) + (1-Q) \cdot j \cdot K(z). \tag{25}$$

Therefore

$$P \cdot [F(w) - F(x)] = (1-P) \cdot [F(t) - F(s)] = P \cdot [F(y) - F(z)], \tag{26}$$

and

$$(1-Q) \cdot [K(w) - K(x)] = Q \cdot [K(v) - K(u)] = (1-Q) \cdot [K(y) - K(z)]. \tag{27}$$

Equations (26) and (27) show that if $F(w) - F(x) = F(y) - F(z)$, then $K(w) - K(x) = K(y) - K(z)$. If the structure $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$ has sufficiently many lotteries like (18), (19), (20), and (21) with which to compare the magnitudes of intervals on the separate scales, then one can show that F and K are linear with respect to each other. It turns out that this is possible if F and K are continuous. The proof starts by showing that F and K are continuous, and then proceeds to show that standard sequences on F correspond to standard sequences on K. The RDU representation theorem for the multiplicative QALY model follows:

THEOREM 11. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *is a RDU structure, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) 50/50 *certainty equivalents are invariant under same valence changes in health state.*

(iv) *The interlocking condition is satisfied.*

Then the multiplicative QALY model (5) *is satisfied.*

The proof of Theorem 11 is given in the Appendix. Miyamoto (1992) and Miyamoto and Wakker (1996) proved similar theorems for better-than-death health states only. The use of an interlocking condition to bind together the representations for better-than-death and worse-than-death health states is new.

The axiomatizations and proofs of the representation theorems for the general exponential model and the general power model are essentially identical to the corresponding EU theorems.

THEOREM 12. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the RDU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant risk posture holds with respect to every* $c \in \mathscr{Q}$.

Then the general exponential model (8) *holds.*

THEOREM 13. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the RDU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Constant proportional risk posture holds with respect to every* $c \in \mathscr{Q}$.

Then the general power model (7) *holds.*

The proof of Theorem 12 is sketched in the Appendix. The proof of Theorem 13 is analogous and will be omitted.

Once again, the axiomatizations of the power multiplicative and the exponential multiplicative model combine axioms for a multiplicative model with axioms for the general power and general exponential models, respectively. Redundant axioms are dropped.

THEOREM 14. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *satisfies the RDU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) 50/50 *certainty equivalents are invariant under same valence changes in health state.*

(iv) *The interlocking condition is satisfied.*

(v) *Constant risk posture holds with respect to every* $c \in \mathcal{Q}$.

Then the exponential multiplicative model (4) *holds.*

THEOREM 15. *Suppose that* $(\mathcal{L}, \mathcal{H}, \succcurlyeq)$ *satisfies the EU axioms, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) 50/50 *certainty equivalents are invariant under same valence changes in health state.*

(iv) *The interlocking condition is satisfied.*

(v) *Constant proportional risk posture holds with respect to every* $c \in Q$.

Then the power multiplicative model (3) *holds.*

The proof of Theorem 14 is straightforward. Conditions (i), (ii), (iii), and (iv) imply that the representation is multiplicative. Conditions (i) and (v) imply that the utility of survival duration is linear or exponential. The invariance of 50/50 certainty equivalents and the interlocking condition preclude the possibility that the exponential parameter could differ for different health states. The proof of Theorem 15 is analogous. It is interesting to note that if one omits the interlocking condition from these axiom systems, then utility functions for better-than-death and worse-than-death health states can have different curvature.

Finally, we will consider the axiomatization of the linear QALY model (1) under RDU assumptions. Bleichrodt and Quiggin (1997) identified a condition which, in combination with the zero condition, yields an axiomatization of the linear QALY model under RDU assumptions. Their analysis is developed under a more general form of RDU theory (state and rank dependent utility theory) than the version of RDU theory adopted here (RDU theory for preference under risk). I will present their axiomatization in the present RDU framework. It is easy to show that under RDU assumptions, the linear QALY model predicts the following relations: for any $b \in \mathcal{Q}$ that is not equal to death and any $bw, bx, by, bz \in \mathcal{Y}$ such that $bw \succcurlyeq bx$ and $by \succcurlyeq bz$,

$$(bw, p; bx, 1-p \sim (by, p; bz, 1-p) \tag{28}$$

iff

$$(b(w+\varepsilon), p; bx, 1-p) \sim (b(y+\varepsilon), p; bz, 1-p) \tag{29}$$

and

$$(bw, p; bx, 1-p) \sim (by, p; bz, 1-p) \tag{30}$$

iff

$$(bw, p; b(x+\varepsilon), 1-p) \sim (by, p; b(z+\varepsilon), 1-p). \tag{31}$$

The only restriction on these relations is that ε must be sufficiently small so that $w+\varepsilon, x+\varepsilon, y+\varepsilon, z+\varepsilon \in \mathscr{Y}$. It is not hard to show that under RDU assumptions, if (28) and (29) both hold, or if (30) and (31) both hold, then

$$U(bw) - U(by) = U[b(w+\varepsilon)] - U[b(y+\varepsilon)]. \tag{32}$$

Therefore (28)–(31) can be interpreted as a characterization of constant of marginal utility.

DEFINITION 13. The preference structure $(\mathscr{L}\downarrow, \mathscr{H}, \succsim)$ satisfies *constant marginal utility* iff for any $b \in \mathscr{Q}$ and $bw, bx, by, bz \in \mathscr{Y}$ such that $bw \succsim bx$ and $by \succsim bz$, (28) holds iff (29) holds, and (30) holds iff (31) holds.

Bleichrodt and Quiggin (1997) show that the equivalences, (28) and (29), and (30) and (31), imply that the utility of survival duration is linear with respect to b, and, in combination with the zero condition, yield an axiomatization of the linear QALY model.

LEMMA 5. *Suppose that* $(\mathscr{L}, \mathscr{H}, \succsim)$ *is an RDU structure, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) *Marginal utility is constant.*

Then the linear QALY model (1) *holds.*

The proof of Lemma 5 is given in Bleichrodt and Quiggin (1997).

The linear QALY model can also be characterized by an invariance of probability equivalents or of 50/50 certainty equivalents. If $x > y > z \geqslant 0$, let us call the ratio, $(y-z)/(x-z)$, the *proportional coverage* of the $[x, z]$ interval by y. Now suppose that $b \in \mathscr{Q}$ is better than death, that $x > y > z \geqslant 0$, and $(bx, p; bz, 1-p) \sim by$. By RDU theory and the linear QALY model, $w(p)\,x + (1-w(p))\,z = y$, so $w(p) = (y-z)/(x-z)$, or $p = w^{-1}[(y-z)/(x-z)]$. Therefore RDU theory and the linear QALY model imply that probability equivalents are a function of proportional coverage. Similarly, if we observe certainty equivalents, $by \sim (bx, 1/2; bz, 1/2)$ and $by' \sim (bx', 1/2; bz', 1/2)$, then $(y-z)/(x-z) = w(1/2) = (y'-z')/(x'-z')$, so linearity implies that 50/50 certainty equivalents always cover the same proportion of the ranges of their respective comparison lotteries. These relationships motivate the following definition.

DEFINITION 14. We say that *probability equivalents are a function of proportional coverage* iff for any $b \in \mathcal{Q}$ that is not equal to death, and any $x > y > z \geqslant 0$ and $x' > y' > z' \geqslant 0$, if $(bx, p; bz, 1-p) \sim by$, $(bx', q; bz', 1-q) \sim by'$, and

$$\frac{y-z}{x-z} = \frac{y'-z'}{x'-z'},$$

then $p = q$. We say that 50/50 certainty equivalents *cover a constant proportion of the lottery range* iff for any $b \in \mathcal{Q}$ that is not equal to death, and any $x > y > z \geqslant 0$ and $x' > y' > z' \geqslant 0$, if $by \sim (bx, 1/2; bz, 1/2)$ and $by' \sim (bx', 1/2; bz', 1/2)$, then $(y-z)/(x-z) = (y'-z')/(x'-z')$.

Either of these conditions can be combined with the zero condition to yield the linear QALY model.

LEMMA 6. *Suppose that* $(\mathcal{L}, \mathcal{H}, \succcurlyeq)$ *is an RDU structure, and the following conditions hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

(iii) 50/50 *certainty equivalents cover a constant proportion of the lottery range.*

Then the linear QALY model (1) *holds.*

The proof of Lemma 6 is given in the Appendix.

Lemmas 5 and 6 give alternative axiomatizations of the linear QALY model. These results are summarized in the following theorem.

THEOREM 16. *Suppose that* $(\mathcal{L}, \mathcal{H}, \succcurlyeq)$ *is an RDU structure, and suppose that* (i) *and* (ii) *hold*:

(i) *Survival duration satisfies restricted solvability and is sign dependent with respect to health state.*

(ii) *The zero condition holds.*

In addition, suppose that either (iii), (iii)$'$, *or* (iii)$''$ *holds*

(iii) *Marginal utility is constant.*

(iii)$'$ 50/50 *certainty equivalents cover a constant proportion of the lottery range.*

(iii)$''$ *Probability equivalents are a function of proportional coverage.*

Then linear QALY model (1) *holds.*

The sufficiency of (i), (ii) and (iii), and of (i), (ii), and (iii)$'$ was established in Lemmas 5 and 6. Conditions (i), (ii), and (iii)$''$ are sufficient because w is strictly monotonic, and therefore (iii)$''$ implies (iii)$'$.

Summary of Axiomatizations under RDU Assumptions

Multiplicative Model (5)

Theorem 11

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. 50/50 certainty equivalents are invariant under same valence changes in health state (Def. 11).
5. The interlocking condition (Def. 12).

General Exponential Model (8)

Theorem 12

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. Constant risk posture (Def. 6).

General Power Model (7)

Theorem 13

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. Constant proportional risk posture (Def. 7).

Exponential Multiplicative Model (4)

Theorem 14

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. 50/50 certainty equivalents are invariant under changes in same valence health state (Def. 11).
5. The interlocking condition (Def. 12).
6. Constant risk posture (Def. 6).

Power Multiplicative Model (3)

Theorem 15

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).
4. 50/50 certainty equivalents are invariant under changes in same valence health state (Def. 11).
5. The interlocking condition (Def. 12).
6. Constant proportional risk posture (Def. 7).

Linear QALY Model (1)

Theorem 16

1. Restricted solvability with respect to survival duration (Def. 1).
2. Survival duration is sign dependent on health state (Def. 3).
3. The zero condition (Def. 2).

(4), (4′), or (4″)

4. Constant marginal utility (Def. 13).

4′. 50/50 certainty equivalents cover a constant proportion of the lottery range (Def. 14).

4″. Probability equivalents are a function of proportional coverage (Def. 14).

SUMMARY AND CONCLUSIONS

QALY utility models are increasingly important in health decision analysis because of the obvious need to incorporate quality as well as quantity of survival in the evaluation of health outcomes. The discussion of EU axiomatizations attempted to show that the axiomatizations of the main QALY models that are currently under investigation can be greatly simplified if one adopts the zero condition in the axiomatizations. The assumption that probability equivalents or 50/50 certainty equivalents are invariant under changes in health state also contributes to simpler QALY axiomatizations. The resulting axiomatizations are generally much simpler than earlier axiomatizations that did not employ these postulates. The EU axiomatizations are of theoretical interest, especially if one were to attempt a normative argument in favor of a QALY utility model, but they are suspect from a descriptive standpoint.

The RDU axiomatizations of QALY utility models are generally rather similar to the EU axiomatizations. Differences between the RDU and EU axiomatizations appeared for two reasons. First, some EU axioms make essential use of linearity in probability, and thus, no longer have the same implications in the RDU framework. Risk neutrality (Definition 8) is an example of such an axiom. If probabilities are transformed nonlinearly as in RDU theory, it is no longer the case that a linear utility function implies that the certainty equivalent of a gamble is its expected value. Marginality is another such property. Whereas violations of marginality can be used to exclude an additive model under EU assumptions, an RDU representation could produce violations of marginality even if the utility model were actually additive. Miyamoto (1988, Corollary 2) identified an alternative property that distinguishes additive from multiplicative models under RDU assumptions. Thus, one characteristic difference between multiattribute utility theory under EU and RDU assumptions is that in the latter framework, one must drop or revise assumptions that make essential use of linearity in probability.

Second, some EU axioms require modification because they assume symmetries that no longer obtain in the RDU framework when the rank order of outcomes shifts. The best example of this phenomenon is the fact that under EU assumptions, a multiplicative QALY model implies that probability equivalents are invariant under changes in health state, whereas under RDU assumptions, the invariance only holds for changes between better-than-death health states or between worse-than-death health states. The reason for this is that if b is better-than-death and c is worse-than-death, the first outcome in $(bx, p; by, 1-p)$ is the superior outcome if and only if the second outcome in $(cx, p; by, 1-p)$ is the superior outcome. Because the multiplicative QALY model does not imply that probability equivalents should be invariant under changes from better-than-death to worse-than-death health states, it was necessary to introduce an additional, interlocking condition in the axiomatizations of the multiplicative QALY model (5), the power multiplicative model (3) and the power exponential model (4). Intriguingly, dropping this condition would lead to a multiplicative utility model with different utility curvatures in the domains of better-than-death and worse-than-death health states.

Because of the central role of QALY models in health utility analysis, it is imperative that we achieve a better understanding of descriptive preference theory in this domain. Miyamoto and Eraker (1988, 1989) attempted to test some of the assumptions discussed in this paper under RDU and prospect theoretic assumptions, but much more research is needed along these lines. The purpose of the present paper is to direct attention to some of the critical properties that distinguish among the various candidate QALY models, but much more work needs to be done. One question which deserves exploration is the generalization of QALY models to cumulative prospect theory (Tversky & Kahneman, 1992; Wakker & Tversky, 1993) or what has been called a rank and sign dependent utility theory (Luce & Fishburn, 1991), i.e., to theories that distinguish between the domains of gains and of losses. Such theories have different utility representations for lotteries whose outcomes are exclusively in the domain of gains, exclusively in the domain of losses, or mixtures of gains and losses. The utility representations posited in these theories reduce to RDU theory in the domain of gains and in the domain of losses, but differ for lotteries that mix gains and losses. Therefore the RDU QALY representation theorems presented in this paper are applicable under cumulative prospect theory assumptions or under rank and sign dependent utility assumptions if the outcomes are exclusively gains or exclusively losses, but new models and axiomatizations must be developed for the case where outcomes include both gains and losses.

APPENDIX

Proofs of Lemmas and Theorems

Proof of Lemma 1. The proofs that (i) implies (ii) and that (iii) implies (i) are straightforward and will be omitted. Assume Lemma 1(ii) and let us prove that Lemma 1(iii) holds. For any $b \in \mathscr{Q}$ let $U(b, \bullet)$ be the conditional utility function on duration. If b is equal to death, then $U(b, \bullet)$ is constant, and hence, continuous. If b is not equal to death, then sign dependence implies that $U(b, \bullet)$: $\mathscr{Y} \to \mathbb{R}$ is either strictly increasing or strictly decreasing. Restricted solvability (Definition 1) implies that $U(b, \bullet)$ is onto its range. Therefore $U(b, \bullet)$ is continuous for all $b \in \mathscr{Q}$.

Choose any $b \in \mathscr{Q}$ that is not equal to death, and define a binary relation, $\succcurlyeq_b$, on $\mathscr{Y} \times \mathscr{Y}$ by the condition: for any wx, $yz \in \mathscr{Y} \times \mathscr{Y}$, $wx \succcurlyeq_b yz$ iff $(bw, 1/2; bx, 1/2) \succcurlyeq (by, 1/2; bz, 1/2)$. Because U is a utility function for $(\mathscr{L}, \mathscr{H}, \succcurlyeq)$, we have for every wx, $yz \in \mathscr{Y} \times \mathscr{Y}$,

$$wx \succcurlyeq_b yz \qquad \text{iff} \quad 0.5 \cdot U(bw) + 0.5 \cdot U(bx) \geqslant 0.5 \cdot U(by) + 0.5 \cdot U(bz) \tag{33}$$

Equation (33) implies that the structure $(\mathscr{Y} \times \mathscr{Y}, \succcurlyeq_b)$ satisfies all of the necessary axioms for an additive conjoint structure as defined in Krantz *et al.* (1971, Chapter 6). The only non-necessary axiom, the additive conjoint restricted solvability assumption, is also satisfied because $U(b, \bullet)$ is continuous. Therefore, for any $b \in \mathscr{Q}$ that is not equal to death, $U(b, \bullet)$ is an interval scale representation for $(\mathscr{Y} \times \mathscr{Y}, \succcurlyeq_b)$.

I claim that if $b, c \in \mathscr{Q}^+$ or $b, c \in \mathscr{Q}^-$, we have $wx \succcurlyeq_b yz$ iff $wx \succcurlyeq_c yz$ for all wx, $yz \in \mathscr{Y} \times \mathscr{Y}$, and if $b \in \mathscr{Q}^+$ and $c \in \mathscr{Q}^-$, we have $wx \succcurlyeq_b yz$ iff $wx \preccurlyeq_c yz$ for all wx, $yz \in \mathscr{Y} \times \mathscr{Y}$. To see this, note first that for any $bw, bx \in \mathscr{H}$, restricted solvability implies that there must exist $by \in \mathscr{H}$ such that $(bw, 1/2; bx, 1/2) \sim bz$. Furthermore z is unique if b is not equal to death (sign dependence). Define a function λ: $\mathscr{Q} \times \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$ by $\lambda(b, w, x) = 0$ if b is equal to death, and $\lambda(b, w, x) = z$ if b is not equal to death and $(bw, 1/2; bx, 1/2) \sim bz$. Sign dependence and the invariance of 50/50 certainty equivalents implies that if $b, c \in \mathscr{Q}^+$ or $b, c \in \mathscr{Q}^-$, we have $wx \succcurlyeq_b yz$ iff $(bw, 1/2; bx, 1/2) \sim b\lambda(b, w, x)$ and $(by, 1/2; bz, 1/2) \sim b\lambda(b, y, z)$ and $b\lambda(b, w, x) \succcurlyeq b\lambda(b, y, z)$ iff $(cw, 1/2; cx, 1/2) \sim c\lambda(c, w, x)$ and $(cy, 1/2; cz, 1/2) \sim c\lambda(c, y, z)$ and $c\lambda(c, w, x) \succcurlyeq c\lambda(c, y, z)$ iff $wx \succcurlyeq_c yz$. A similar argument shows that if $b \in \mathscr{Q}^+$ and $c \in \mathscr{Q}^-$, we have $wx \succcurlyeq_b yz$ iff $wx \preccurlyeq_c yz$ for all wx, $yz \in \mathscr{Y} \times \mathscr{Y}$.

Now choose any $c \in \mathscr{Q}$ that is not equal to death, and define F: $\mathscr{Y} \to \mathbb{R}$ by $F(y) = U(cy)$ for any $y \in \mathscr{Y}$. For any $b \in \mathscr{Q}$ that is not equal to death, we either have

$$wx \succcurlyeq_b yz \quad \text{iff } wx \succcurlyeq_c yz \quad \text{iff } 0.5 \cdot F(w) + 0.5 \cdot F(x) \geqslant 0.5 \cdot F(y) + 0.5 \cdot F(z)$$

or we have

$$wx \succcurlyeq_b yz \quad \text{iff } wx \preccurlyeq_c yz \quad \text{iff } 0.5 \cdot F(w) + 0.5 \cdot F(x) \leqslant 0.5 \cdot F(y) + 0.5 \cdot F(z).$$

Therefore, either F and $U(b, \bullet)$ both represent the additive ordering of $(\mathscr{Y} \times \mathscr{Y}, \succcurlyeq_b)$, or $-F$ and $U(b, \bullet)$ both represent the additive ordering of $(\mathscr{Y} \times \mathscr{Y}, \succcurlyeq_b)$. In either case, there exist $\alpha(b), \beta(b) \in \mathbb{R}$, which depend in general on b, such that $U(bx) = \alpha(b) \cdot F(x) + \beta(b)$ for all $x \in \mathscr{Y}$. We have only derived this equation for b that are not equal to death, but if b is equal to death, then set $\alpha(b) = 0$. Hence $U(bx) = \alpha(b) \cdot F(x) + \beta(b)$ for all $b \in \mathscr{Q}$ and all $x \in \mathscr{Y}$. This proves that Lemma 1(ii) implies Lemma 1(iii). Q.E.D

Proof of Theorem 2. By Lemma 2, for every $c \in \mathscr{Q}$ either $[U \mid c](x) = \alpha(c) \cdot \exp(\lambda(c) \cdot x) + \beta(c)$ or $[U \mid c](x) = \alpha(c) \cdot x + \beta(c)$. Define a new function $\beta^*\colon \mathscr{Q} \to \mathbb{R}$ by the conditions:

$$\beta^*(c) = \beta(c) \qquad \text{if} \quad \alpha(c) = 0 \quad \text{or} \quad [U \mid c](x) = \alpha(c) \cdot x + \beta(c);$$

$$\beta^*(c) = \beta(c) + \alpha(c) \qquad \text{if} \quad [U \mid c](x) = \alpha(c) \cdot \exp(\lambda(c) \cdot x) + \beta(c).$$

Then for every $c \in \mathscr{Q}$ either $[U \mid c](x) = \alpha(c) \cdot [\exp(\lambda(c) \cdot x) - 1] + \beta^*(c)$ or $[U \mid c](x) = \alpha(c) \cdot x + \beta^*(c)$. Note that $[U \mid c](0) = \beta^*(c)$ for any c. The zero condition implies that for any $b, c \in \mathscr{Q}$ $\beta^*(b) = [U \mid b](0) = [U \mid c](0) = \beta^*(c)$. Therefore β^* is a constant. Redefine U as $U - \beta^*(c)$. Then for every $c \in \mathscr{Q}$ $U(cx) = \alpha(c) \cdot [\exp(\lambda(c) \cdot x) - 1]$ or $U(cx) = \alpha(c) \cdot x$. Q.E.D

Proof of Theorem 3. By Lemma 3, $[U \mid c](x) = \alpha(c) \cdot x^{\theta(c)} + \beta(c)$ for every $c \in \mathscr{Q}$. The zero condition implies that $U(c0) = U(b0)$. But then $\beta(c) = \alpha(c) \cdot 0^{\theta(c)} + \beta(c) = \alpha(b) \cdot 0^{\theta(c)} + \beta(b) = \beta(b)$, so β is a constant. Redefine U as $U - \beta(c)$. Then $U(cx) = \alpha(c) \cdot x^{\theta(c)}$ for every $c \in \mathscr{Q}$ and $x \in \mathscr{Y}$. Q.E.D

Proof of Theorem 9. By Theorem 7, there exists $J\colon \mathscr{Q} \to \mathbb{R}^+$ and $K\colon \mathscr{Y} \to \mathbb{R}^+$ such that

$$U(by) = J(b) \cdot K(y) \tag{34}$$

for every $b \in \mathscr{Q}$ and $y \in \mathscr{Y}$. Condition (ii) implies that survival duration is sign dependent on health state (special case where $\mathscr{Q}^0 = \mathscr{Q}^- = \varnothing$). As shown in the proof of Lemma 1, K must be continuous. Therefore K is onto an interval of real numbers. By condition (iii), J must also be onto an interval of real numbers.

Now choose any $\bar{b}, \underline{b} \in \mathscr{Q}$ and $\bar{y}, \underline{y} \in \mathscr{Y}$ such that $J(\bar{b})$, $J(\underline{b})$, $K(\bar{y})$, and $K(\underline{y})$ and neither maximal or minimal in $J(\mathscr{Q})$ and $K(\mathscr{Y})$, respectively, and $J(\bar{b}) > J(\underline{b})$, $K(\bar{y}) > K(\underline{y})$, and $J(\bar{b}) \cdot K(\underline{y}) = J(\underline{b}) \cdot K(\bar{y})$. Let $\Omega = (\bar{y}, \underline{y})$, and let $\Gamma = \{t \in \mathbb{R}^+$ such that $t \cdot \bar{y} \in \mathscr{Y}$ and $t \cdot \underline{y} \in \mathscr{Y}\}$. Γ is nonempty because $1 \in \Gamma$, and there must exist an $\varepsilon > 0$ such that $(1 + \varepsilon, 1 - \varepsilon) \subseteq \Gamma \subseteq K(\mathscr{Y})$ because $K(\bar{y})$, and $K(\underline{y})$ are neither maximal or minimal in $K(\mathscr{Y})$. Choose any $w, x \in \Omega$. By the choice of $\bar{b}$, $\underline{b}$, $\bar{y}$, and $\underline{y}$, we must have $J(\bar{b})/J(\underline{b}) = K(\bar{y})/K(\underline{y}) > K(w)/K(x) > K(\underline{y})/K(\bar{y}) = J(\underline{b})/J(\bar{b})$. Therefore we can find $a, c \in \mathscr{Q}$ such that $J(a)/J(c) = K(w)/K(x)$, hence $ax \sim cw$. By constant proportional time trade-off, $a(t \cdot x) \sim c(t \cdot w)$ for all $t \in \Gamma$. Hence $J(a)/J(c) = K(t \cdot w)/K(t \cdot x)$ for all $t \in \Gamma$. Therefore for all $w, x \in \Omega$ and $t \in \Gamma$,

$$K(w)/K(x) = K(t \cdot w)/K(t \cdot x). \tag{35}$$

Aczel (1965) shows that the only increasing functions that satisfy (35) on a real interval are the functions $K(x) = \alpha \cdot x^{\theta}$ for $\alpha, \theta > 0$ or $\alpha, \theta < 0$. Therefore, there exists

$\theta \neq 0$ such that (3) holds for every $x \in \Omega$ and every $b \in \mathcal{Q}$ such that $J(\bar{b}) \geqslant J(b) \geqslant J(\underline{b})$. Clearly, (3) holds also for every $b \in \mathcal{Q}$ because $\mathcal{Q}$ is utility independent from $\mathcal{Y}$. We can extend the representation to every $x \in \mathcal{Y}$, because for every x we can choose $\bar{x}, \underline{x} \in \mathcal{Y}$ and $\bar{b}, \underline{b} \in \mathcal{Q}$ such that $\bar{x} > x > \underline{x}$ and $K(\bar{x})/K(\underline{x}) = J(\bar{b})/J(\underline{b})$. A repetition of the argument shows that (3) holds on $(\bar{x}, \underline{x})$ with possibly a different value of θ, but in fact, θ must be the same for every x because the intervals $(\bar{x}, \underline{x})$ cover $\mathcal{Y}$ and they overlap. Q.E.D

Proof of Theorem 10. Because $\mathcal{Y}$ and $\mathcal{Q}$ are mutually utility independent, the utility function over $\mathcal{Q} \times \mathcal{Y} - \{0\}$ must be additive or multiplicative. Because of risk neutrality, the additive function must have the form, $U(bx) = H(b) + \alpha \cdot x + \beta$, but this additive form is impossible because it would violate constant proportional time trade-off. Therefore, U is multiplicative. Given that U is linear in duration, the general form of a multiplicative utility is $U(bx) = H(b) \cdot (\alpha \cdot x + \beta)$. If H is a constant function, rescale $U = U - H(b) \cdot \beta$ and $\alpha = H(b) \cdot \alpha$. Then $U(bx) = \alpha \cdot x$, and we have a linear QALY model. If H is not a constant function, choose $b, c \in \mathcal{Q}$ such that $H(b) \neq H(c)$, and choose x and y such that $bx \sim cy$. Then $H(b) \cdot (\alpha \cdot x + \beta) = H(c) \cdot (\alpha \cdot y + \beta)$. By constant proportional time trade-off, $H(b) \cdot (t \cdot \alpha \cdot x + \beta) = H(c) \cdot (t \cdot \alpha \cdot y + \beta)$, so

$$t \cdot [H(b) \cdot \alpha \cdot x - H(c) \cdot \alpha \cdot y] = [H(c) - H(b)] \cdot \beta. \tag{36}$$

The right side of (36) does not contain t, so the left side of (36) must be constant. Therefore, $[H(b) \cdot \alpha \cdot x - H(c) \cdot \alpha \cdot y] = 0$, so $\beta = 0$ because $H(c) \neq H(b)$. Hence $U(bx) = H(b) \cdot \alpha \cdot x$. This proves that $U(bx) = \alpha \cdot H(b) \cdot x$ for all $bx \in \mathcal{Q} \times \mathcal{Y} - \{0\}$. Q.E.D

Proof of Lemma 4. The proof that (i) implies (ii), and that (iii) implies (i) are straightforward and will be omitted. The proof that (ii) implies (iii) is exactly like the corresponding proof for Lemma 1, with the following alterations: Whenever an expression of the form, "$0.5 \cdot U(bw) + 0.5 \cdot U(bx)$" appears in the proof of Lemma 1, replace it with an expression of the form, "$w(1/2) \cdot U(bw) + [1 - w(1/2)] \cdot U(bx)$"; apply the argument of Lemma 1 separately to $\mathcal{Q}^+$ and $\mathcal{Q}^-$. Within $\mathcal{Q}^+$, the argument of Lemma 1 yields $U(bx) = \alpha^+(b) \cdot F(x) + \beta^+(b)$ for all $b \in \mathcal{Q}^+$ and all $x \in \mathcal{Y}$, and within $\mathcal{Q}^-$, the argument of Lemma 1 yields $U(bx) = \alpha^-(b) \cdot F(x) + \beta^-(b)$ for all $b \in \mathcal{Q}^-$ and all $x \in \mathcal{Y}$. Q.E.D

Proof of Theorem 11. The argument used to prove Theorem 1 can be applied separately to health states in $\mathcal{Q}^+$ and in $\mathcal{Q}^-$ to yield multiplicative representations: $U(bx) = H(b)\, F(x) + k$ for every $b \in \mathcal{Q}^+$ and every $x \in \mathcal{Y}$, and $U(cx) = J(c)\, K(x) + m$ for every $c \in \mathcal{Q}^-$ and every $x \in \mathcal{Y}$. These scales can be constructed such that $H(b) > 0$ and $J(c) < 0$ for every $b \in \mathcal{Q}^+$ and $c \in \mathcal{Q}^-$, so by sign dependence F and K are both strictly increasing functions. Restricted solvability implies that $F(\mathcal{Y})$ and $K(\mathcal{Y})$ are both dense in real intervals. Therefore F and K are continuous.

Suppose that $x_0, x_1, x_2, x_3, \ldots$ is any increasing standard sequence in $\mathcal{Y}$ starting at 0; in other words, $0 = x_0$ and $F(x_{k+1}) - F(x_k) = \varepsilon > 0$ for all k in the sequence.

Choose any $p \in (0, 1)$ such $0.5 > w(p) > 0$. Then $w(p)[F(x_{k+1}) - F(x_k)] < (1 - w(p))[F(x_1) - F(0)]$ for every $k > 0$, so by continuity of F, there exists s such that $x_1 > s > 0$ and $w(p)[F(x_{k+1}) - F(x_k)] = (1 - w(p))[F(s) - F(0)]$ for every $k > 0$. Therefore, $w(p)\, H(b)\, F(x_{k+1}) + (1 - w(p))\, H(b)\, F(0) = w(p)\, H(b)\, F(x_k) + (1 - w(p))\, H(b)\, F(s)$ for every $k > 0$. Therefore $(bx_{k+1}, p; b0, 1 - p) \sim (bx_k, p; bs, 1 - p)$ for every $k > 0$.

Now choose any $c \in \mathscr{Q}^-$. By continuity of w, we can choose $q \in (0, 1)$ such that $w(q)\, J(c)[K(0) - K(x_1)] > (1 - w(q))\, J(c)[K(x_1) - K(x_2)]$, so by continuity of K we can choose v such that $x_1 > v > 0$ and $w(q)\, J(c)[K(0) - K(v)] = (1 - w(q)) \times [K(x_1) - K(x_2)]$. Therefore $w(q)\, J(c)\, K(0) + (1 - w(q))\, J(c)\, K(x_2) = w(q)\, J(c)\, K(v) + (1 - w(q))\, J(c)\, K(x_1)$, so

$$(c0, q; cx_2, 1 - q) \sim (cv, q; cx_1, 1 - q). \tag{37}$$

The interlocking condition yields

$$(c0, q; cx_{k+1}, 1 - q) \sim (cv, q; cx_k, 1 - q) \tag{38}$$

for every $k > 1$. These equivalences establish that

$$(1 - w(1 - q))\, J(c)[K(x_{k+1}) - K(x_k)] = w(q)\, J(c)[K(v) - K(u)] \tag{39}$$

for every $k > 1$. This establishes that $x_1, x_2, x_3, \ldots$ is equally spaced on the K scale as well as on the F scales. We have not established that $K(x_1) - K(0) = K(x_{k+1}) - K(x_k)$ for $k > 0$. Therefore F and K are linear with respect to each other over their entire domain except for an initial interval $(x_1, 0]$. As we choose an increasingly fine grained standard sequence such that $x_1 \to 0$, we will have $F(x_1) \to 0$ and $K(x_1) \to 0$ by continuity.

Therefore, F and K must be linear with respect to each other over their entire range. As we have $F(0) = K(0) = 0$, there must exist $\alpha > 0$ such that $F = \alpha K$. If we define $H^*\colon \mathscr{Q} \to \mathbb{R}$ by $H^*(b) = H(b)$ if $b \in \mathscr{Q}^+$, $H^*(b) = 0$ if $b \in \mathscr{Q}^0$, and $H^*(b) = \alpha J(b)$ if $b \in \mathscr{Q}^-$, we have $U(bx) = H^*(b)\, F(x)$ for all $bx \in \mathscr{H}$. Q.E.D

Proof of Theorem 12. As demonstrated in the proof of Theorem 11, if $c \in \mathscr{Q}$ is not equal to death, then $U \mid c$ is a continuous function that is onto a real interval. Miyamoto and Wakker (1996) showed that under RDU assumptions, constant risk posture and continuity of $U \mid c$ imply that $U \mid c$ is linear or exponential. The remainder of the proof is exactly like the proof of Theorem 2. Q.E.D

Proof of Lemma 6. Condition (i) implies that utility is continuous with respect to survival duration. Condition (iii) implies that $by \sim (bx, 1/2; bz, 1/2)$ iff $b(s + y) \sim [b(s + x), 1/2; b(s + z), 1/2]$ and $b(ty) \sim [b(tx), 1/2; b(tz), 1/2]$ for any real s and positive real t such that $s + y, s + x, s + z, ty, tx, tz \in \mathscr{Y}$. Therefore, constant risk posture and constant proportional risk posture are satisfied with respect to 50/50 lotteries. Miyamoto and Wakker (1996) showed that continuity and constant risk posture with respect to 50/50 lotteries is sufficient under RDU theory for linear or exponential utility functions, and that continuity and constant proportional risk posture with respect to 50/50 lotteries is sufficient under RDU theory for logarithmic or power utility functions. As the linear function is the only function in the intersection of these two classes, utility must be linear with respect to survival duration. The zero condition establishes that all utility functions intersect at duration 0. Q.E.D

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