

Astr 509: Astrophysics III: Stellar Dynamics

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Lecture 9: Equilibria of Collisionless Systems. III

Applications of the Jeans Eqs., the Virial Theorem

The Jeans Equations

Assuming axially symmetric system in a steady state:

$$\begin{aligned}\frac{\partial(\nu \overline{v_R^2})}{\partial R} + \frac{\partial \nu \overline{v_R v_z}}{\partial z} + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) &= 0, \\ \frac{\partial(\nu \overline{v_R v_\phi})}{\partial R} + \frac{\partial(\nu \overline{v_\phi v_z})}{\partial z} + \frac{2\nu}{R} \overline{v_\phi v_R} &= 0, \\ \frac{\partial(\nu \overline{v_R v_z})}{\partial R} + \frac{\partial(\nu \overline{v_z^2})}{\partial z} + \frac{\nu \overline{v_R v_z}}{R} + \nu \frac{\partial \Phi}{\partial z} &= 0.\end{aligned}\tag{1}$$

Some Applications of the Jeans Equations

- Asymmetric drift
- The local mass density
- The shape of local velocity ellipsoid
- Spheroidal components with isotropic velocity dispersion

Asymmetric drift

From the v_R Jeans equation at $z = 0$, with an assumed symmetry around the equatorial plane, $\partial\nu/\partial z = 0$, and definitions $\sigma_\phi^2 = \overline{v_\phi^2} - \overline{v_\phi}^2$ and $v_c^2 = R(\partial\Phi/\partial R)$:

$$\overline{v_\phi} = v_c - \frac{\overline{v_R^2}}{2v_c}\zeta, \quad (2)$$

where

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu \overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}} \frac{\partial(\overline{v_R v_z})}{\partial z} \quad (3)$$

How large is each of these terms?

Asymmetric drift

$$\zeta = \frac{\sigma_\phi^2}{v_R^2} - 1 - \frac{\partial \ln(\nu \overline{v_R^2})}{\partial \ln R} - \frac{R}{v_R^2} \frac{\partial(\overline{v_R v_z})}{\partial z} \quad (4)$$

1. We know that locally $\overline{v_z^2}/\overline{v_R^2} \approx \sigma_\phi^2/\overline{v_R^2} \approx 0.45$
2. $R(\partial(\overline{v_R v_z})/\partial z)/\overline{v_R^2}$ is somewhere between 0 and 0.55
3. The largest term is $\partial \ln(\nu \overline{v_R^2})/\partial \ln R \approx 2(\partial \ln \nu/\partial \ln R) \approx R_\odot/R_d \approx 2.4$, where it was assumed that $v_R^2 \propto \nu$ and that $\nu(R) \propto \exp(-R/R_d)$.

Asymmetric drift

Hence,

$$\zeta = 0.45 - 1 - 4.8 - x = -5.35 - x \quad (5)$$

where $0 < x < 0.55$. That is, ζ is uncertain to within only 10%.

These arguments can be inverted, and the measured value of ζ (from asymmetric drift slope) can be used to infer R_{\odot}/R_d (or, more generally, $\partial \ln \nu / \partial \ln R$).

If there were no density gradient, there would be no asymmetric drift!

The Local Mass Density

The v_z Jeans equation (steady-state):

$$\frac{\partial(\nu \overline{v_R v_z})}{\partial R} + \frac{\partial(\nu \overline{v_z^2})}{\partial z} + \frac{\nu \overline{v_R v_z}}{R} + \nu \frac{\partial \Phi}{\partial z} = 0.$$

Drop the first and third terms because they are a factor of $\approx z^2/(RR_d)$ smaller than the second and fourth terms:

$$\frac{1}{\nu} \frac{\partial(\nu \overline{v_z^2})}{\partial z} = - \frac{\partial \Phi}{\partial z} \quad (6)$$

Near the plane of a highly flattened system, Poisson's equation becomes

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho \quad (7)$$

The Local Mass Density

$$\frac{\partial}{\partial z} \left[\frac{1}{\nu} \frac{\partial(\nu \overline{v_z^2})}{\partial z} \right] = -4\pi G \rho \quad (8)$$

If we can measure ν and $\overline{v_z^2}$ (as functions of z), then we can determine the local mass density ρ , which also includes *dark matter* component, if any. This ρ is called the Oort limit.

Oort (1932) estimated $\rho(R_\odot, z = 0) = 0.15 \text{ M}_\odot/\text{pc}^3$.

Bahcall (1984) estimated $\rho(R_\odot, z = 0) = 0.18 \pm 0.03 \text{ M}_\odot/\text{pc}^3$. This appeared as a significant result because the local density of the luminous matter (stars, gas and white dwarfs) is estimated at $0.11 \text{ M}_\odot/\text{pc}^3$, and thus suggests the existence of dark matter in the disk (the halo dark matter contribution to local ρ is less than $0.01 \text{ M}_\odot/\text{pc}^3$).

However, Kuijken & Gilmore (1989, MNRAS 239, 651) showed that previous samples and analysis were flawed: **there is no evidence that the dynamical mass density is larger than the local density of the luminous matter – both are around $0.10 \text{ M}_{\odot}/\text{pc}^3$.**

The shape of local velocity ellipsoid: multiply the collisionless Boltzmann equation by $v_R v_{\phi}$ and assume that the velocity ellipsoid is aligned with the azimuthal direction: can show that $\sigma_{\phi}^2 \sigma_R^2 = -B/(A - B) \approx 0.45$, which was used to derive the asymmetric drift equation.

Spheroidal components with isotropic velocity dispersion: connect the ellipticity of the surface brightness distribution and rotational velocity. The measured rotation speed of giant elliptical galaxies is much smaller than predicted by assuming isotropic velocity dispersion, while they agree for low-luminosity spheroidal galaxies.

The Virial Theorem

Zwicky's derivation: (Ap. J. 1937, 86, 217)

$$m_i \frac{d^2 \vec{x}_i}{dt^2} = \vec{F}_i \quad (9)$$

where \vec{F}_i is the total forces on galaxy i .

Scalar multiplication with \vec{x}_i gives:

$$\frac{1}{2} \frac{d^2}{dt^2} (m_i x_i^2) = \vec{x}_i \cdot \vec{F}_i + m_i \left(\frac{d\vec{x}_i}{dt} \right)^2 \quad (10)$$

Summing over all cluster members gives the virial theorem. For Newtonian gravity only, the virial can be converted into the potential energy of the system.

The Virial Theorem

In a steady state,

$$\frac{1}{2} \frac{d^2}{dt^2} (m_i x_i^2) = 0, \quad (11)$$

and, for self-gravitating system,

$$2K + \Phi = 0 \quad (12)$$

where $K = M \langle v^2 \rangle$ is the kinetic energy. Thus,

$$E = K + \Phi = -K = \frac{1}{2} \Phi \quad (13)$$

If a system collapses from infinity, half of the potential energy will end up in kinetic energy, and the other half will be disposed of!

Self-gravitating systems have negative heat capacity!