Astr 509: Astrophysics III: Stellar Dynamics

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Lecture 3: Potential Theory II

Multipole Expansion, Potential of Disks

Multipole Expansions

How do we solve Poisson's equation for a general density distribution?

Poisson's equation is a linear equation – find a complete set of orthogonal functions ψ_n such that

$$\nabla^2 \psi_n = \lambda_n \psi_n, \tag{1}$$

where orthogonal means

$$\int \psi_n^* \psi_m w(\mathbf{x}) \, \mathrm{d}^3\mathbf{x} = \delta_{nm}, \tag{2}$$

and $w(\mathbf{x})$ is some weight function.

This is an eigenvalue problem and the potential for any arbitrary mass distribution is easily found. The density is decomposed into a sum over the eigenfunctions:

$$\rho(\mathbf{x}) = \sum_n C_n \psi_n(\mathbf{x}), \tag{3}$$

where

$$C_n = \int \psi_n^*(\mathbf{x})\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}. \tag{4}$$

The potential is now just

$$\Phi(\mathbf{x}) = 4\pi G \sum_n \frac{C_n}{\lambda_n}\psi_n(\mathbf{x}). \tag{5}$$

and we are done.

For example, in cartesian coordinates,

$$\psi_{\mathbf{k}}(\mathbf{x}) = \rho_{\mathbf{k}} e^{\mathrm{i}(k_x x + k_y y + k_z z)} \tag{6}$$

are eigenfunctions of the Laplacian, with eigenvalues $-|\mathbf{k}|^2$. So that if we decompose the density field into its Fourier modes, $\rho(\mathbf{k})$, then the potential is just

$$\Phi = \sum_{\mathbf{k}} -\frac{4\pi G\rho(\mathbf{k})e^{\mathrm{i}(k_x x + k_y y + k_z z)}}{k^2}. \tag{7}$$

Multipole Expansions

This procedure is practical only if the given density distribution can be approximated by a small number of eigenfunctions.

It works well for nearly spherical mass distributions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}=\lambda\psi. \tag{8}$$

Using the method of separation of variables we can split this up into three equations, one in each of the independent variables. To do this we assume ψ will be a product of eigenfunctions of the individual equations.

$$\psi(r,\theta,\phi)=R(r)P(\theta)Q(\phi). \tag{9}$$

Substituting this in, we have

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\lambda r^2\sin^2\theta+\frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)=-\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2}. \tag{10}$$

The left side of the equation does not depend on ϕ, and the right side does not depend on r or θ, so both sides must be equal to a constant which we will, with foresight, guess as m^2. So

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = -m^2 Q, \tag{11}$$

whose eigenfunctions are easily seen as

$$Q(\phi) = \sum_m e^{\mathrm{i}m\phi}. \tag{12}$$

Where m must be an integer for single valued functions.

Rewriting the other side of the equation, we have

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right). \tag{13}$$

Again, we can set both sides equal to a constant, $l(l+1)$. In terms of $x \equiv \cos\theta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right] - \frac{m^2}{1-x^2}P = -l(l+1)P, \tag{14}$$

the eigenfunctions of which are associated Legendre functions, $P_l^{|m|}$, with eigenvalues, $l(l+1)$. The combination of Legendre functions and circular functions are referred to as spherical harmonics, $Y_l^m(\theta, \phi)$, where

$$\int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi Y_l^{m*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \qquad (15)$$

The eigenfunctions of the radial equation are spherical Bessel functions, $j_l(kr)$ with eigenvalues $-k^2$.

In practice, the radial eigenfunctions are rarely used since they do a very poor job of representing the radial density distribution of a galaxy, but since many galaxies are nearly spherical, the spherical harmonics can be of considerable practical use.

Multipole Expansions

A more practical way to proceed is to consider the potential of a spherical shell. In this case, everywhere except on the shell, we have Laplace's equation: $\nabla^2 \Phi = 0$. The solution to this equation is very similar to Poisson's equation except that the radial equation is now:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0. \tag{16}$$

The simplest solutions of which are

$$R(r) = Ar^l \quad \text{and} \quad Br^{-(l+1)}. \tag{17}$$

Note that spherical bessel functions reduce to these in the limit of large k.

Now we expect the potential for a thin shell to be finite in its interior so it should be comprised of terms that look like:

$$\Phi_{\mathrm{int}}(r, \theta, \phi) = Ar^l Y_l^m(\theta, \phi). \tag{18}$$

On the other hand externally, the field should fall to zero. So

$$\Phi_{\text{ext}}(r,\theta,\phi) = D r^{-(l+1)} Y_l^m(\theta,\phi). \tag{19}$$

The interior and the exterior solutions can be matched at the shell by requiring that the potential be continuous and matching the gradient of the potential using Gauss's theorem on a small volume around the shell. We can therefore find the potential at any point by summing the exterior potentials of all shells interior to that point, and summing the interior potentials of all shells exterior to that point:

$$\begin{aligned}\Phi(r,\theta,\phi) \;&=\; -4\pi G \sum_{l,m} \frac{Y_l^m(\theta,\phi)}{2l+1} \left[\frac{1}{r^{(l+1)}} \int_0^r \rho_{lm}(r') r'^{(l+2)} \mathrm{d}r' \right. \\ &\qquad \left. + r^l \int_r^\infty \rho_{lm}(r') \frac{\mathrm{d}r'}{r'^{(l-1)}} \right].\end{aligned}$$

Note in particular the exterior potential: the monopole declines as $1/r$, the dipole ($l = 1$) as $1/r^2$ and the quadrapole ($l = 2$) as $1/r^3$. If we expand about the center of mass, the dipole will be zero, and the correction to the monopole potential gets very small with increasing r. Direct analogy with E&M!

Potentials of Disks

Multipoles don't do well for disks because of the wrong coordinate system.

Consider a thin disk such that $\rho = 0$ for $z \neq 0$. Laplace's equation in cylindrical coordinates (ignoring ϕ) is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{20}$$

Just as in the spherical case, we can use separation of variables to find solutions of this equation:

$$\Phi(R, z) = J(R)Z(z). \tag{21}$$

This gives

$$\frac{1}{J(R)R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}J}{\mathrm{d}R}\right) = \frac{-1}{Z(z)}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -k^2, \tag{22}$$

with k an arbitrary constant.

So we have

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - k^2 Z = 0, \tag{23}$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}J}{\mathrm{d}R}\right) + k^2 J(R) = 0. \tag{24}$$

The potential is therefore

$$\Phi_k(R, z) = e^{-k|z|} J_0(kR). \tag{25}$$

where $J_0(kR)$ is a Bessel function.

If we match the solutions at $z = 0$, from Gauss's theorem we have

$$\Sigma_k(R) = -\frac{k}{2\pi G} J_0(kR). \tag{26}$$

Potentials of Disks

Thus we can find the potential of a disk with surface density $\Sigma(R)$ if we can find a function $S(k)$ such that

$$\Sigma(R) = \int_0^\infty S(k)\Sigma_k(R)\mathrm{d}k = -\frac{1}{2\pi G}\int_0^\infty S(k)J_0(kR)k\mathrm{d}k. \qquad (27)$$

This is a Hankel transform, (which is completely analogous to the Fourier transform) and the inverse gives

$$S(k) = -2\pi G\int_0^\infty J_0(kR)\Sigma(R)R\mathrm{d}R. \qquad (28)$$

The potential is

$$\Phi(R,z) = -2\pi G\int_0^\infty \mathrm{d}k e^{-k|z|}J_0(kR)\int_0^\infty \Sigma(R')J_0(kR')R'\mathrm{d}R', \qquad (29)$$

and the circular speed (R times centripetal force) is

$$v_c^2 = R\left(\frac{\partial\Phi}{\partial R}\right)_{z=0} = -R\int_0^\infty S(k)J_1(kR)k\mathrm{d}k. \qquad (30)$$

Potentials of Disks – Examples

Mestel's disk where

$$\Sigma(R) = \frac{\Sigma_0 R_0}{R}. \tag{31}$$

In this case

$$S(k) = -\frac{2\pi G \Sigma_0 R_0}{k} \tag{32}$$

and

$$v_c^2(R) = 2\pi G \Sigma_0 R_0 R \int_0^\infty J_1(kR)\mathrm{d}k = 2\pi G \Sigma_0 R_0, \tag{33}$$

which is constant!

This is the same as a spherical mass distribution with $M(r) \sim r$. If the disk had a sharp edge, there would be greater than keplerian fall off.

Potentials of Disks – Examples

Exponential disk where

$$\Sigma(R) = \Sigma_0 \exp \frac{-R}{R_d}. \tag{34}$$

In this case

$$S(k) = -\frac{2\pi G \Sigma_0 R_d^2}{(1 + k^2\, R_d^2)^{3/2}} \tag{35}$$

and (after some integration and algebra)

$$v_c^2(R) = 4\pi G \Sigma_0 R_d\, l y^2 \,[I_0(y)\, K_0(y) - I_1(y)\, K_1(y)], \tag{36}$$

where $y = R/(2R_d)$, and I and K are modified Bessel functions of the first and second kinds.

v_c increases to $y \sim 1$ and then decreases (slightly faster than Keplerian, see fig. 2-17).

Potentials of Disks – Examples

In both examples disks are infinitely thin – what about "real" disks, say exponential disk

$$\rho(R) = \rho_0 \exp \frac{-R}{R_d} \exp -|z|/H? \tag{37}$$

The solution is given in Kuijken & Gilmore (1989, MNRAS, 239, 571).

Potentials of Disks

Can we determine $\Sigma(R)$ from a given $v_c(R)$?

In principle, yes (see eq. 2-174). $\Sigma(R)$ can be inferred from the value of the gradient of v_c^2 in the neighborhood of R.

However, in practice this is a very noisy method and more robust $\Sigma(R)$ is obtained by fitting a model to the v_c curve.