

Astr 509: Astrophysics III: Stellar Dynamics

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Lecture 2: Potential Theory I

Spherical Systems and Potential–Density Pairs

Potential Theory I

- If we know the mass distribution, how do we find the gravitational potential?
- If we know the gravitational potential, how do we find the mass distribution?
- Gauss' theorems
- Some famous potential-density pairs

Poisson's equation

Gravitational force ($F = GMm/r^2$):

$$\mathbf{F}(\mathbf{x}) = GMm \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \quad (1)$$

To calculate the force on a **unit mass** ($M=1$) at a point \mathbf{x} , we sum over all the contributions from each element $\delta^3\mathbf{x}'$:

$$\delta\mathbf{F}(\mathbf{x}) = G \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \delta m(\mathbf{x}') = G \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \rho(\mathbf{x}') \delta^3\mathbf{x}'. \quad (2)$$

This gives

$$\mathbf{F}(\mathbf{x}) = G \int \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \rho(\mathbf{x}') \delta^3\mathbf{x}'. \quad (3)$$

Poisson's equation

Introduce the **gravitational potential**, defined as:

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}', \quad (4)$$

note that

$$\nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}' - \mathbf{x}|} \right) = \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3}, \quad (5)$$

(so $\nabla_{\mathbf{x}}\Phi$ looks like \mathbf{F}) we have

$$\mathbf{F}(\mathbf{x}) = \nabla \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}' = -\nabla\Phi. \quad (6)$$

Poisson's equation

$$\mathbf{F}(\mathbf{x}) = \nabla \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}' = -\nabla\Phi. \quad (7)$$

Deriving the force from a potential field has several advantages:

- It constrains the force field to be conservative. (The work required to get a mass from one position to another is independent of the path, or $\int \mathbf{F} d\mathbf{x}$ is path independent.) Also note that we can't choose an arbitrary \mathbf{F} .
- The scalar field, Φ is easier to visualize than a vector field.
- A scalar field is often easier to calculate than a vector field.
(1/3 the work).

Poisson's equation

The divergence of the force is:

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = G \int \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \rho(\mathbf{x}') d^3 \mathbf{x}'. \quad (8)$$

But (from the product rule)

$$\nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}. \quad (9)$$

which is 0 for $\mathbf{x}' \neq \mathbf{x}$. So we can restrict the volume of integration to an arbitrarily small sphere (of radius h) about \mathbf{x} .

Poisson's equation

We can take $\rho(\mathbf{x}')$ out of the integral and we have

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) &= G\rho(\mathbf{x}) \int_{|\mathbf{x}' - \mathbf{x}| \leq h} \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x}) \int_{|\mathbf{x}' - \mathbf{x}| \leq h} \nabla_{\mathbf{x}'} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) d^3\mathbf{x}' \\ &= -G\rho(\mathbf{x}) \int_{|\mathbf{x}' - \mathbf{x}| = h} \frac{(\mathbf{x}' - \mathbf{x}) \cdot d^2\mathbf{S}'}{|\mathbf{x}' - \mathbf{x}|^3},\end{aligned}$$

where we have replaced a divergence with respect to \mathbf{x} with a divergence with respect to \mathbf{x}' , and used the divergence theorem to replace a volume integral with a integral over the enclosing surface. Now on the surface, we have $|\mathbf{x}' - \mathbf{x}| = h$ and $d^2\mathbf{S}' = (\mathbf{x}' - \mathbf{x})hd^2\Omega$. So

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = -G\rho(\mathbf{x}) \int d^2\Omega = -4\pi G\rho(\mathbf{x}). \quad (10)$$

Poisson's equation

Substituting $-\nabla\Phi$ for \mathbf{F} we have **Poisson's equation**: Hoorray!!

$$\nabla^2\Phi = 4\pi G\rho. \quad (11)$$

The derivation of the Poisson's equation stems from the facts that

1. the gravitational force is conservative (so we can define a Φ) and
2. the central $1/r^2$ nature of the force between two mass elements, so that the divergence of the force is zero in a vacuum.

Poisson's equation

Integrating Poisson's equation over an arbitrary volume gives

$$4\pi G \int \rho d^3\mathbf{x} = 4\pi G M = \int \nabla^2 \Phi d^3\mathbf{x} = \int \nabla \Phi \cdot d^2\mathbf{S}. \quad (12)$$

where the divergence theorem is used.

This is **Gauss's theorem**: *the integral of the normal component of $\nabla \Phi$ over any closed surface equals $4\pi G$ times the mass contained within that surface.*

The Potential Energy

The force between two point masses is conservative, so the **total work required to assemble a configuration of mass**, $\rho(\mathbf{x})$ is independent of the path taken to assemble it, and is defined as the **potential energy**. That is, for any $\rho(\mathbf{x})$, there exists a well defined W , the work required to assemble that distribution, and is given by

$$W = -\frac{1}{8\pi G} \int |\nabla\Phi|^2 d^3\mathbf{x} = \frac{1}{2} \int \rho(\mathbf{x})\Phi(\mathbf{x}) d^3\mathbf{x}. \quad (13)$$

When discussing equilibrium models of stellar systems, relating observed velocity dispersions to the mass distribution of a model will involve a tensor

$$W_{jk} = - \int \rho(\mathbf{x}) x_j \frac{\partial\Phi}{\partial x_k} d^3\mathbf{x}, \quad (14)$$

known as the **Chandrasekhar potential energy tensor**.

The Potential Energy Tensor

It is “easy” to show that

$$W_{jk} = -\frac{1}{2}G \int \int \rho(\mathbf{x})\rho(\mathbf{x}') \frac{(x'_j - x_j)(x'_k - x_k)}{|\mathbf{x} - \mathbf{x}'|^3} d^3\mathbf{x}' d^3\mathbf{x}. \quad (15)$$

i.e., \mathbf{W} is symmetric.

Note that

$$\begin{aligned} \text{trace}(\mathbf{W}) &= -\frac{1}{2}G \int \rho(\mathbf{x}) \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' d^3\mathbf{x} \\ &= \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3\mathbf{x}. \end{aligned}$$

i.e. $\text{trace}(\mathbf{W})$ is W , the gravitational potential energy.

The Potential Energy Tensor

For spherical matter distribution:

$$W = -4\pi G \int \rho M(r) r dr. \quad (16)$$

Allows a definition of a characteristic size of a system that does not have a sharp boundary:

$$r_g \equiv \frac{GM^2}{|W|}. \quad (17)$$

Also note that for a spherical system W_{jk} is diagonal ($W_{jk} = 0$ for $j \neq k$), and isotropic:

$$W_{jk} = \frac{1}{3} W \delta_{jk}. \quad (18)$$

Spherical Systems – Newton's theorems

Newton's first theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

Newton's second theorem: *The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*

Since the potentials add linearly, we can easily calculate the potential at any point in a spherical density distribution by separately calculating contributions from the interior and exterior parts:

$$\Phi(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') r'^2 dr' + \int_r^\infty \rho(r') r' dr' \right]. \quad (19)$$

The first term is the interior mass taken to be at the center, and the second is a sum over the potentials due to exterior shells.

Spherical Systems – Important Quantities

The velocity of a test particle on a circular orbit is the **circular speed**, v_c . Setting the centripetal acceleration equal to the force we get

$$v_c^2 = r \frac{d\Phi}{dr} = r|\mathbf{F}| = \frac{GM(r)}{r}. \quad (20)$$

So the circular speed is a measure of the mass interior to r . Now we have something we can use: if you tell me what v_c is as a function of r for a galaxy, and I can assume it is spherical, I can tell you what the mass is as a function of r . (Not the case for a non-spherical distribution.)

Another important quantity is the **escape speed**, v_e , defined by

$$v_e(r) = \sqrt{2|\Phi(r)|}. \quad (21)$$

This definition comes from setting the kinetic energy of a star equal to the absolute value of its potential energy. That is, stars with positive total energy are not bound to the system. In order for a star to escape from the gravitational field represented by Φ , it is necessary that its speed be greater than v_e . This can be used to get the local Φ of the galaxy.

Spherical Systems – Simple Examples

Point mass:

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}. \quad (22)$$

Whenever the circular speed declines as $r^{1/2}$ it is referred to as **Keplerian**. It usually implies that there is no significant mass at that radius.

Homogeneous sphere:

$$M = \frac{4}{3}\pi r^3 \rho \quad ; \quad v_c = \sqrt{\frac{4\pi G \rho}{3}} r. \quad (23)$$

The equation of motion for a particle in such a body is

$$\frac{d^2 r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G \rho}{3} r, \quad (24)$$

which describes a harmonic oscillator with period

$$T = \sqrt{\frac{3\pi}{G\rho}}. \quad (25)$$

Spherical Systems – Simple Examples

Independent of r , if a particle is started at r , it will reach the center in a time

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}}, \quad (26)$$

known as the **dynamical time**. Although this result is only true for a homogeneous sphere, it is common practice to use this definition with any system of density ρ .

By integrating the density for a homogeneous sphere, we can get the potential:

$$\Phi = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\frac{4\pi G\rho a^3}{3r}, & r > a. \end{cases}$$

One would expect the center of a galaxy to have a potential of this type if there is no cusp in the central density (implying a linear rise in v_c).

Spherical Systems – Simple Examples

Isochrone potential:

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}. \quad (27)$$

This has the nice property of going from a harmonic oscillator in the middle to a Keplerian potential at large r , with the transition occurring at a scale b .

The circular speed is

$$v_c^2 = \frac{GM r^2}{(b + a)^2 a}, \quad (28)$$

where $a \equiv \sqrt{b^2 + r^2}$.

Using Poisson's equation, we can find the density:

$$\rho(r) = \frac{1}{4\pi G r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = M \left[\frac{3(b + a)a^2 - r^2(b + 3a)}{4\pi(b + a)^3 a^3} \right]. \quad (29)$$

So the central density is

$$\rho(0) = \frac{3M}{16\pi b^3}, \quad (30)$$

and the asymptotic density is

$$\rho(r) \approx \frac{bM}{2\pi r^4}. \quad (31)$$

See also **modified Hubble profile** and **power-law profile**.

Potential–Density Pairs

Simple models can be used to illustrate the dynamics of axisymmetric galaxies.

Plummer's (1911) model: spherically symmetric

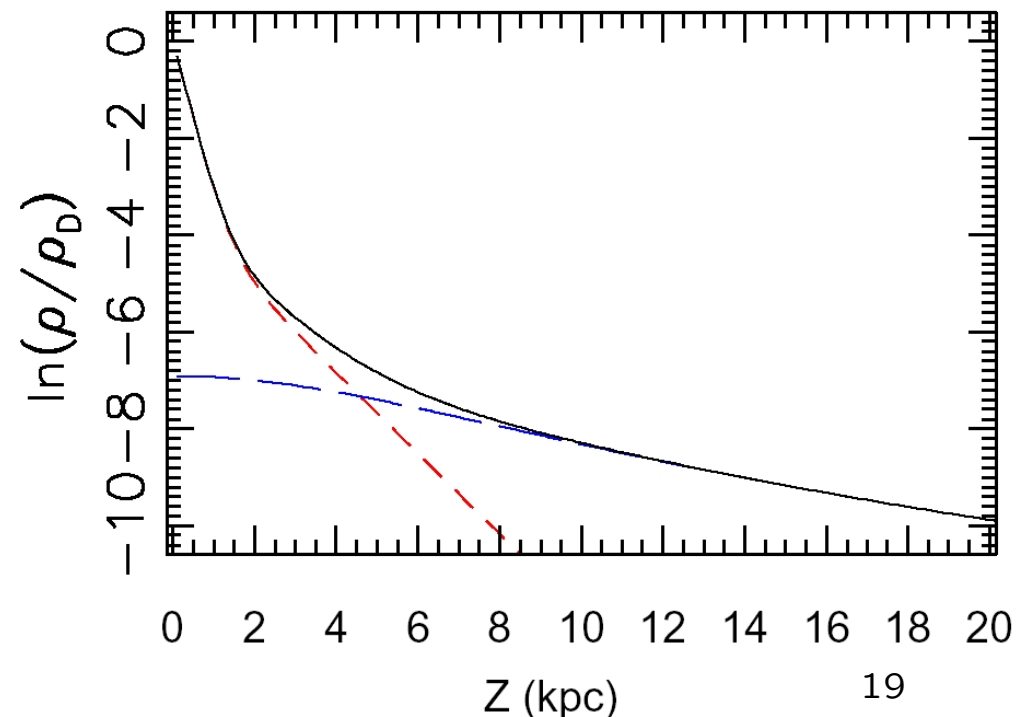
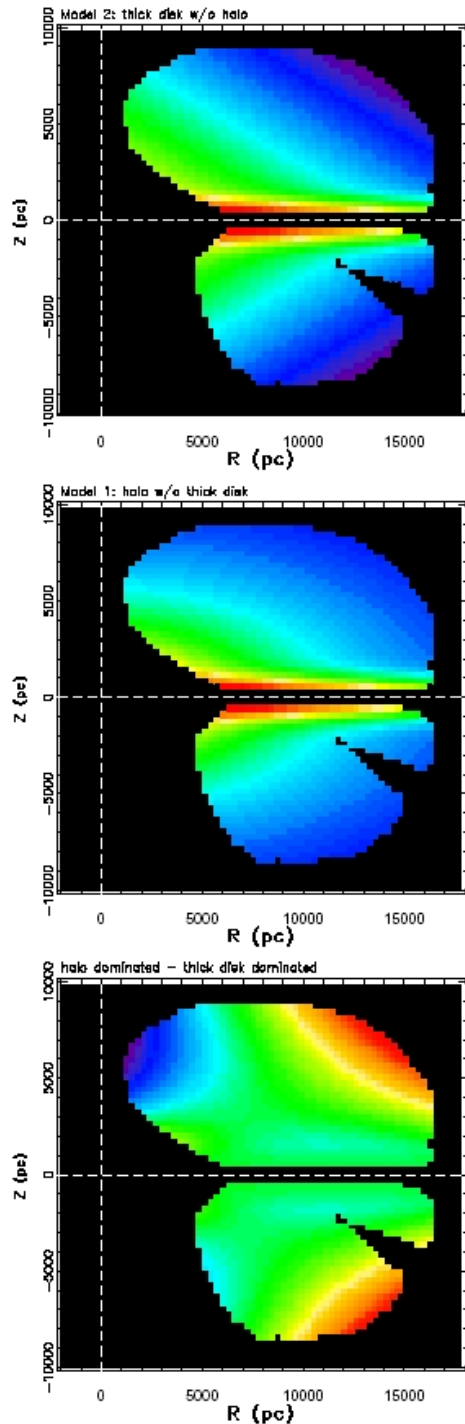
Kuzmin's (1956) model: infinitely thin disk (aka *Toomre's model 1*)

Plummer–Kuzmin models': introduced by Miyamoto & Nagai (1975), smooth transition from Plummer's to Kuzmin's models

Logarithmic potentials: the circular speed is a constant at large radii

Simple Models

- Very different models (top: thin and thick disk without halo; middle: single disk and halo, bottom: the difference) can produce the same $\rho(z|R = R_\odot)$
- Observationally, $\rho(z|R = R_\odot)$ is well fit by a sum of double exponential (thin and thick disk) and power-law profiles.



Plummer–Kuzmin models

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}. \quad (32)$$

For $a = 0$, this reduces to a Plummer model. For $b = 0$, it reduces to the Kuzmin disk. By varying the ratio b/a we have a series of models that go from a thin disk to a spherical model. Using Poisson's equation to calculate ρ , we have

$$\rho(R, z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2 + b^2})(a + \sqrt{z^2 + b^2})^2}{[R^2 + (a + \sqrt{z^2 + b^2})^2]^{5/2}(z^2 + b^2)^{3/2}}. \quad (33)$$

For $b/a = 0.2$, this density is qualitatively the same as disk galaxies, but the asymptotic behaviour is different: ρ falls off like $1/r^3$ whereas in real disks, the light falls off exponentially.

Logarithmic potentials

If $v_c = v_0$ is a constant, then $d\Phi/dR \propto 1/R$, and therefore $\Phi \propto v_0^2 \ln R + C$. So consider

$$\Phi = \frac{1}{2}v_0^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2} \right) + \text{constant}. \quad (34)$$

where $q_\Phi \leq 1$ for oblate potentials. Poisson's equation gives:

$$\rho = \left(\frac{v_0^2}{4\pi G q_\Phi^2} \right) \frac{(2q_\Phi^2 + 1)R_c^2 + R^2 + 2(1 - \frac{1}{2}q_\Phi^{-2})z^2}{(r_c^2 + R^2 + z^2 q_\Phi^{-2})^2}. \quad (35)$$

The density asymptotes to R^{-2} or z^{-2} . Note that this implies an infinite mass.

This potential also gives a drastic example of a general phenomenon: *the density distribution is much flatter than the potential distribution*. In this case, the density can even go negative if $q_\Phi < 1/\sqrt{2}$ (giving unphysical Φ).

Poisson's equation for thin disks

In an axisymmetric system Poisson's equation is

$$\frac{1}{R} \frac{\partial}{\partial R} R \left(\frac{\partial \Phi}{\partial R} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho. \quad (36)$$

Note that if the density is concentrated in the plane, both ρ and the second derivative w.r.t. z will get very large while the second derivative w.r.t R remains well behaved. For a thin disk, therefore, Poisson's equation simplifies to

$$\frac{\partial^2 \Phi(R, z)}{\partial z^2} = 4\pi G \rho(R, z). \quad (37)$$

So in the thin disk approximation one can first determine the potential in the plane of the disk $\Phi(R, 0)$, and then at each radius solve for the vertical structure.