Astr 509: Astrophysics III: Stellar Dynamics

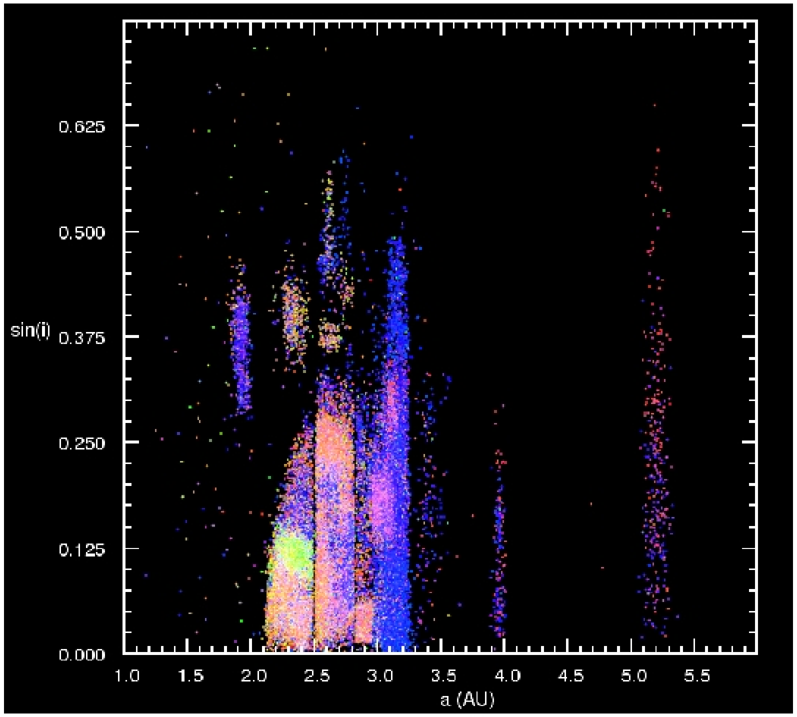
Winter Quarter 2005, University of Washington, Željko Ivezić

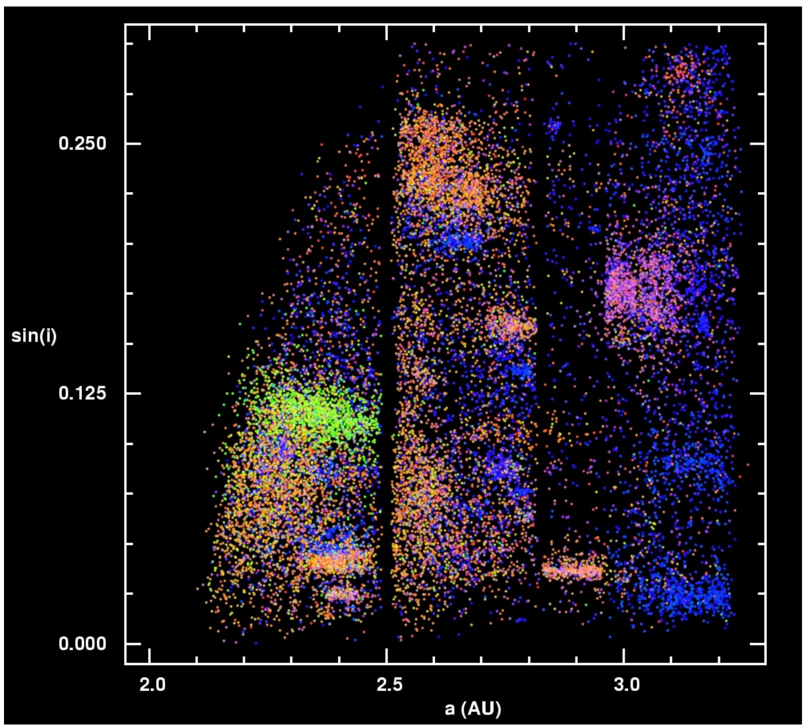
Lecture 11:

A note on Solar System Dynamics,

The Isothermal Sphere and Slab,

King Models, the Jeans Instability





A note on Solar System Dynamics

- For example, main belt asteroids (top osculating elements, bottom proper elements): rich structure (note resonances, e.g. Kirkwood gaps) – perturbations by planets and other asteroids are important.
- Binney's notes on Solar System dynamics are linked to the class web page.
- Modern developments: modeling of potentially hazardous asteroids, non-radial forces (so-called Yarkovsky effect), etc. Currently very active field!

Distribution functions, $f(\mathcal{E})$

One can either assume f and solve for Φ (which also gives ρ), or assume Φ (or ρ) and solve for f.

It will be convenient to define the relative potential and the relative energy by

$$\Psi \equiv -\Phi + \Phi_0 \quad \text{and} \quad \mathcal{E} \equiv -E + \Phi_0 = \Psi - \frac{1}{2}v^2. \tag{1}$$

Φ_0 is chosen to be the value of Φ at the edge of the galaxy, where $f = 0$, and so $\Psi = 0$, too.

In spherical symmetry:

If $f = f(\mathcal{E}) = f(\Psi - \frac{1}{2}v^2)$ then the velocity dispersion in the radial direction is given by

$$\overline{v_r^2} = \frac{\int v_r^2 f(\mathbf{x}, \mathbf{v}) d^3\mathbf{v}}{\int f d^3\mathbf{v}} = \frac{1}{\rho}\int v_r^2 f[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)] \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi. \tag{2}$$

Likewise we have for v_θ^2:

$$\overline{v_\theta^2} = \frac{1}{\rho}\int v_\theta^2 f[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)]\mathrm{d}v_r\mathrm{d}v_\theta\mathrm{d}v_\phi. \tag{3}$$

Note that these only differ by the labeling of the variables. We can therefore conclude that $v_r^2 = v_\theta^2 = v_\phi^2$. In other words, the velocity dispersion tensor is everywhere isotropic.

Using spatial spherical symmetry as well, Poisson's equation becomes

$$\begin{aligned}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= -16\pi^2 G\int_0^{\sqrt{2\Psi}} f(\Psi - \frac{1}{2}v^2)v^2\mathrm{d}v \\ &= -16\pi^2 G\int_0^{\Psi} f(\mathcal{E})\sqrt{2\Psi - \mathcal{E}}\mathrm{d}\mathcal{E}\end{aligned}$$

Now we are talking: give me $f(\mathcal{E})$ and I give you ρ and velocity distribution!

Example 1: the Isothermal Sphere

Assume that DF is given by

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\mathcal{E}/\sigma^2}. \qquad (4)$$

where ρ_1 and σ are constants. The parameter σ sets the systems "temperature" according to $\sigma^2 = k_B T/m$. Since σ is assumed to be a constant, the solution is called "the isothermal sphere".

What ρ and velocity distribution correspond to this DF?

The distribution of velocities is everywhere Maxwellian ($\propto \exp(-v^2)$). The mean-square speed of stars is

$$\overline{v^2} = \frac{\int_0^\infty \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right) v^4 \mathrm{d}v}{\int_0^\infty \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right) v^2 \mathrm{d}v} = 2\sigma^2 \frac{\int_0^\infty e^{-x^2} x^4 \mathrm{d}x}{\int_0^\infty e^{-x^2} \mathrm{d}x} = 3\sigma^2 = \text{const.}$$

Integrating over velocities, we find

$$\rho = \rho_1 e^{\Psi/\sigma^2}, \tag{5}$$

and Poisson's equation can be written

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -4\pi G \rho_1 r^2 e^{\Psi/\sigma^2}. \tag{6}$$

This can be put into dimensionless form by defining the **King radius** by

$$r_0 \equiv \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}} \tag{7}$$

and scaling $\tilde{\rho} \equiv \rho/\rho_0$ and $\tilde{r} \equiv r/r_0$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^2 \frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}\tilde{r}}\right) = -9\tilde{r}^2\tilde{\rho}, \tag{8}$$

which can be solved numerically. For large r the solution asymptotes to

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}, \tag{9}$$

known as the **singular isothermal sphere**.

This provides a very good fit to dark halos since the circular velocity is constant.

Unfortunately, it has infinite density in the center, and an infinite total mass.

The problem of infinite density in the center can be fixed: start integrating (numerically) from the center. The resulting density profile (see Fig. 4-7 in BT) is well behaved, and approaches $\rho(r) \propto r^{-2}$ for large r. For small r, $\rho(r) \propto (1 + r^2)^{-3/2}$.

However, the total mass is still infinite.

King Models

We can fix up the isothermal sphere by truncating it at finite radius. A natural way to do this in terms of the DF is to "lower" f:

$$f(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2} \left(e^{\mathcal{E}/\sigma^2} - 1 \right) & \mathcal{E} > 0; \\ 0 & \mathcal{E} \leq 0. \end{cases} \tag{10}$$

This defines **King models** (a subset of **Michie models**, see 4.4-4b) or **lowered isothermal spheres**. The spatial structure can be determined by integrating over $\mathbf{v}$, then numerically integrating starting from $r = 0$ with $\frac{\mathrm{d}\Psi}{\mathrm{d}r} = 0$ and a given $\Psi(0)$. At some point the density will drop to zero. This is the **tidal radius**. The **concentration** is defined by

$$c \equiv \log_{10}(r_t/r_0). \tag{11}$$

King models can also be paramaratized by $\Psi(0)/\sigma^2$.

In summary, King models are parametrized by one free dimensionless parameter (and physical scales, of course) and have finite mass. This makes them much more useful for fitting profiles of globular clusters and galaxies than isothermal sphere solution.

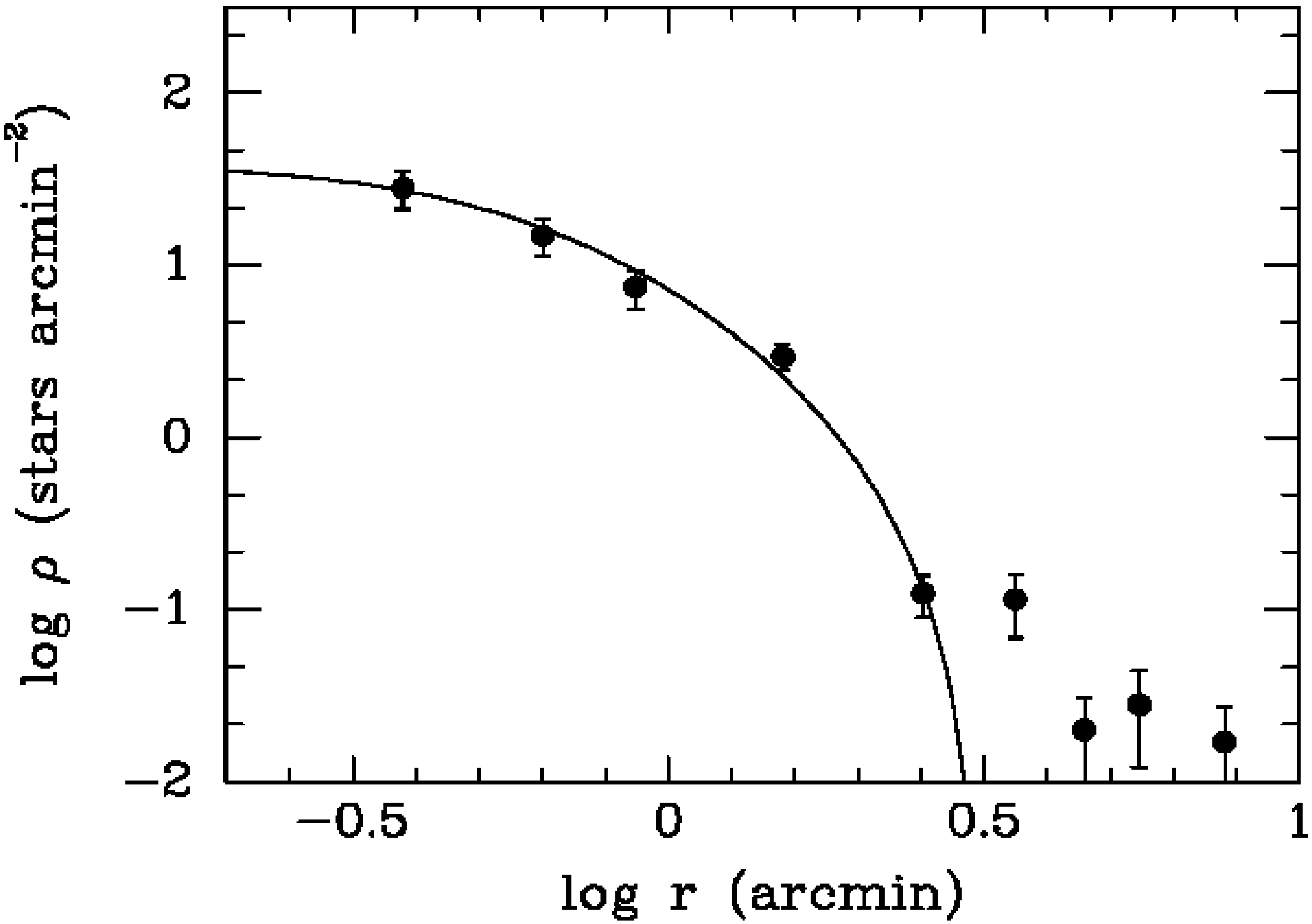


FIG. 7.—Radial star count profile of Pal 13 for stars with membership probabilities $\geq 50\%$. The line is the best-fit King profile to the cluster. Note the member stars outside the classical limiting radius.

Dying globular cluster Pal 13 (Siegel et al. 2001, AJ 121, 935)

Example 2: the Isothermal Slab

For studying local vertical dynamics of a disk, assume that it is a) self-gravitating, b) velocity distribution is Gaussian, and c) the disk is stratified in layers parallel to its plane, so everything is a function of the vertical coordinate, z, only.

These assumptions lead to (verify this at home):

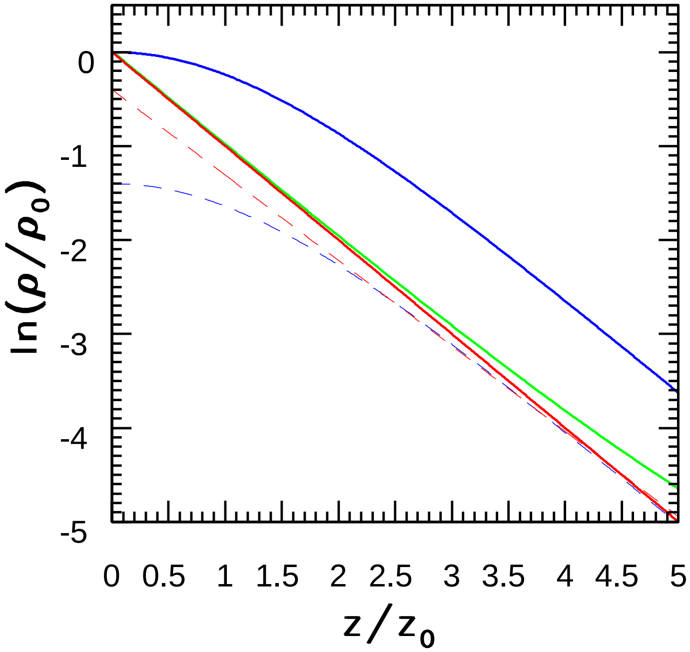
$$\rho = \rho_0 \, \mathrm{sech}^2(z/2z_0), \text{ with } z_0 = \sqrt{\frac{\sigma^2}{8\pi G \rho_0}} \tag{12}$$

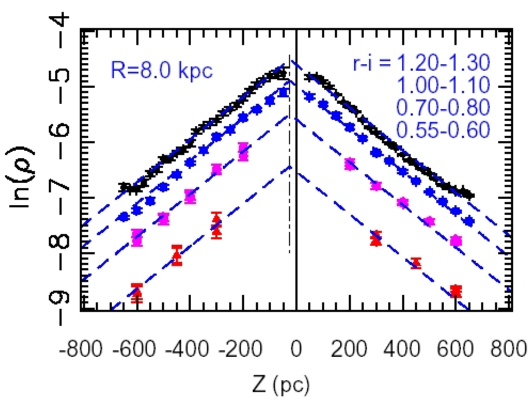
For large r, $\rho(r) \propto \exp(-z/z_0)$. Note: sech is hyperbolic secant of an angle:

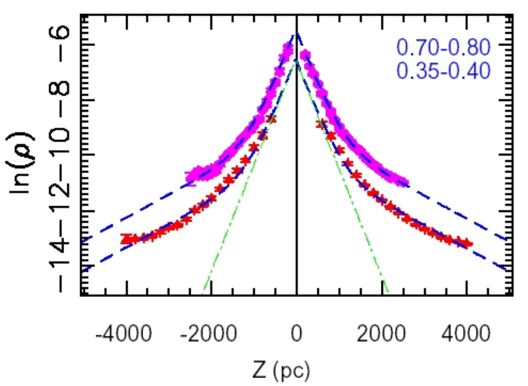
$$\mathrm{sech}(x) = \frac{1}{\cosh(x)}, \tag{13}$$

where

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{14}$$







Isothermal vs. Exponential Disk

- Top: isothermal (solid blue) vs. exponential (solid red) comparison. Green: added 1% of thick disk with 4 times larger scale height. Dashed blue: isothermal scaled to match exponential for large z. Dashed red: exponential matched to fit the isothermal (requires 10% larger scale height). Conclusion: it is hard to distinguish exponential from isothermal profile!
- Bottom: SDSS observations – favor exponential over isothermal profile with the **same** population (and provide the best measurement ever of thick disk parameters).

Eddington inversion

Given a $\rho(r)$ (and hence, a $\Psi(r)$), can we find an f that generates it? Since Ψ is a monotonic function of r we can write the density as

$$\frac{1}{\sqrt{8}\pi}\rho(\Psi) = 2\int_0^{\Psi} f(\mathcal{E})\sqrt{\Psi-\mathcal{E}}\,\mathrm{d}\mathcal{E}. \tag{15}$$

If we differentiate both sides, we have an Abel integral equation, which has the solution

$$\begin{aligned} f(\mathcal{E}) &= \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^{\mathcal{E}}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}} \\ &= \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}}\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right] \end{aligned}$$

In order for the DF to be physical, the integral has to be a monotonically increasing function of $\mathcal{E}$.

The Jeans Instability

Will a self-gravitating dynamical system collapse if slightly perturbed? Fly apart? Does the answer depend on the rotational state? Chemical composition?

Important for a wide variety of astrophysical environments, from star and planet formation, to the formation of galaxy clusters and large-scale structure.

Intuitive answer: Large systems are more likely to collapse, rotation can help against collapsing.

Analysis and results similar to self-gravitating gaseous spheres.

The Jeans Instability

Basic result: a self-gravitating dynamical system will be unstable if

$$k^2 < k_J^2 = \frac{4\pi G\rho_0}{\sigma^2}, \tag{16}$$

where ρ_0 is unperturbed density, and σ is the velocity dispersion. That is, is the characteristic *size* of pertubation is *larger* than the critical value called **Jeans length**, $\lambda_J = 2\pi/k_J$.

Very similar result is obtained for a self-gravitating gaseous sphere

$$k_J^2 = \frac{4\pi G\rho_0}{v_s^2}, \tag{17}$$

where v_s is the speed of sound.

The Jeans Instability: the works!

We wish to study perturbations in an infinite homogeneous self-gravitating system.

Problem: There is no equilibrium configuration for such a system.

Poisson's equation says:

$$\nabla^2 \Phi = 4\pi G \rho \tag{18}$$

but by translational invarience, both ρ and Φ must be constant.

Hence, $\Rightarrow \rho = 0$.

Therefore, we perpetrate the **Jeans swindle** by assuming that Poisson's equation only describes the relation between the perturbed potential and density.

This is justified as long as the scale for changes in Φ in the equilibrium system is much larger than the scale of the perturbations.

Linearized equations of motion

To linearize, we divide variables into an equilibrium part and a perturbed part which is assumed small. Hence for a collisionless system,

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}, t) &= f_0(\mathbf{x}, \mathbf{v}) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t), \\ \Phi(\mathbf{x}, t) &= \Phi_0(\mathbf{x}) + \epsilon \Phi_1(\mathbf{x}, t). \end{aligned}$$

Plugging these into the CBE and Poisson's equation, and dropping all terms proportional to ϵ^2, we have the **linearized collisionless Boltzmann equation**

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} - \nabla \Phi_0 \cdot \frac{\partial f_1}{\partial \mathbf{v}} - \nabla \Phi_1 \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{19}$$

and

$$\nabla^2 \Phi_1 = 4\pi G \int f_1 \mathrm{d}^3\mathbf{v}. \tag{20}$$

Similarly for a self gravitating barytropic fluid ($p = p(\rho)$), the continuity equation becomes

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) + \nabla \cdot (\rho_1 \mathbf{v}_0) = 0. \tag{21}$$

Euler's equation becomes

$$\frac{\partial v_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_0 = \frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1 - \nabla \Phi_1, \tag{22}$$

and the equation of state is

$$p_1 = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 \rho_1 \equiv v_s^2 \rho_1. \tag{23}$$

Here we have introduced the sound speed defined by

$$v_s^2 \equiv \left(\frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho}\right)_{\rho_0}. \tag{24}$$

Jeans Instability for a fluid

If we assume ρ_0 is constant and $\mathbf{v}_0 = 0$ and swindle Φ_0 to be zero, we have

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \\
&\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{1}{\rho_0}\nabla p_1 - \nabla \Phi, \\
&\nabla^2 \Phi_1 = 4\pi G \rho_1, \\
&p_1 = v_s^2 \rho_1.
\end{aligned}$$

Taking the time derivative of the conservation equation and the divergence of the Euler equation, and eliminating in favor of ρ_1 give us

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0. \tag{25}$$

This is a wave equation, so we expect solutions of the form:

$$\rho_1 = C e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{26}$$

This satisfies the equation provided the dispersion relation holds:

$$\omega^2 = v_s^2 k^2 - 4\pi G \rho_0. \tag{27}$$

For large k (small wavelength), this reduces to just sound waves, and things are stable, but for some k

$$k^2 < k_J^2 \equiv \frac{4\pi G \rho_0}{v_s^2} \tag{28}$$

the system will be unstable. We define the **Jeans length** to be $\lambda_J = 2\pi/k_J$ or

$$\lambda_J^2 = \frac{\pi v_s^2}{G\rho_0}. \tag{29}$$

The **Jeans Mass** is the mass contained within a sphere of diameter λ_J:

$$M_J = \frac{4\pi}{3}\rho_0(\frac{1}{2}\lambda_J)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi v_s^2}{G\rho_0}\right)^{3/2}. \tag{30}$$

Jeans Instability for a stellar system

The CBE and Poisson's equation also admit wave solutions, $f_1 = f_a(\mathbf{v})\,\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)], \Phi_1 = \Phi_a\,\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$, giving the relations:

$$(\mathbf{k}\cdot\mathbf{v}-\omega t)f_a - \Phi_a\mathbf{k}\cdot\frac{\partial f_1}{\partial \mathbf{v}} = 0$$

$$-k^2\Phi_a = 4\pi G\int f_a \mathrm{d}^3\mathbf{v}$$

Combining these gives a dispersion relation

$$1 + \frac{4\pi G}{k^2}\int\frac{\mathbf{k}\cdot\partial f_0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-\omega}\mathrm{d}^3\mathbf{v} = 0. \tag{31}$$

The stability is determined by the dependence of f_0 on $\mathbf{v}$. If we assume a Maxwellian,

$$f_0(\mathbf{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}e^{-\frac{1}{2}v^2/\sigma^2}. \tag{32}$$

If we choose a coordinate system such that v_x lies along the $\mathbf{k}$ axis then the dispersion relation becomes

$$1 - \frac{2\sqrt{2\pi}G\rho_0}{k\sigma^3}\int_{-\infty}^{\infty}\frac{v_x e^{-\frac{1}{2}v_x^2/\sigma^2}}{kv_x - \omega}\mathrm{d}v_x = 0. \tag{33}$$

The point that divides stability from instablility is where $\omega = 0$, and this where the integral can be evaluated:

$$k_J^2 = \frac{4\pi G\rho_0}{\sigma^2}. \tag{34}$$

And we are done!