Astr 323: Extragalactic Astronomy and Cosmology

Spring Quarter 2014, University of Washington, Željko Ivezić

Lecture 7:

Introduction to Cosmology

Outline:

- Hubble's Expansion of the Universe
- Newtonian and Relativistic Cosmology
- Observational (astronomer's) Cosmology

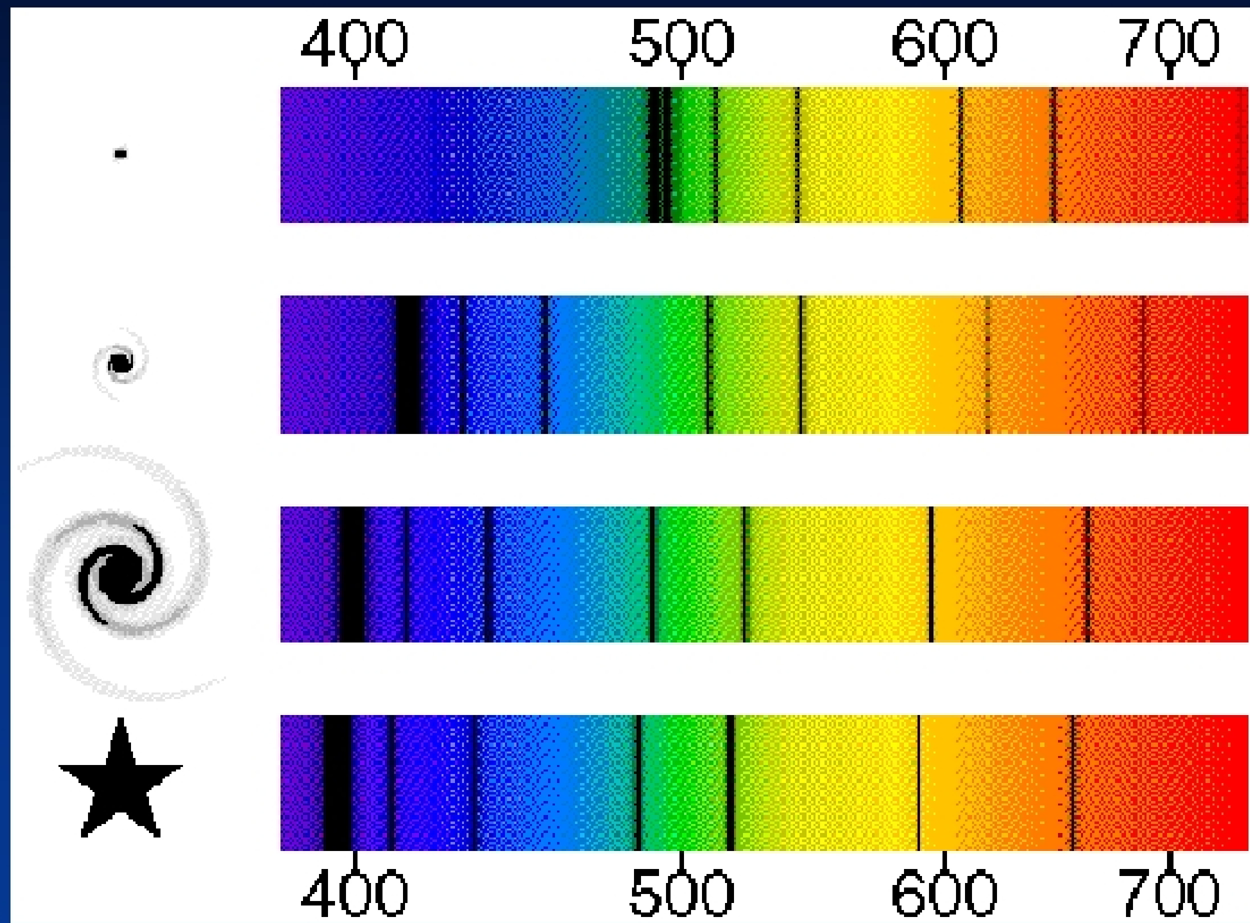
redshift:

$$1+z=\sqrt{\frac{1+v/c}{1-v/c}}$$

$z=0$: not moving

$z=2$: $v=0.8c$

$z=\infty$: $v=c$



Redshift, z, Distance D, and Relative Radial Velocity v

Redshift is **defined** by the shift of the spectral features, relative to their laboratory position (in wavelength space)

$$z = \frac{\Delta\lambda}{\lambda} \tag{1}$$

(n.b. for negative $\Delta\lambda$ this is effectively *blueshift*).

When interpreted as due to the Doppler effect,

$$z = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \tag{2}$$

where v is the *relative* velocity between the source and observer, and c is the speed of light. This is the correct relativistic expression! For nearby universe, $v << c$, and

$$\frac{1}{1 - v/c} \approx 1 + v/c, \text{ and thus } z \approx \frac{v}{c} \tag{3}$$

E.g. at $z = 0.1$ the error in implied v is 5% (and 17% for $z = 0.3$)

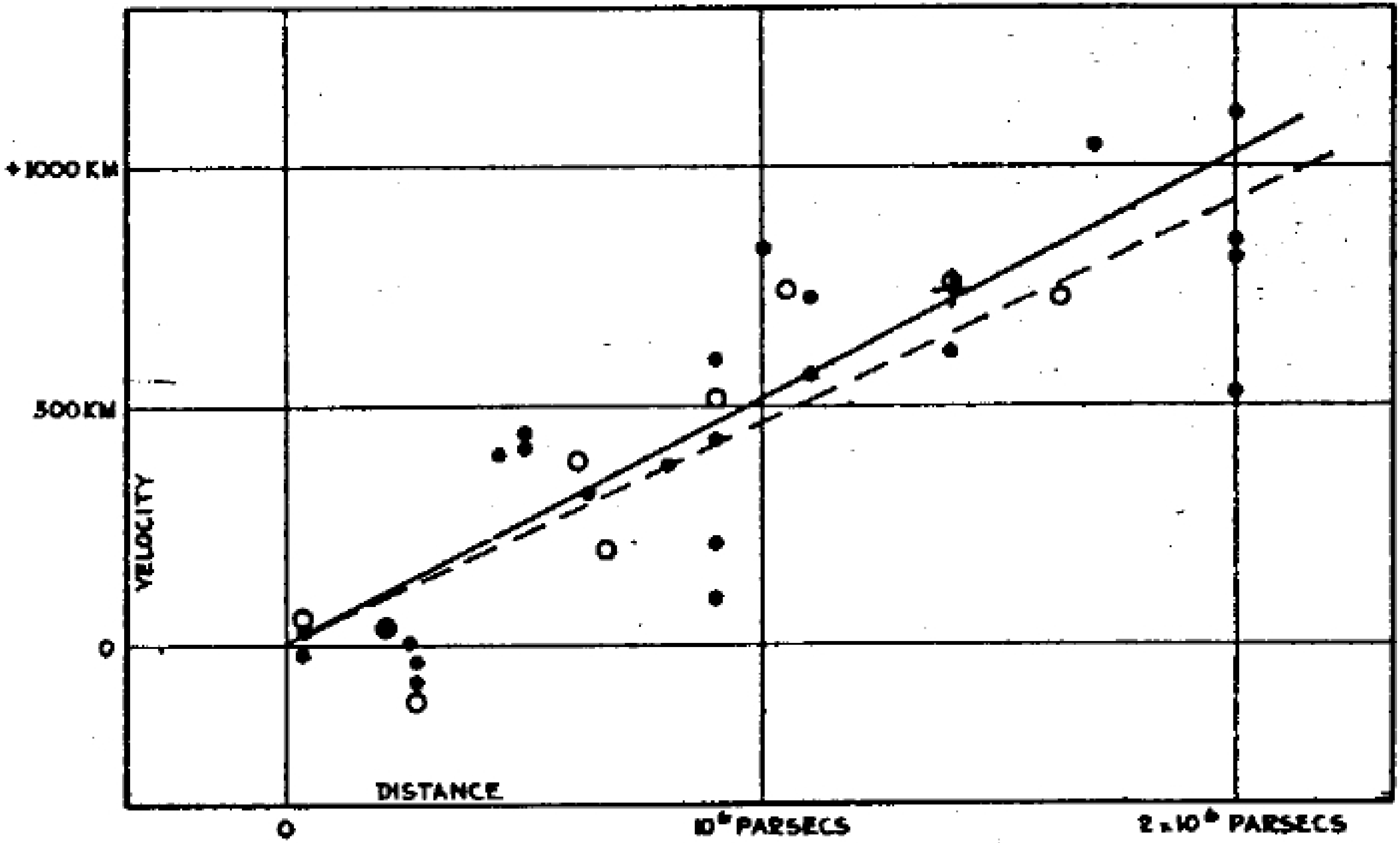
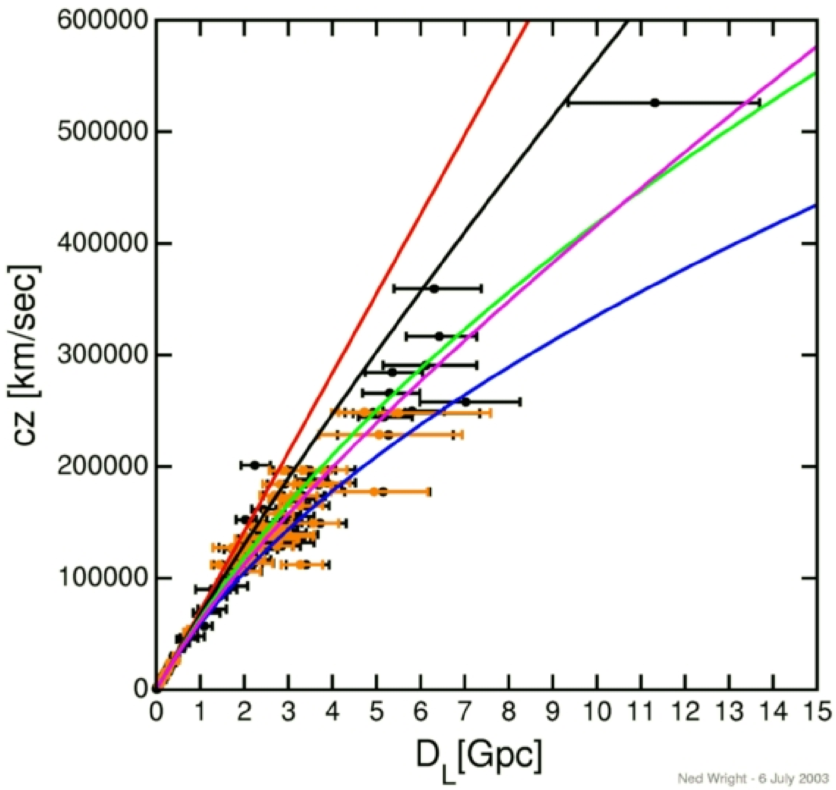
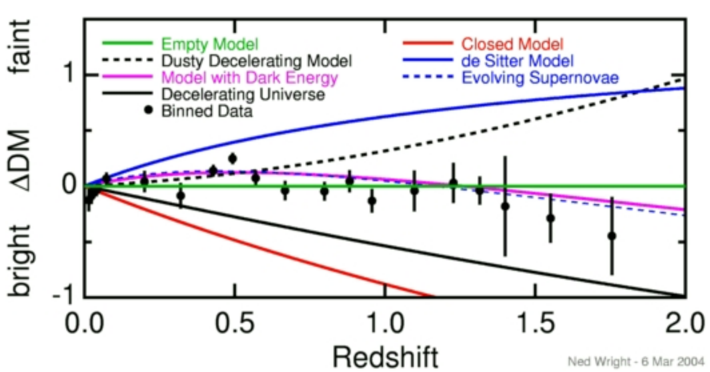


FIGURE 1

Hubble's redshift*c vs. distance diagram (1929)





Expansion of the Universe

- Discovered as a linear law ($v = H\,D$) by Hubble in 1929.
- With distant SNe, today we can measure the deviations from linearity in the Hubble law due to cosmological effects
- The curves in the top panel show different models for the expansion of the Universe: a closed Universe ($\Omega = 2$) in red, the critical density Universe ($\Omega = 1$) in black, the empty Universe ($\Omega = 0$) in green, the steady state model in blue, and the WMAP based concordance model with $\Omega_m = 0.27$ and $\Omega_\Lambda = 0.73$ in purple.
- The data imply an accelerating Universe at low to moderate redshifts but a decelerating Universe at higher redshifts: the explanation involves both dark matter and dark energy: the topics for the rest of this course.

Introduction to Cosmology: just one stone

- Lots of math – but can't have quantitative science without it, ey?

- Before we start thinking about the whole universe, let's first review the mechanics of a stone thrown vertically up.

- At some time t after it was thrown up, the stone of mass m is at height h away from the ground, and moving up with a speed v. Will it leave the Earth, or come back and fall on our head?

- As the stone moves up, its total energy $E = K + U$, doesn't change with time (neglect the deceleration by the atmosphere). However, its kinetic energy K must decrease because its potential energy U is increasing

$$E = \frac{1}{2}mv^2 - G\frac{M_E m}{R_E + h} \tag{4}$$

- Here M_E and R_E are the mass and radius of the Earth. Note that $R_E + h$ is simply the distance between the stone and the Earth's center. As h increases the negative potential energy becomes less negative, i.e. it increases (at the expense of kinetic energy – v becomes smaller, or the stone is decelerating). Note also that, as h goes to infinity, the potential energy goes to 0.

- The stone will reach infinity if its v is large enough so that there is enough kinetic energy to bring the potential energy from its negative level all the way to 0. In other words, E has to be positive (this is what happens with rockets). For a plain stone thrown up by a feeble astronomy professor, E is negative: the stone is *bound* and it eventually stops ($v = 0$) and then returns back.

- If a bystander sees a stone going up, (s)he can find out whether it is bound or not even if the initial act of stone being thrown up was not observed – simply evaluate (measure) kinetic and potential energy, add them up and thus get the total energy E; is it < 0 or > 0?

Introduction to Cosmology: the whole Universe

- The Cosmological Principle: on the largest scales, the universe is isotropic and homogeneous, appearing the same in all directions and at all locations

- Imagine a universe filled with some uniform "stuff" with mass density $\rho(t)$. How does $\rho(t)$ vary with time as this universe expands?

- Imagine a thin shell of radius r, that contains "stuff" with total mass m; this shell expands with the rest of the universe, and thus always contains the same material; the total energy of the shell, $E = K + U$, doesn't change with time, but the kinetic energy K and potential energy U do. If the shell's recessional velocity is v, then the total energy is

$$E = \frac{1}{2}mv^2 - G\frac{M_r m}{r} \tag{5}$$

- Here M_r is the mass enclosed by the shell:

$$M_r = \frac{4}{3}\pi r^3 \rho \tag{6}$$

Now let's introduce (for future convenience)

$$E = -\frac{1}{2} m\, c^2\, k\, a^2 \tag{7}$$

You'll see in a minute why $mc^2/2$; note that only k can be negative. Instead of ka^2, we could have introduced only a single constant, but bear with me for now. Using these definitions we get

$$v^2 - \frac{8}{3}\pi G r^2 \rho = -k\, a^2\, c^2 \tag{8}$$

Note that the shell's mass m has disappeared. Also, the sign of the right-hand side (and thus of the left-hand side, too) is determined solely by k.

- Now we can see that k encapsulates the fate of the universe.

$$v^2 - \frac{8}{3}\pi G r^2 \rho = -k\, a^2\, c^2 \tag{9}$$

  - If $k > 0$ the total energy is negative, and the universe if *bounded* or **closed**. The expansion will someday halt and reverse itself (just like the stone falling back)

  - If $k < 0$ the total energy is positive, and the universe if *unbounded* or **open**. The expansion will continue forever (but at a decreasing rate)

  - If $k = 0$, the total energy is zero. Such universe is called **flat**. The expansion will at a decreasing rate and will come to a halt as time goes to infinity.

- Now, let's invoke the cosmological principle: the expansion must proceed in the same way for all shells (think of raisin bread: the time to double distance from some arbitrary point must be the same for all raisins). Therefore, we can express the radius of a shell as

$$r(t) = R(t)\,a \qquad (10)$$

where $R(t)$ is some general dimensionless **scale factor** of the universe (at some specified time), which is the same for all shells. The expansion is fully described by $R(t)$. Here a specifies a particular shell – think of it as the present radius of the shell; i.e. $R(t_o) = 1$. The shell coordinate a is called a **comoving coordinate**.

- Using the concept of scale factor and comoving coordinate, recall the Hubble law:

$$v(t) = H(t)\,r(t) = H(t)\,R(t)\,a \qquad (11)$$

- Inserting this to the energy conservation equation (6), we get

$$\left(H^2 - \frac{8}{3}\pi G\rho\right) R^2 = -k\, c^2 \tag{12}$$

Note that the right-hand side is the same for all positions and all times.

- As we will see in a second, the left-hand side relates the scale factor and density of the universe. But let's first introduce the concept of **critical density**.

- For a **flat** universe $k = 0$. The value of density (as a function of time) which produces a flat universe is

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \tag{13}$$

Evaluating this at the present time ($H_o = 70$ km/s/Mpc), we get

$$\rho_c(t_o) = 0.92 \times 10^{-29} \text{gcm}^{-3} \tag{14}$$

This corresponds to only about five hydrogen atoms per cubic meter!

- A 'fate of the universe'' question is then what is the present density of the universe $\rho(t_o)$, i.e. is it smaller, equal or larger than $\rho_c(t_o)$?

- But let's first see why the above equation relates density and *the scale factor* of the universe. With

$$v(t) = \frac{\mathrm{d}r(t)}{\mathrm{d}t} = a\frac{\mathrm{d}R(t)}{\mathrm{d}t} \tag{15}$$

we get for the Hubble's constant (well, it's not really a constant since it depends on time – *constant* refers to the lack of spatial variation at a given time)

$$H(t) = \frac{1}{R(t)}\frac{\mathrm{d}R(t)}{\mathrm{d}t} \tag{16}$$

- Inserting this to the energy conservation equation (6, or the above equation 9), we get

$$\left[\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -k\,c^2 \tag{17}$$

We can obtain yet another form of the same equation by noting that mass M_r enclosed by the shell doesn't depend on time. Therefore,

$$R^3(t)\,\rho(t) = R^3(t_o)\,\rho(t_o) = \rho_o \tag{18}$$

which, together with eq.(14) gives

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8\pi G\rho_o}{3R} = -k\,c^2 \tag{19}$$

The density parameters Ω

- The ratio of the present density of the universe and the critical density determines the fate of the universe. Instead of working with numbers like 10^{-29}, it is convenient to define

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)} \qquad (20)$$

  which has the present value Ω_o. So, $\Omega <>= 1$ is the same condition as $k <>= 0$ (flat, open and closed universe).

- The contributions to Ω_o from baryons, dark matter and dark energy are treated separately.

- The current best estimates (from a combination of CMB measurements by WMAP and other data; we will discuss later why CMB measurements can constrain these parameters) are:

- Only baryons: $\Omega_b = 0.043 \pm 0.002$
- Stable neutrinos: $\Omega_\nu < 0.01$
- Total matter: $\Omega_m = 0.260 \pm 0.002$
- Total all: $\Omega_o = 1.02 \pm 0.02$

- Therefore, the universe appears to be **flat**, and its energy content is **NOT** dominated by matter.

- The difference between Ω_o and Ω_m is contributed to dark energy ($\Omega_m + \Omega_\Lambda = \Omega_o$), with $\Omega_\Lambda \sim 0.75$. Here,

$$\Omega_\Lambda = \frac{\Lambda\, c^2}{3H^2(t)}, \tag{21}$$

where Λ is **the cosmological constant**, originally introduced by Einstein (which he called "his greatest blunder").

Introduction

So far, we derived an equation that relates the time evolution of scale factor $R(t)$ and density $\rho(t)$:

$$\left[\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -k\,c^2 \tag{22}$$

But we didn't discuss the fact that this equation cannot be solved by itself – because it contains two independent quantities. We need another equation!

Another problem was that the underlying meaning of k wasn't obvious – in Newtonian derivation it is simply an unspecified integration constant, that determines the fate of the Universe according to

- If $k > 0$ the total energy is negative, and the universe if *bounded* or **closed**. The expansion will someday halt and reverse itself

- If $k < 0$ the total energy is positive, and the universe if *unbounded* or **open**. The expansion will continue forever (but at a decreasing rate)

- If $k = 0$, the total energy is zero. Such universe is called **flat**. The expansion will at a decreasing rate and will come to a halt as time goes to infinity.

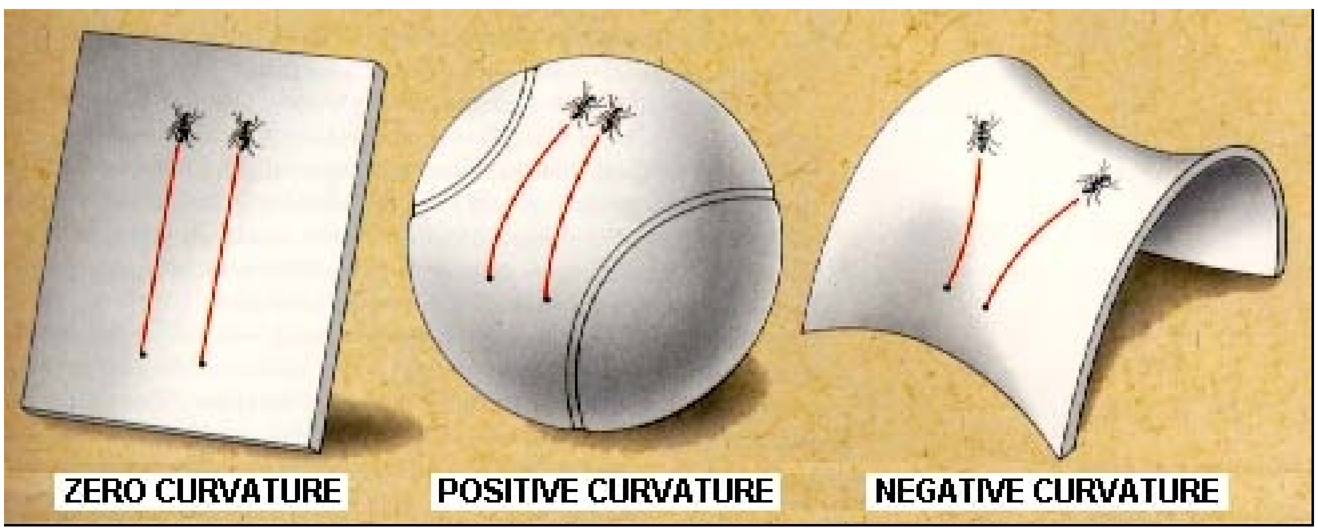
Let's first see what the relativity theory has to say about k, and then we will see how thermodynamical considerations give us another equation (actually two, since we need to introduce a third quantity).

Space-time Curvature

- From a geometrical point of view, there are 3 qualitatively different classes of curvature, as illustrated for a two-dimensional surface. If you can describe your surface mathematically, then it is straightforward to compute the curvature. For example, a circle of radius r has a curvature of size $1/r$ (the curvature of a line, or a circle with infinite r, is 0).
- Formally, we define curvature k of a two-dim. surface as

$$k = \frac{3}{\pi} \lim_{R \to 0} \frac{2\pi R - C_{meas}}{R^3} \tag{23}$$

where R is the radius of an infinitesimally small circle, and C_{meas} is its measured circumference.



Space-time Curvature

- General Relativity asserts that space itself (not just an object in space) can be curved, and furthermore, the space of General Relativity has 3 space-like dimensions and one time dimension, not just two as in our example above.
- While this **is** difficult to visualize, the math proceeds in the same way in 4-dimensional case as it does in 2-dimensional case
- We don't know a priori what is the curvature of the universe. One of the most profound insights of General Relativity was the conclusion that mass caused space to curve. Thus, the curvature of the universe is tied to the amount of mass (and thus to the total strength of gravitation) in the universe.
- How do we measure distances in the space-time continuum of General Relativity?

The Robertson-Walker Metric

The distance between the two neighbouring events in the space-time is given by the Robertson-Walker metric:

$$(\mathrm{d}s)^2 = (c\mathrm{d}t)^2 - R^2(t) \left[\left(\frac{\mathrm{d}a}{\sqrt{1-ka^2}} \right) + (a\mathrm{d}\theta)^2 + (a\sin(\theta)\mathrm{d}\phi)^2 \right] \tag{24}$$

Here ds is the line element of the space time, or the distance between the two neighbouring events in the space-time. $R(t)$ is the scale factor and k is a constant which specifies the curvature. Comoving spherical coordinates are a, θ and ϕ and t is the proper (time between two events at the same place) time.

The Robertson-Walker metric is the most general metric possible for describing an isotropic and homogeneous universe filled with matter.

For completeness, compare to the Schwarzschild metric, which describes the curving of space-time continuum around a massive object (note that here the radial coordinate is r, instead of a)

$$(\mathrm{d}s)^2 = (c\mathrm{d}t\alpha)^2 - \left(\frac{\mathrm{d}r}{\alpha}\right) + (r\mathrm{d}\theta)^2 + (r\sin(\theta)\mathrm{d}\phi)^2 \tag{25}$$

where

$$\alpha = \sqrt{1 - 2GM/rc^2} \tag{26}$$

describes the curvature induced by the massive object.

The Friedmann Equation

Just for illustration, here are Einstein's field equations:

$$R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu} R - \lambda g^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu} \tag{27}$$

Note the λ term that was introduced to allow for a static solution. Here g^{μ}_{ν} "hides" the Robertson-Walker metric.

From these equations, one can derive the Friedmann equation

$$\left[\left(\frac{1}{R} \frac{\mathrm{d}R}{\mathrm{d}t} \right)^2 - \frac{8}{3} \pi G \rho \right] R^2 = -k\, c^2 + \frac{\Lambda c^2}{3} \tag{28}$$

The equation that describes the expansion of the Universe, and is the most important equation in cosmology, the Friedmann equation, is the same whether derived using General Relativity or Newtonian mechanics!

The Friedmann Equation: Einstein vs. Newton

The Friedmann equation

$$\left[\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -k\, c^2 + \frac{\Lambda c^2}{3} \tag{29}$$

The differences between the two derivations are that k is interpreted by General Relativity as the curvature of time-space continuum, and that ρ, which is mass density in Newtonian mechanics, becomes total energy density (divided by c^2) in relativistic cosmology.

The Closure Equation

Let's now use thermodynamical considerations to derive an equation that can be used with the Friedmann equation to solve for the size of the Universe as a function of time.

The first law of thermodynamics says that the change in energy of an expanding system must be equal to the work done by the pressure:

$$\frac{dE}{dt} = -p\frac{dV}{dt} \tag{30}$$

(n.b. the right hand side also includes the TdS/dt term, where S is the entropy, but it drops out since we assume a *reversible* expansion).

Here we have introduced a new quantity: **pressure**. The dependence of pressure on density is specified through an **equation of state**: $p = p(\rho)$. E.g. for ideal gas $p = \rho k_B T$ (here k_B is the Boltzmann constant), and for radiation $p = \rho c^2/3$.

We will relate energy of an expanding shell to its mass using the famous $E = m\,c^2$, which gives us

$$E = V\rho\,c^2 \tag{31}$$

where volume V is

$$V = \frac{4}{3}\pi r^3 \tag{32}$$

The change of energy with time is

$$\frac{dE}{dt} = \rho\,c^2\frac{dV}{dt} + V\,c^2\,\frac{d\rho}{dt}, \tag{33}$$

and the change of volume with time is

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = V\,\frac{3}{r}\frac{dr}{dt} \tag{34}$$

By inserting (30) and (31) in (27), and some rearranging, we get

$$\frac{d\rho}{dt} + \frac{3}{R}\frac{dR}{dt}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{35}$$

The Expansion of the Universe

In summary, we can describe the expansion of a homogeneous isotropic Universe using a scale factor $R(t)$, and a set of equations:

The Friedmann equation:

$$\left[\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -k\,c^2 \tag{36}$$

The "fluid" equation:

$$\frac{d\rho}{dt} + \frac{3}{R}\frac{dR}{dt}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{37}$$

and an equation of state:

$$p = p(\rho) \tag{38}$$

with boundary conditions: at $t = t_o$: $R = 1$, $dR/dt = H_o$

The Size of the Universe as a Function of Time

The Age of the Universe is 13.7$\pm$0.2 Gyr, as measured by WMAP. How do we get age? By solving first for $R(t)$ and then setting $R(t_o) = 1$. This is especially easy in case of an open ($k = 0$) matter dominated universe (so that $p = 0$). In this case $\rho(t)R^3(t)=$ const. from the fluid equation, and when this is inserted into the Friedmann equation (note that the same results can be obtained using eq. 16):

$$R^{1/2}\mathrm{d}R = \left(\frac{8\pi G\rho_{c,o}}{3}\right)^{1/2}\mathrm{d}t \qquad (39)$$

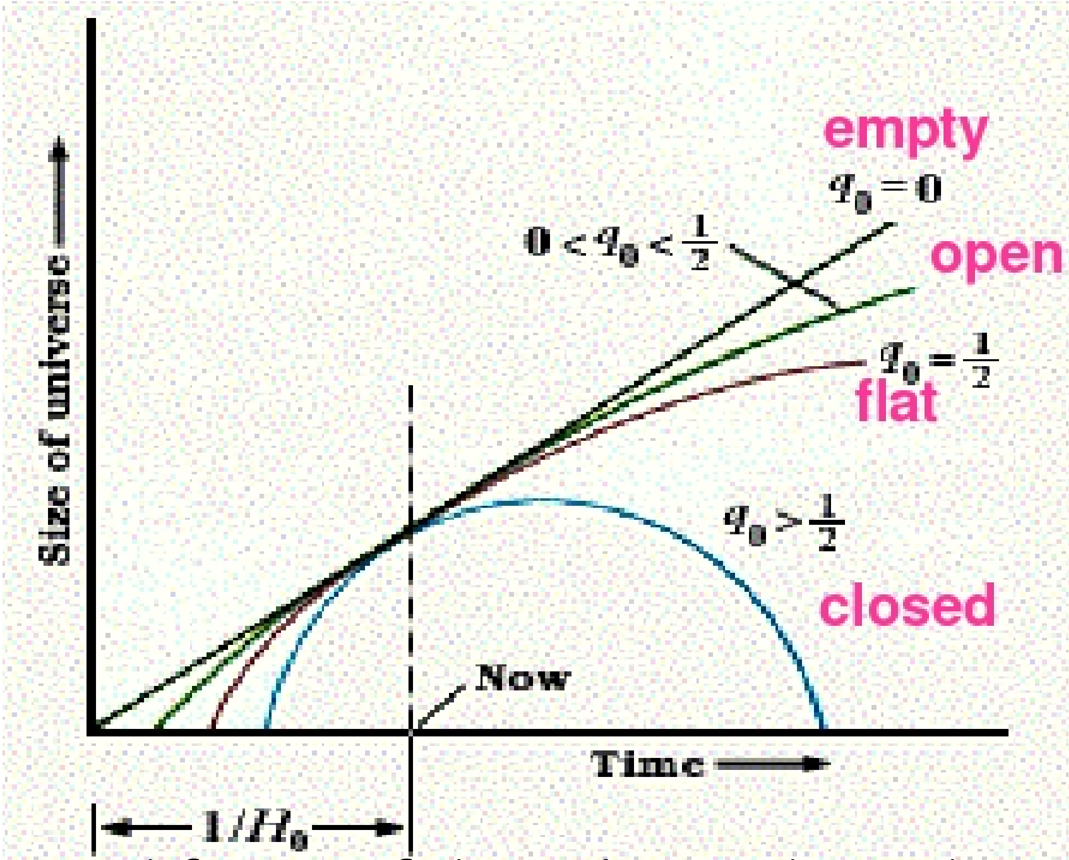
and by simple integration

$$R(t) = (6\pi G\rho_{c,o})^{1/3}\, t^{2/3} = \left(\frac{3H_o\, t}{2}\right)^{2/3} \qquad (40)$$

Because $k = 0$ (and $\Lambda = 0$), the only parameter needed to run the expansion backwards in time is H_o.

The age of the universe, t_o, is usually expressed in terms of $t_H = 1/H_o = 13.8$ Gyr. Thus, for a $k = 0$ matter dominated universe, $t_o/t_H = 2/3$ (because $R(t_o) = 1$ by definition), which gives t_o=9.2 Gyr, which is younger that the age of the oldest stars!

The Size of the Universe as a Function of Time



The past and future of the universe depends on k: open, flat and closed.

Note that the slope $H = 1/R\, dR/dt$ is the same for all models at the present time (this is simply the measured value of the Hubble constant H_o).

In a universe without cosmological constant (as in the models shown above), the age of the universe is **always smaller than** t_H**!**.

The Size of the Universe as a Function of Time

The WMAP estimate for the age of the universe is also obtained by integrating the Friedmann equation (with $k = 0$ and $H_o = 71 \pm 5$ km/s/Mpc), but the measurement of $\Omega_\Lambda = 0.75$ is also taken into account (you'll repeat this calculation in your homework!)

FYI: the seminal WMAP paper is Spergel et al. 2003, The Astrophysical Journal Supplement Series, 148, 175–194.

The latest ESO Planck results: Ade et al. 2013, arXiv:1303.5075

The Expansion of the Universe

To recap, so far we derived equations that relate the time evolution of the scale factor $R(t)$ and density $\rho(t)$ of a homogeneous isotropic Universe:

The Friedmann equation:

$$\left[\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -k\,c^2 \tag{41}$$

The "fluid" equation:

$$\frac{d\rho}{dt} + \frac{3}{R}\frac{dR}{dt}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{42}$$

and an equation of state:

$$p = p(\rho) \tag{43}$$

with boundary conditions: at $t = t_o$: $R = 1$, $dR/dt = H_o$

Astronomers use a slightly different form of these equations.

The Expansion of the Universe: an Astronomer's View

Redshift is used to describe size (or time) instead of scale factor $R(t)$, and densities are normalized by the critical density at the present time

$$\rho_c = \frac{3H_o^2}{8\pi G} \sim 0.92 \times 10^{-29} \mathrm{gcm}^{-3} \tag{44}$$

(using $H_o = 70$ km/s/Mpc), leading to $\Omega = \rho/\rho_c$ parameters.

The scale factor is related to redshift, z, as

$$R(t) = \frac{1}{1+z} \tag{45}$$

which simply reflects the fact that the photon wavelength expands in the same fashion as the whole universe. Note that

$$\frac{dR}{dz} = -\frac{1}{(1+z)^2} = -R^2 \tag{46}$$

The Contributors to the Energy-mass Density

We will assume that the density has contributions from several components with different equations of state:

$$p = w(t)\,\rho\,c^2 \tag{47}$$

The following dominant components are considered:

- Matter: includes both baryonic and dark matter. Matter is non-relativistic and can be approximated by zero pressure ($p = 0$). Therefore, for matter $w = 0$. The amount of matter is specified by Ω_m (~0.26, with 0.04 in baryons – of which only 1/10 are emitting light, and the rest is in the form of dark matter).

- Radiation: today it is energetically negligible, but it was dominant in the early universe. The equation of state for radiation is $p = \rho\,c^2/3$ and hence, for radiation $w = 1/3$. The contribution of radiation to Ω is specified by Ω_r (< 0.0001).

- Dark Energy: since we don't know much about it, we'll retain, at least for now, it's most general description as $w(t)$, (or, equivalently, $w(z)$). When w is *assumed* constant, then observations indicate that $w \sim -1$, which is equivalent to cosmological constant. The contribution of cosmological constant to Ω is specified by Ω_Λ (~ 0.7).

- We can formally treat k term in the Friedmann equation as "energy density" in curvature, with $w = -1/3$, and contribution Ω_k. By definition, Ω_k and all other Ω add up to 1:

$$\Omega_m + \Omega_\Lambda + \Omega_k + \Omega_r = 1 \tag{48}$$

Before we can write down the resulting Friedmann equation, we have to find out how the density of a component described by a particular equation of state depends on redshift.

The Equation of State and the Fluid Equation

For a given equation of state, we can transform the fluid equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{3}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\rho\,[1 + w(R)] = 0 \qquad (49)$$

into

$$\frac{\mathrm{d}\rho}{\rho} = -3\,[1 + w(R)]\,\frac{\mathrm{d}R}{R} = 3\frac{\mathrm{d}z}{1+z}\,[1 + w(z)] \qquad (50)$$

and integrate to obtain

$$\frac{\rho(z)}{\rho(t_o)} = \exp\left(3\int_0^z \frac{1 + w(z')}{1 + z'}\mathrm{d}z'\right) \qquad (51)$$

If $w(t) = w_o =$const., then

$$\frac{\rho(z)}{\rho(t_o)} = (1+z)^{3(1+w_o)} = \frac{1}{R^{3(1+w_o)}} \qquad (52)$$

Hence, for $w_o = 0$ (matter) $\rho_m \propto \Omega_m(1+z)^3$, for $w_o = -1/3$ (radiation) $\rho_r \propto \Omega_r(1+z)^4$, and for $w_o = -1$ (cosmological constant) $\rho_\Lambda \propto \Omega_\Lambda$=const.

The Astronomer's Friedmann Equation

We can write the Friedmann equation in a general form as

$$\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{H^2(z)}{H_o^2} = E^2(z) = \sum_i \Omega_i \exp\left(3\int_0^z \frac{1+w(z')}{1+z'}\mathrm{d}z'\right) \quad (53)$$

Given the solutions of the fluid equation for the dominant components to the energy-mass density, we can write the Friedmann equation as

$$E^2(z) = \Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_{DE} f(z), \quad (54)$$

where

$$f(z) = \exp\left(3\int_0^z \frac{1+w(z')}{1+z'}\mathrm{d}z'\right) \quad (55)$$

For $w(z) = w_o$, $f(z) = (1+z)^{3(1+w_o)}$. In particular, for $w_o = -1$ $f(z) = 1$, and $\Omega_{DE} = \Omega_\Lambda$ (dark energy as cosmological constant).

The function $E(z)$ has fundamental importance for interpreting cosmological observations.

Lookback time and the age of the universe

We can compute the time that took a photon to get to us from an object at redshift z as

$$t_L = \int_t^{t_o} \mathrm{d}t = -\int_0^z \frac{\mathrm{d}t}{\mathrm{d}z}\mathrm{d}z'. \tag{56}$$

Since

$$E(z) = \frac{1}{H_o\,R}\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{t_H}{1+z}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{57}$$

we get for the lookback time

$$t_L = t_H \int_0^z \frac{\mathrm{d}z'}{(1+z')\,E(z')}. \tag{58}$$

where $t_H = 1/H_o \sim 13.7$ Gyr. Thus, given the cosmological parameters, we can compute the age of the universe as the lookback time for $z \to \infty$.

Lookback time and the age of the universe

For a **flat** universe $\Omega_m + \Omega_\Lambda = 1$, favored by the CMB observations and theoretically by inflation, the implied age of the universe increases with Ω_Λ. For $\Omega_\Lambda = 0$, the age of the universe is equal to $2t_H/3 \sim 10$ Gyr smaller than the age of the oldest stars ($\sim 13-14$ Gyr), but for $\Omega_\Lambda = 0.7$, the universe is slightly older than the oldest stars. Observations of distant supernovae *also* favor $\Omega_\Lambda = 0.7$, while dynamical observations of galaxy clusters favor $\Omega_m = 0.3$.

Because of this *simultaneous* success in explaining fundamentally different observations, the $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ model is called the **concordance model**.

By the way: until recently, a very popular parameter for describing cosmology was the *deceleration parameter*

$$q_o = -\frac{dR^2}{dt^2}\left(\frac{dt}{dR}\right)^2 R = \Omega_m/2 - \Omega_\Lambda \qquad (59)$$

An interesting point is that $\Omega_\Lambda > 0$ doesn't necessarily imply a deceleration: $q_o < 0$ **only if** $\Omega_\Lambda > \Omega_m/2$ (assumed $w_o = -1$).

The Luminosity and Angular Diameter Distances

In curved space-time, there is no preferred notion of the distance between two objects. There are several distance measures which can be extracted from observations. The most important are **luminosity distance**

$$d_L = \sqrt{\frac{L}{4\pi F}} = (1+z)D_M, \tag{60}$$

where L is the luminosity and F is the measured flux, and **angular diameter distance**

$$d_A = \frac{D}{\theta} = \frac{D_M}{1+z} \tag{61}$$

where D is the metric (proper) size of an object, and θ is the measured angular size.

Here D_M is *the transverse comoving distance*, given by

$$D_M = D_C\, f(D_C/D_H) \tag{62}$$

When $\Omega_k = 0$, $f = 1$, and it is easily computable otherwise. Here D_C is comoving (line-of-sight) distance

$$D_C = D_H \int_0^z \frac{dz'}{E(z')} \tag{63}$$

and $D_H = c\,t_H = 4.2$ Gpc is the Hubble distance.

Thus, when we know the absolute magnitude of an object, such as supernova, we can compute its apparent magnitude as

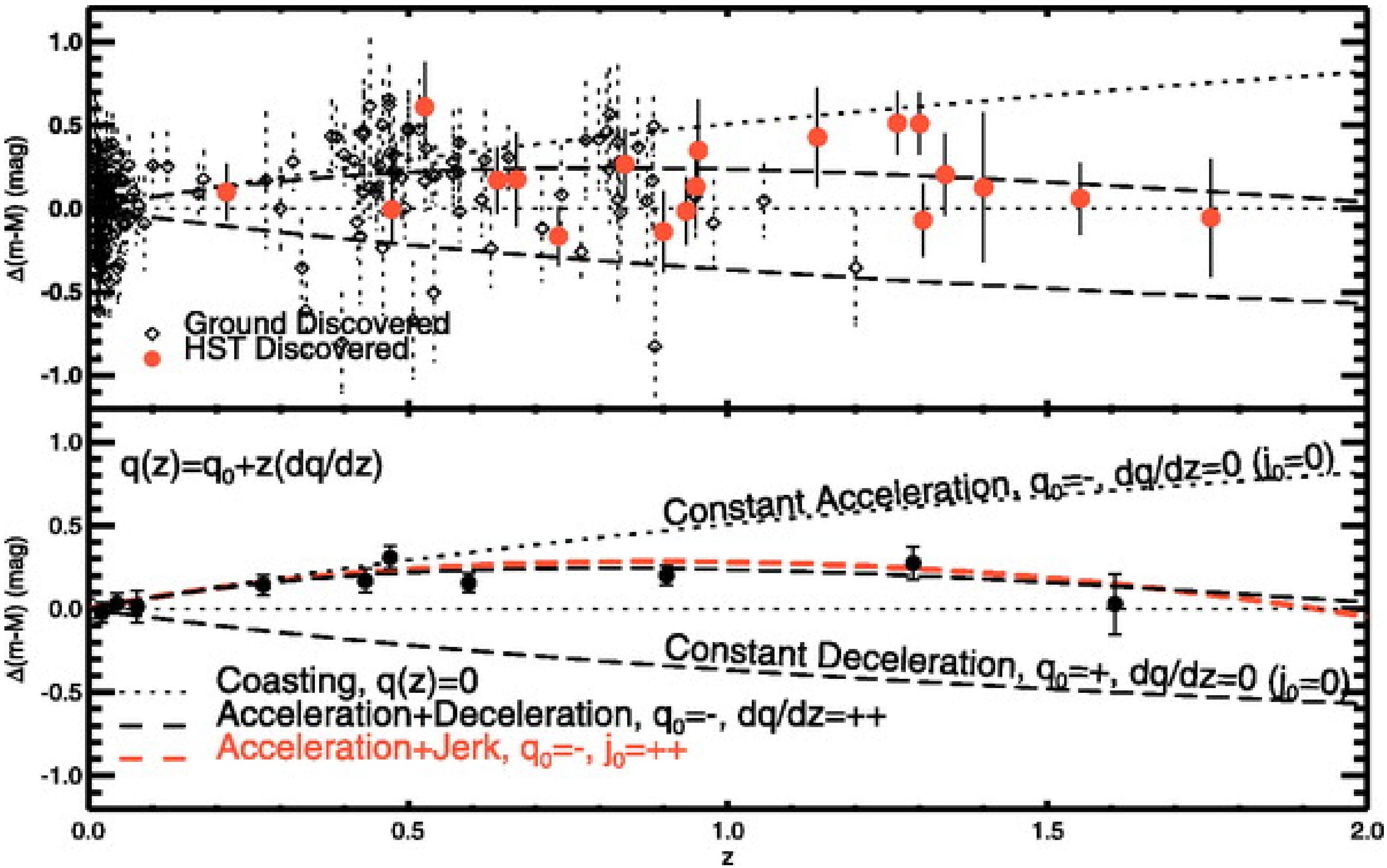
$$m = M + 5 \log(\frac{D_L}{10pc}) + K + E \tag{64}$$

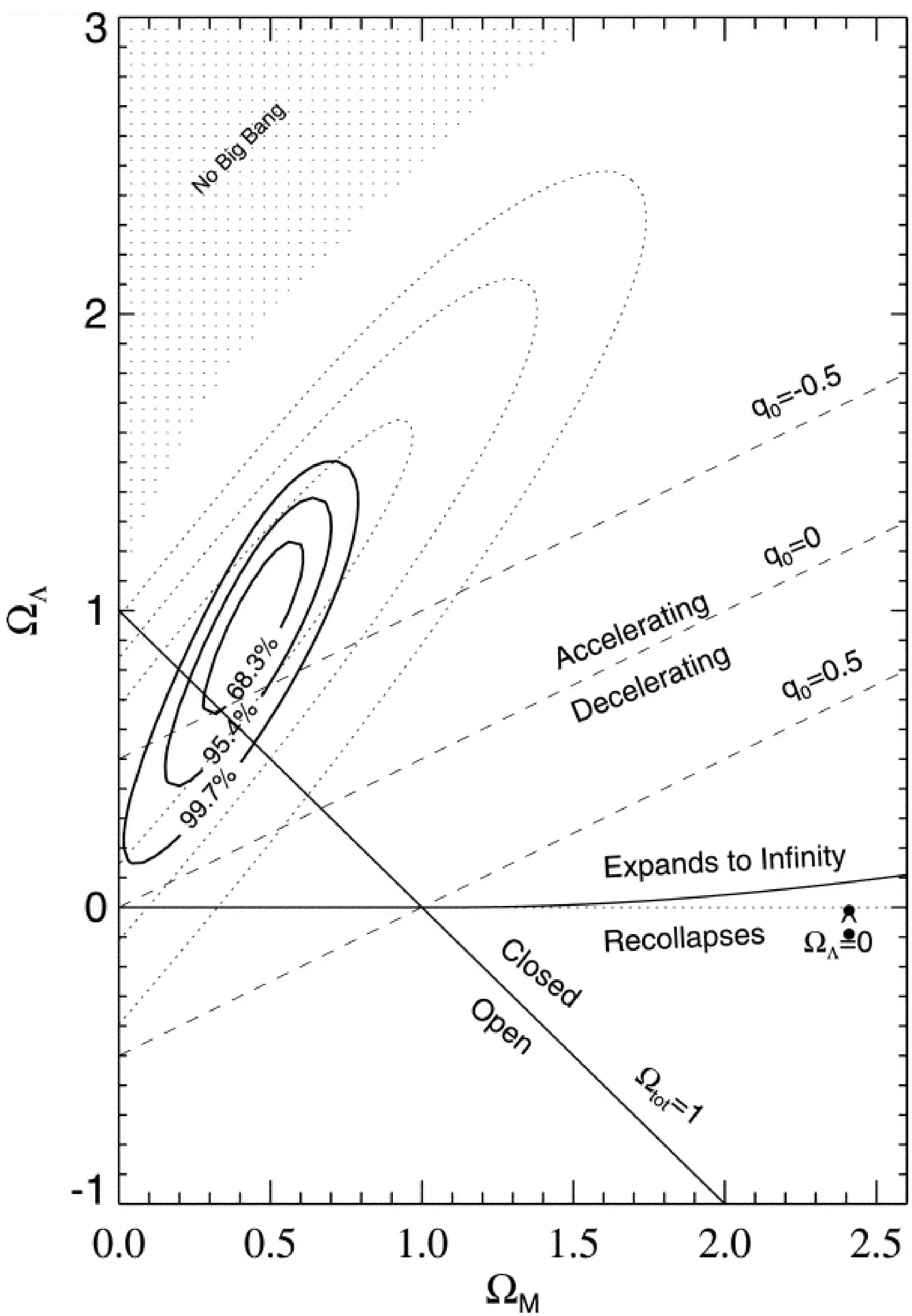
Here K and E are corrections to account for the redshift of spectral energy distribution and for evolutionary effects. For $\Omega_k = K = E = 0$, the distance module is

$$DM = m - M = 5 \log(\frac{D_H}{10pc}) + 5 \log(1 + z) + 5 \log(\int_0^z \frac{dz'}{E(z')}) \tag{65}$$

$$DM = 43.1 + 5 \log(1 + z) + 5 \log(\int_0^z \frac{dz'}{E(z')}) \tag{66}$$

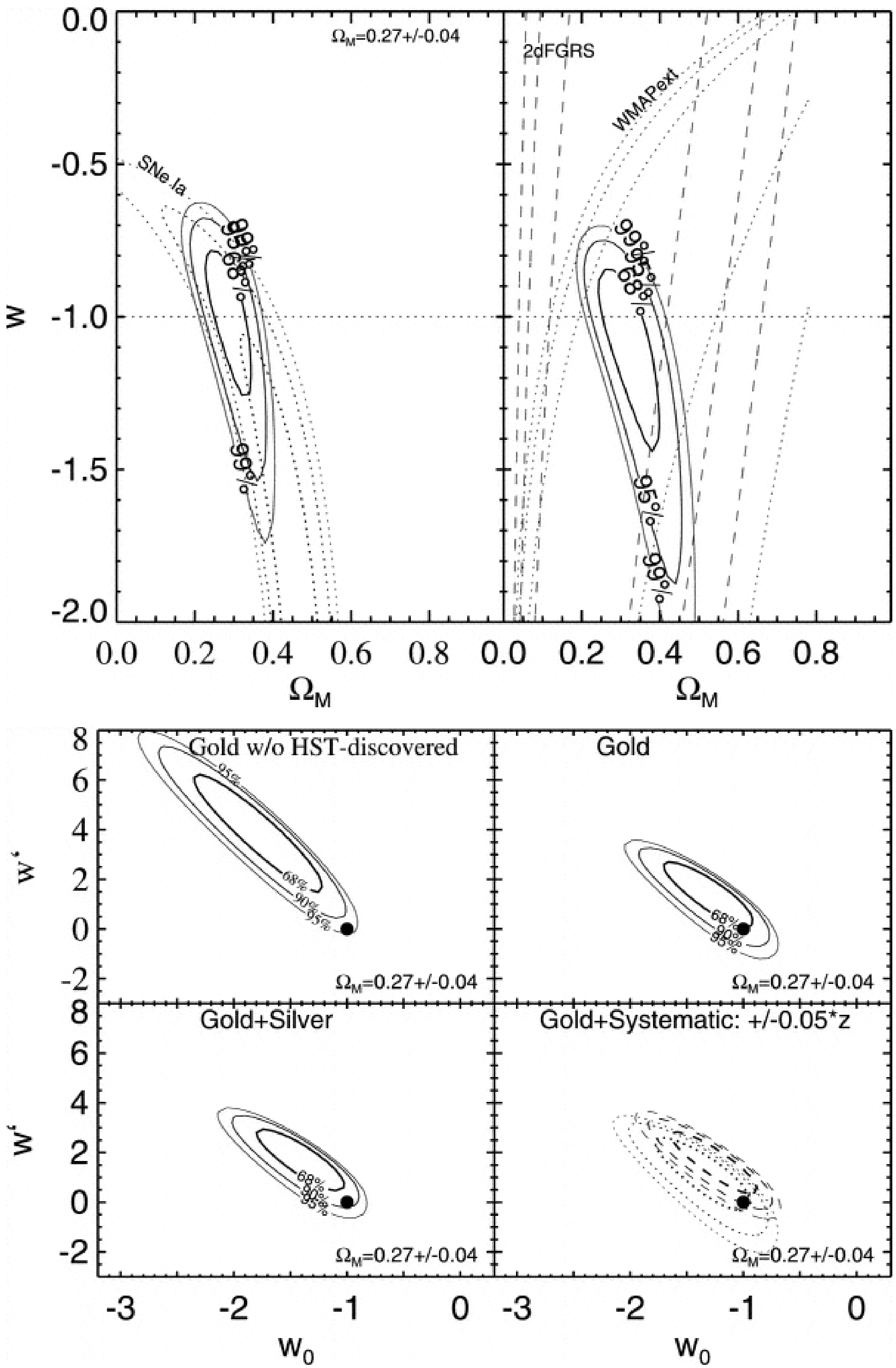
SNe by Riess et al. (2004): extremely convincing evidence for cosmic jerk (deceleration turning into acceleration around $z \sim 0.5$), in amazing agreement with the concordance model! **Nobel Prize in physics for 2012!**





Constraints on Ω_m and Ω_Λ from SNe

- From Riess et al. (2004, ApJ 607, 665)
- SNe constraints are shown by the solid contours; note that they are above the line separating accelerating and decelerating universes
- The $\Omega_{tot} = \Omega_m + \Omega_\Lambda = 1$ condition, favored by the CMB measurements, and theoretically by inflation, is in good agreement with the SNe constraint.
- SNe (i.e. luminosity distance) also provide good constraints on the dark energy equation of state



Constraints on the dark energy equation of state

- From Riess et al. (2004, ApJ 607, 665); *gold* and *silver* refer to subsamples of different quality
- Top: assuming $w(z) = w =$ const.
- Bottom: assuming $w(z) = w_o + w' z$
- Conclusions: if $w(z)$ is assumed constant, then $w_o = -1$ cannot be ruled out: it is -1 to within the measurement error of $\sigma_w \sim 0.2$; if w' different from 0 iz allowed, then perhaps $w_0 < -1$
- Last few years have produced extremely exciting cosmological measurements!

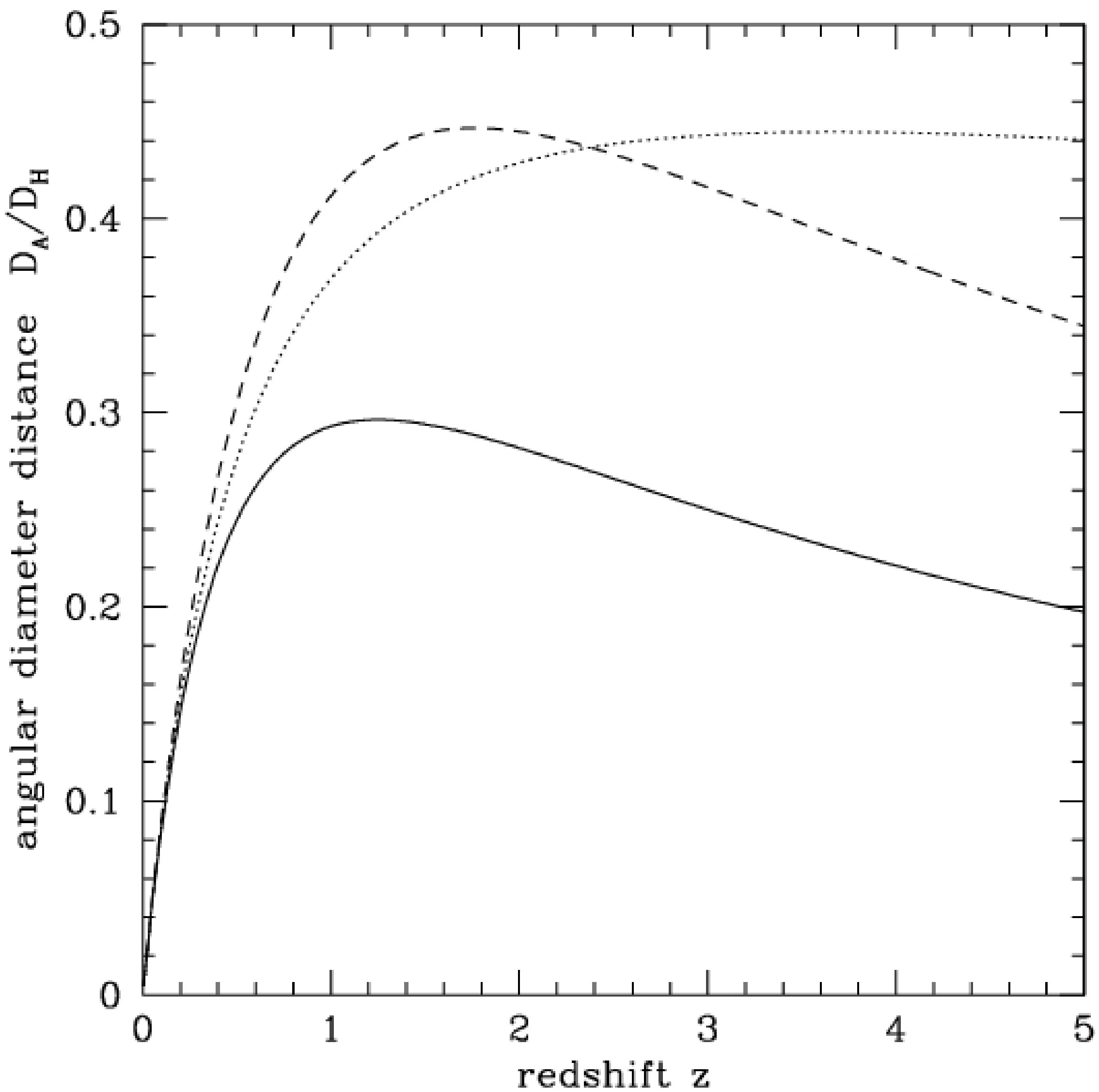
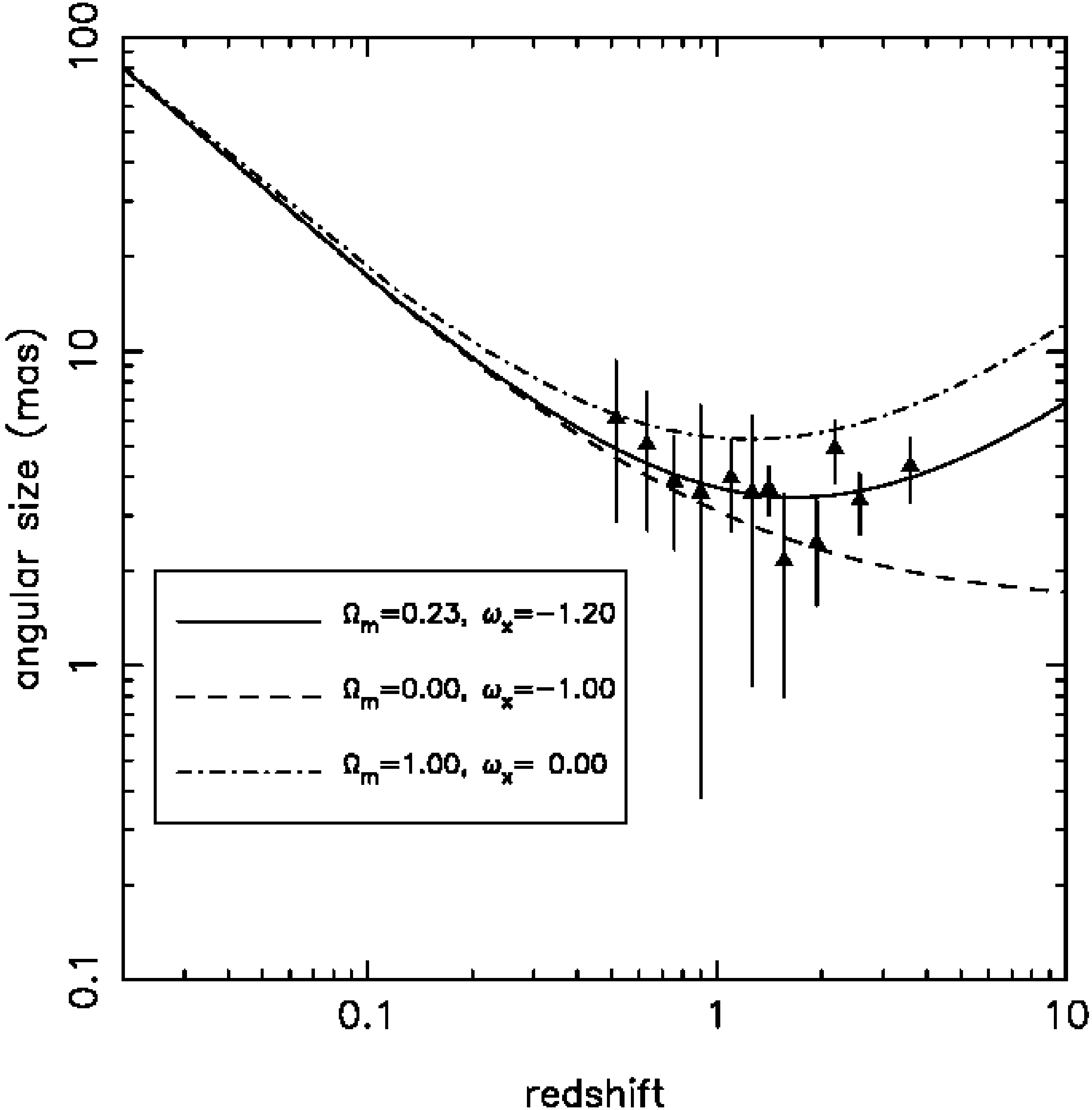


Figure 2: The dimensionless angular diameter distance D_A/D_H. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

The Angular Diameter Distance

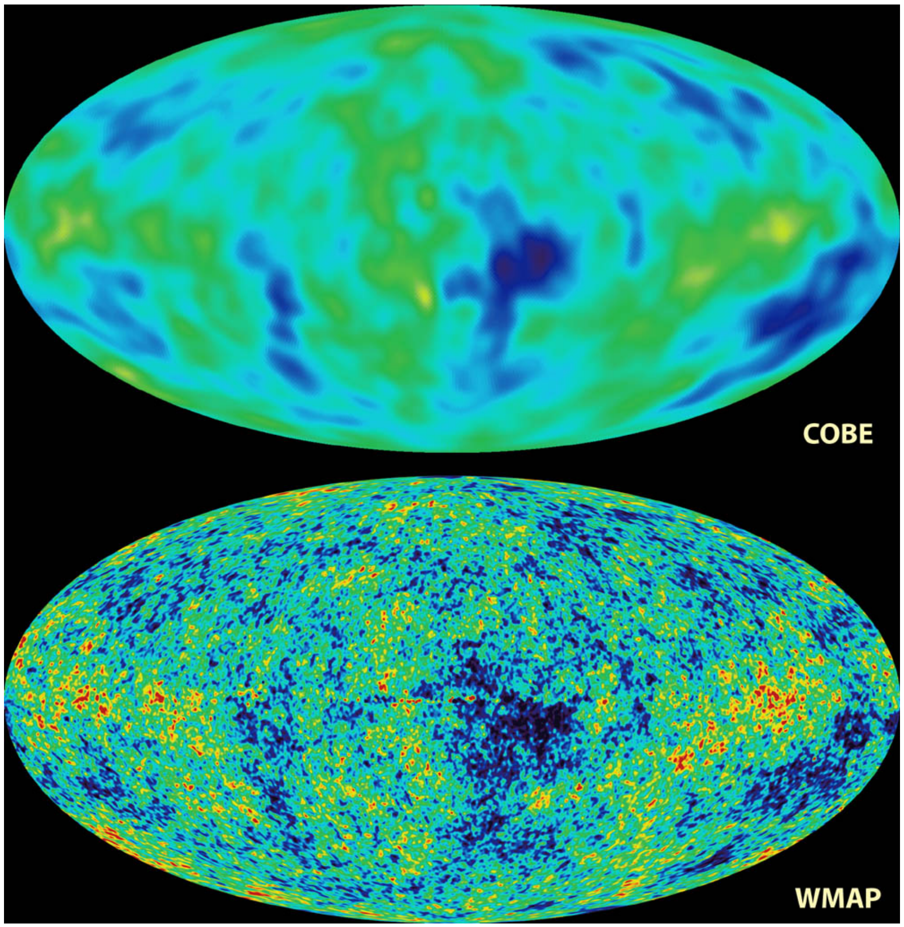
- An interesting/amusing fact: the angular diameter distance has a local maximum around $z \sim 1.5$. Thus, an object becomes apparently smaller as we move it out to that distance, but then its size **increases** with distance!
- Check out a nice compendium of cosmological formulae by Hogg (2000, astro-ph/9905116).

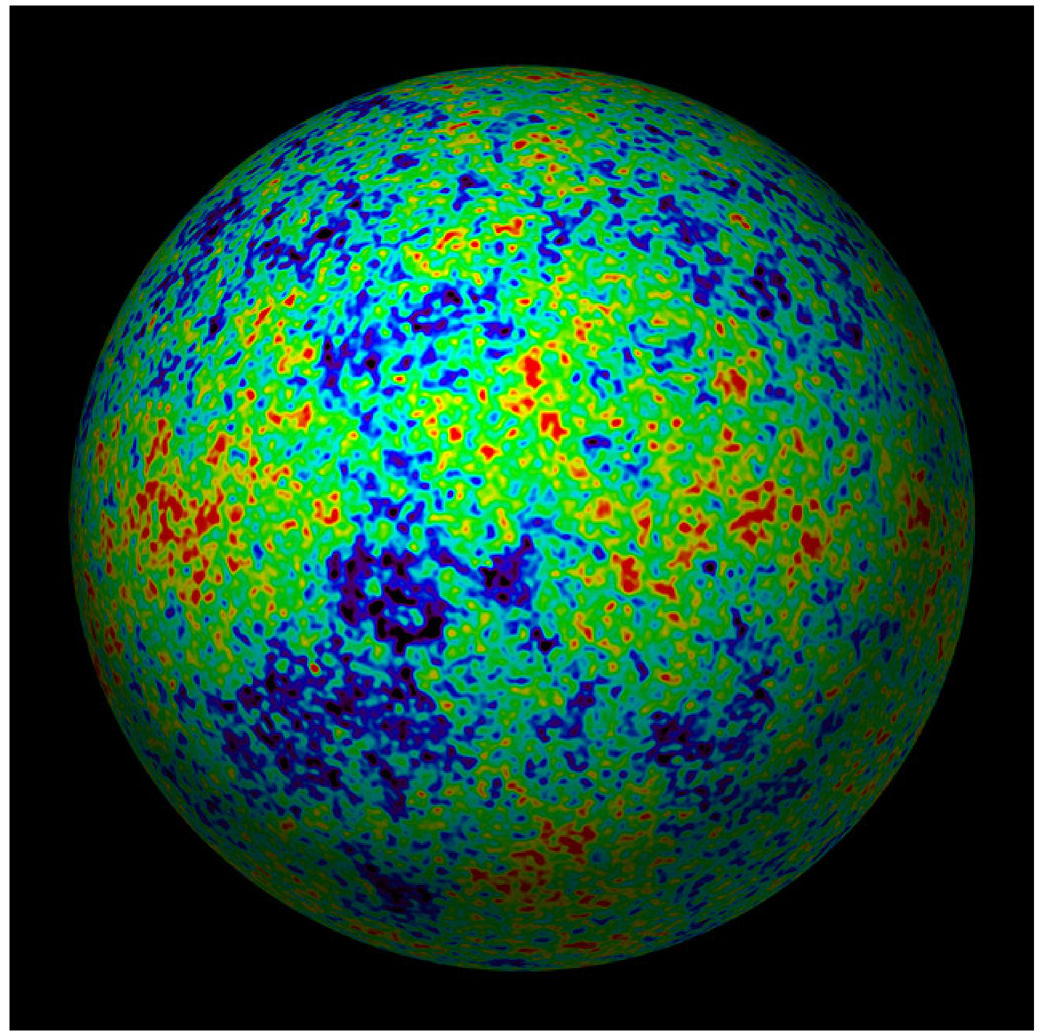


100
10
1
0.1
angular size (mas)
0.1
1
10
redshift
Ω_m=0.23, ω_x=−1.20
Ω_m=0.00, ω_x=−1.00
Ω_m=1.00, ω_x= 0.00

Cosmic Microwave Background (CMB)

- The CMB fluctuations, recently observed by WMAP at a high angular resolution, show a characteristic size of $\sim 1^\circ$
- How do we mathematically describe this behavior? How do we compare models to these observations?





WMAP Power Spectrum

Basic flat WMAP parameters: $W_L = 0.71$, $W_m = 0.29$ ($W_c = 0.24$, $W_b = 0.047$), $n = 0.93$, $h = 0.71$.

WMAP + other: $W_L = 0.71$, $W_m = 0.27$ ($W_c = 0.23$, $W_b = 0.044$), $n = 0.93$, $h = 0.71$.