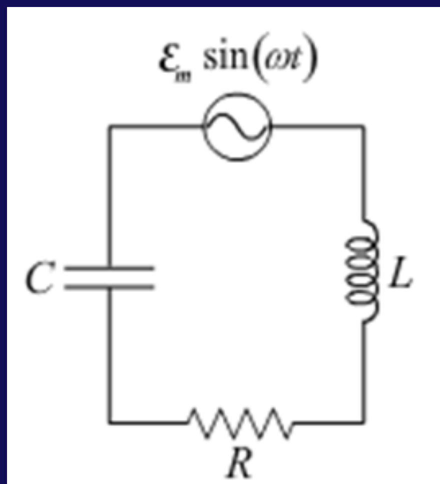
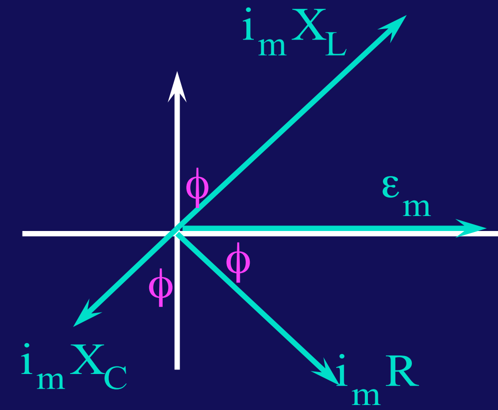
AC Circuits
Resonance





$$X_L \equiv \omega L$$

$$X_C \equiv \frac{1}{\omega C}$$

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

Your thoughts: “I honestly think this is the most confused I've been all quarter. Could you go over all the relationships in the circuits?”

My advice: “Yeah, this part is dreadful. Try to follow qualitatively more than quantitatively.”

Physics 122 Lecture 26
G. Rybka

Core Concepts

- **Here is what I'd like you all to understand**
  - Why do LRC circuits resonate
  - How is LRC resonance like a pendulum
  - What does "phase" mean?
  - What's going on inside a wall-wart transformer

- **If you plan on doing electrical engineering**
  - You'd better learn your phasors, but I highly recommend looking up the complex impedance formalism after class.

Last Time: LC Oscillations

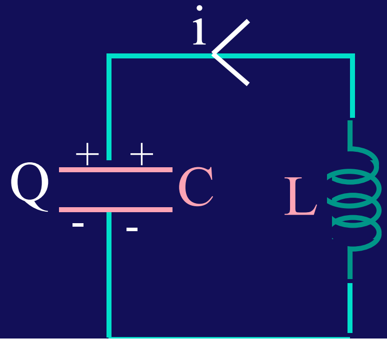
Energy conserved in this ideal world

Energy in E field: $$U_E(t) = \frac{1}{2C} Q_0^2 \cos^2(\omega_0 t + \phi)$$

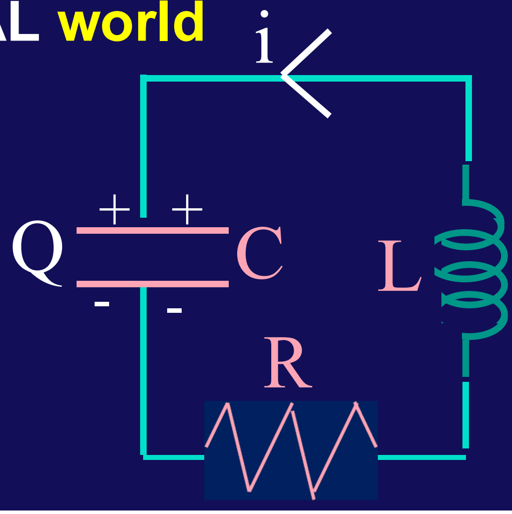
Energy in B field: $$U_B(t) = \frac{1}{2} L\omega_0^2 Q_0^2 \sin^2(\omega_0 t + \phi)$$

$$U_E(t) + U_B(t) = \frac{Q_0^2}{2C}$$

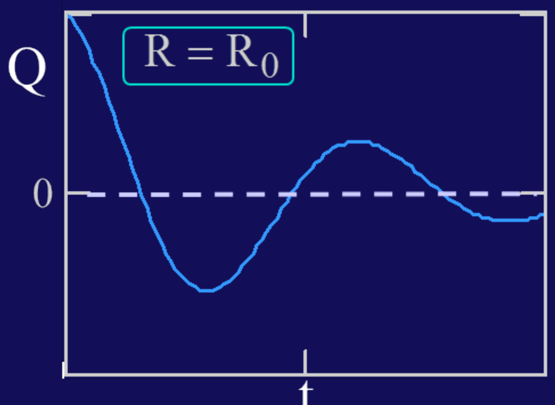
$$\omega_0 = \frac{1}{\sqrt{LC}}$$



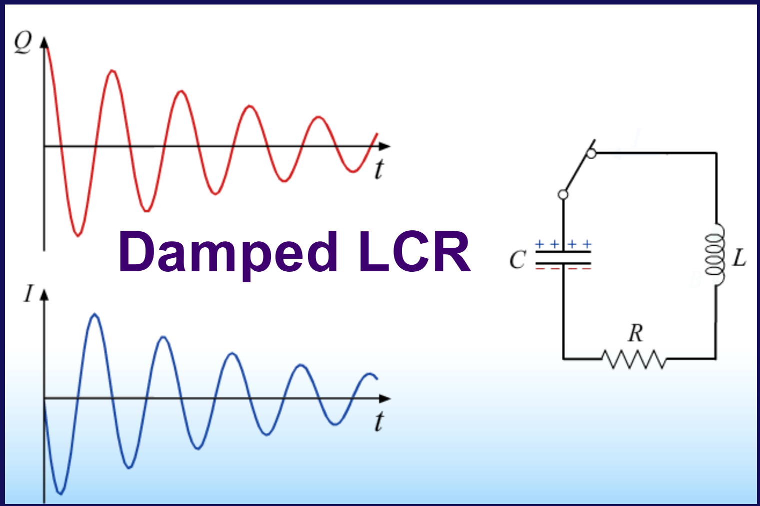
The REAL world



Energy dissipated in finite resistance



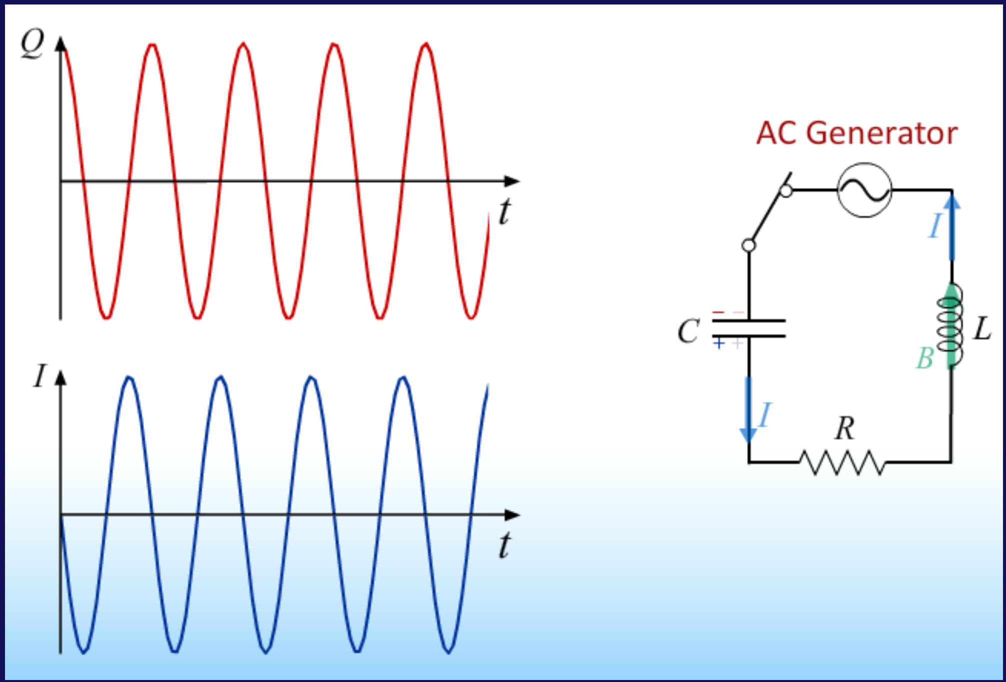
To sustain the oscillations, we must **drive** the circuit at frequency ω. *Your AC wall outlet is an example*



$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}+R\frac{dQ}{dt}=0$$

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}+R\frac{dQ}{dt}=\varepsilon_m \sin\omega t$$

Series LCR with an oscillating DRIVER source



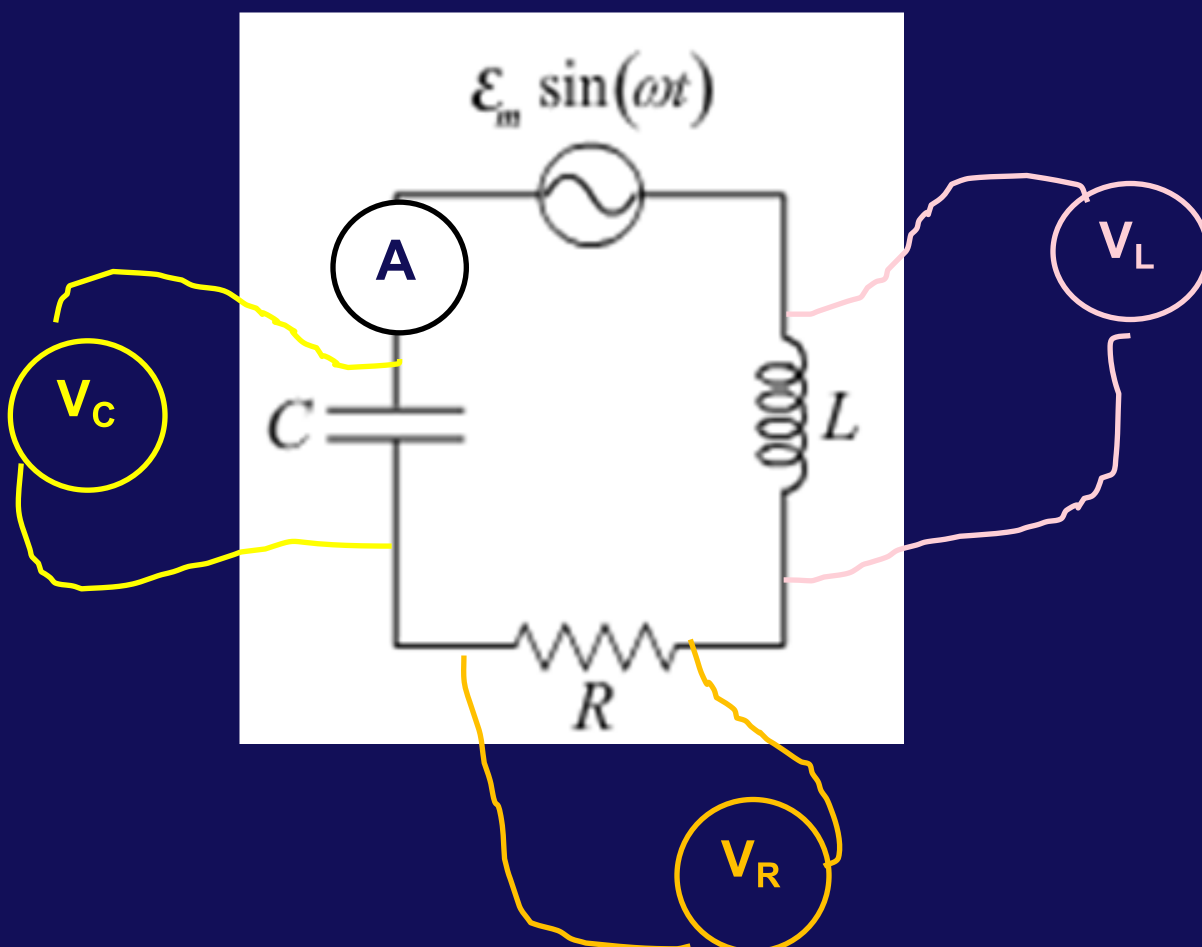
Note: ω is <u>not necessarily</u> ω_0

Next:

"Voltage Across" vs. "Current Through"

Can be out of phase

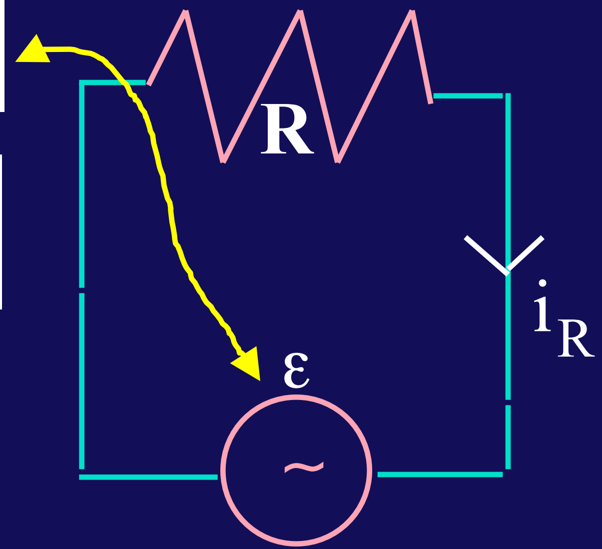
Always the same for each element at same time



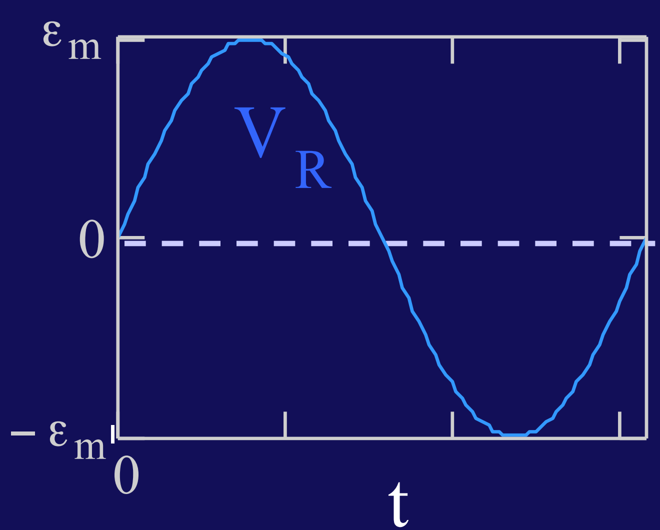
Consider 3 simple cases: ε & R Only

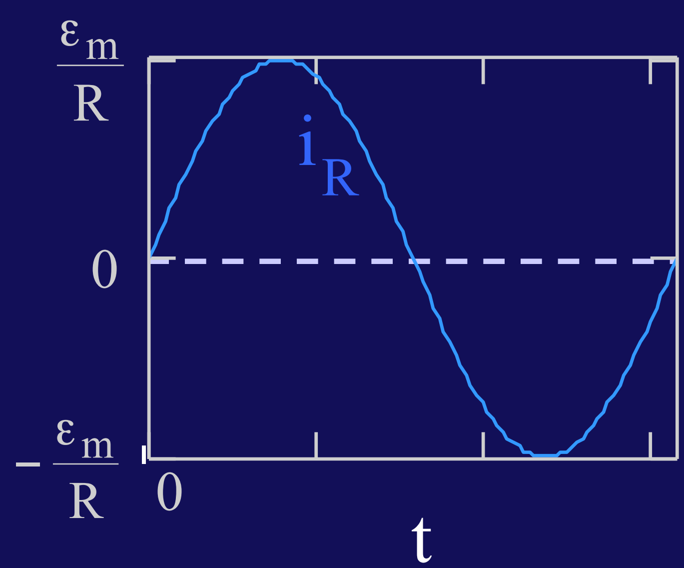
- **Loop eqn gives:** $V_R = Ri_R = \varepsilon_m \sin \omega t$

Þ $i_R = \dfrac{\varepsilon_m}{R} \sin \omega t$



Voltage across R is in phase with the current through R





Note: this is always true

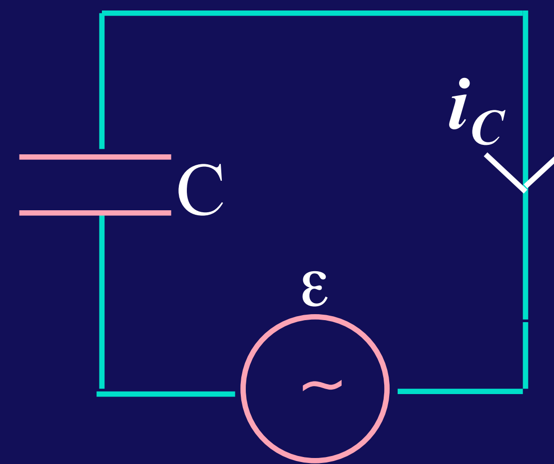
… ***always***

Now, ε & C only

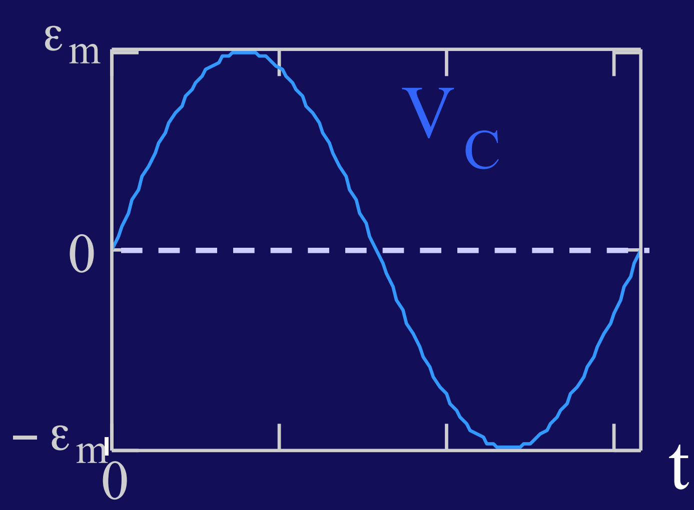
$$V_C = \frac{Q}{C} = \varepsilon_m \sin \omega t$$

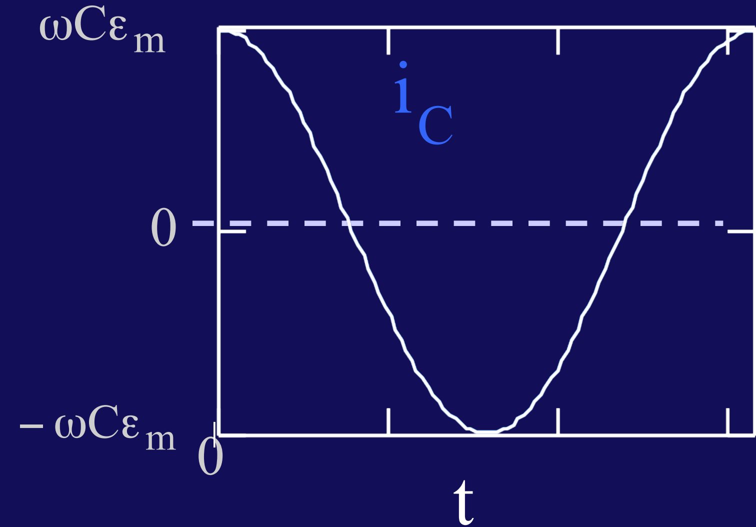
þ $$Q = C\varepsilon_m \sin \omega t$$

þ $$i_C = \frac{dQ}{dt} = \omega C \varepsilon_m \cos \omega t$$



- Voltage across C (V_C) is <u>not in phase</u> with current.
- It "lags" by one-quarter cycle (90°).





Is this always true?

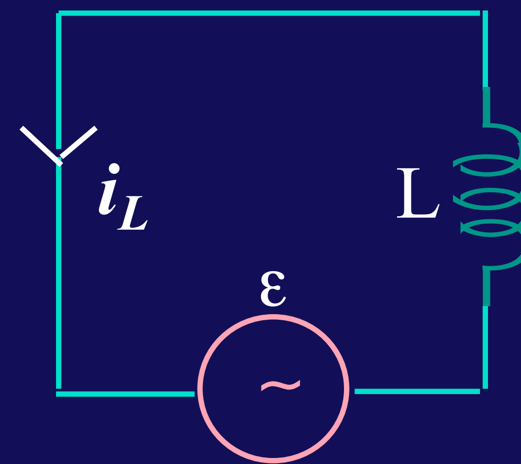
YES

Finally, ε & L only

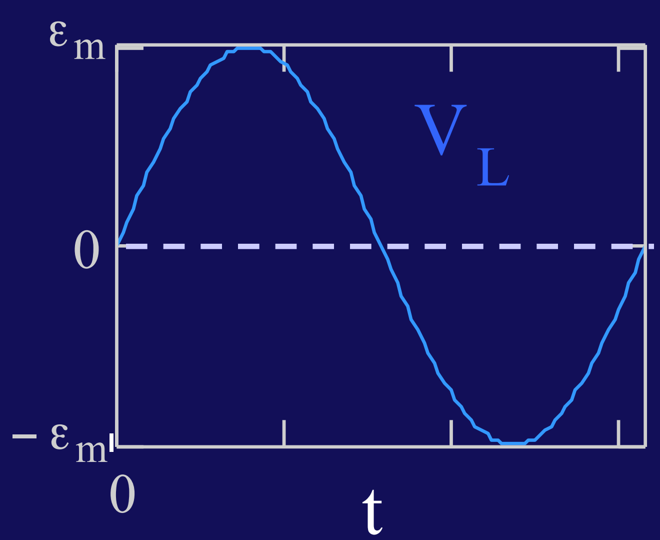
$$V_L = L\frac{di_L}{dt} = \varepsilon_m \sin\omega t$$

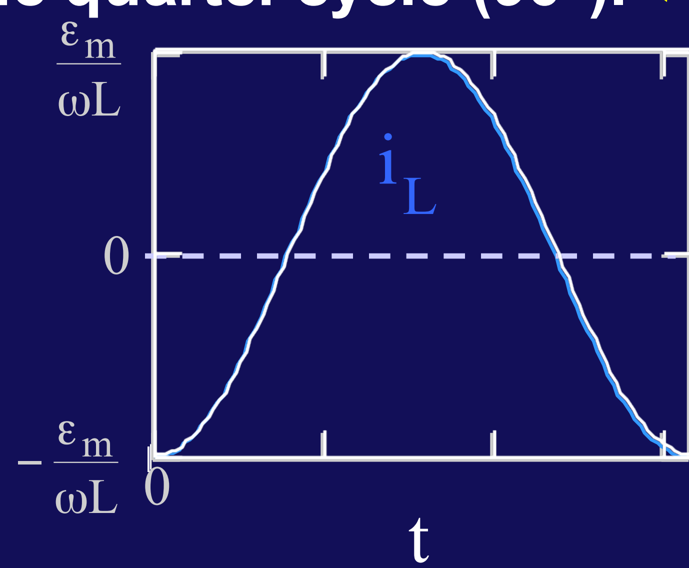
þ $$di_L = \frac{\varepsilon_m}{L}\sin\omega t\,dt$$

þ $$i_L = \int di_L = -\frac{\varepsilon_m}{\omega L}\cos\omega t$$



- Voltage across L (V_L) not in phase with current
- Voltage "leads" current by one quarter cycle (90°).





Yes, true, but how to remember?

How to represent all these equations?

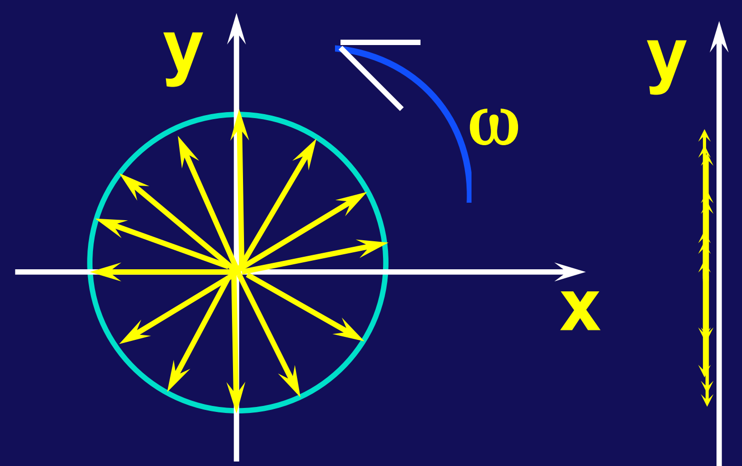
- **R:** V in phase with i $V_R = Ri_R = \varepsilon_m \sin\omega t$ þ $i_R = \frac{\varepsilon_m}{R}\sin\omega t$
- **C:** V lags i by 90° (behind) $V_C = \frac{Q}{C} = \varepsilon_m \sin\omega t$ þ $i_C = \omega C\varepsilon_m \cos\omega t$
- **L:** V leads i by 90° (ahead) $V_L = L\frac{di_L}{dt} = \varepsilon_m \sin\omega t$ þ $i_L = -\frac{\varepsilon_m}{\omega L}\cos\omega t$

PHASORS: A vector whose magnitude is the maximum value of a quantity, which rotates counterclockwise with angular velocity ω.

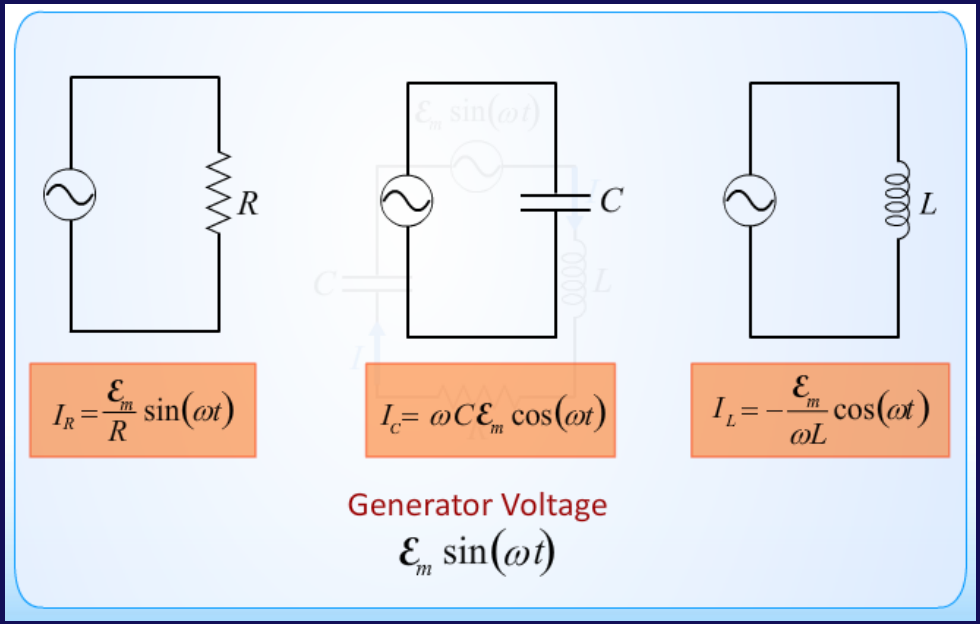
"Sorry, but phasors belong in star trek, not in physics"

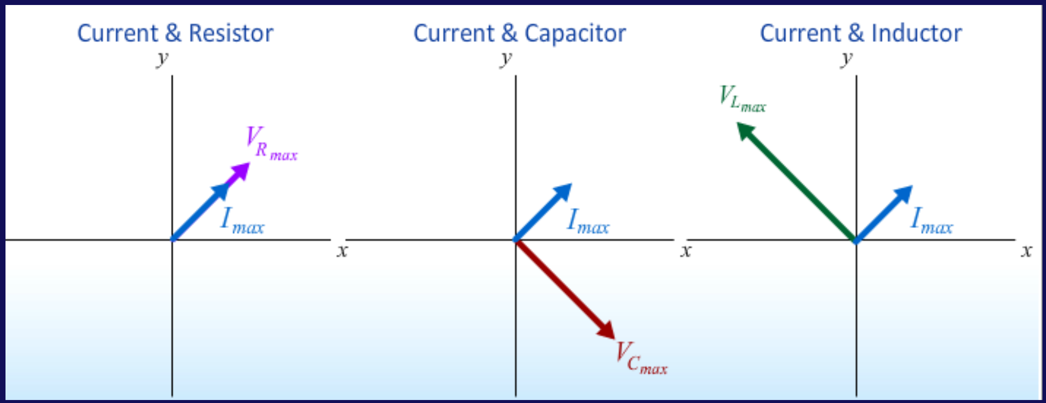
The projections of r (on the vertical y axis) execute sinusoidal oscillation.

$$x = r\cos\omega t$$
$$y = r\sin\omega t$$



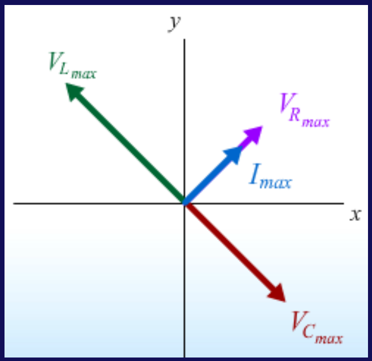
Apply this graphical tool to our 3 simple circuits

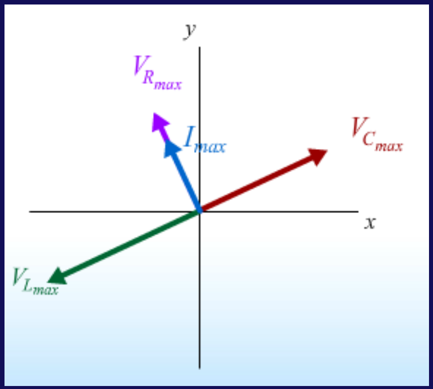


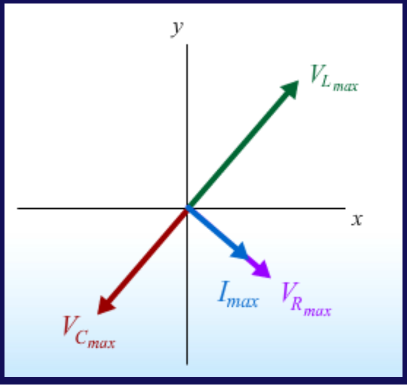


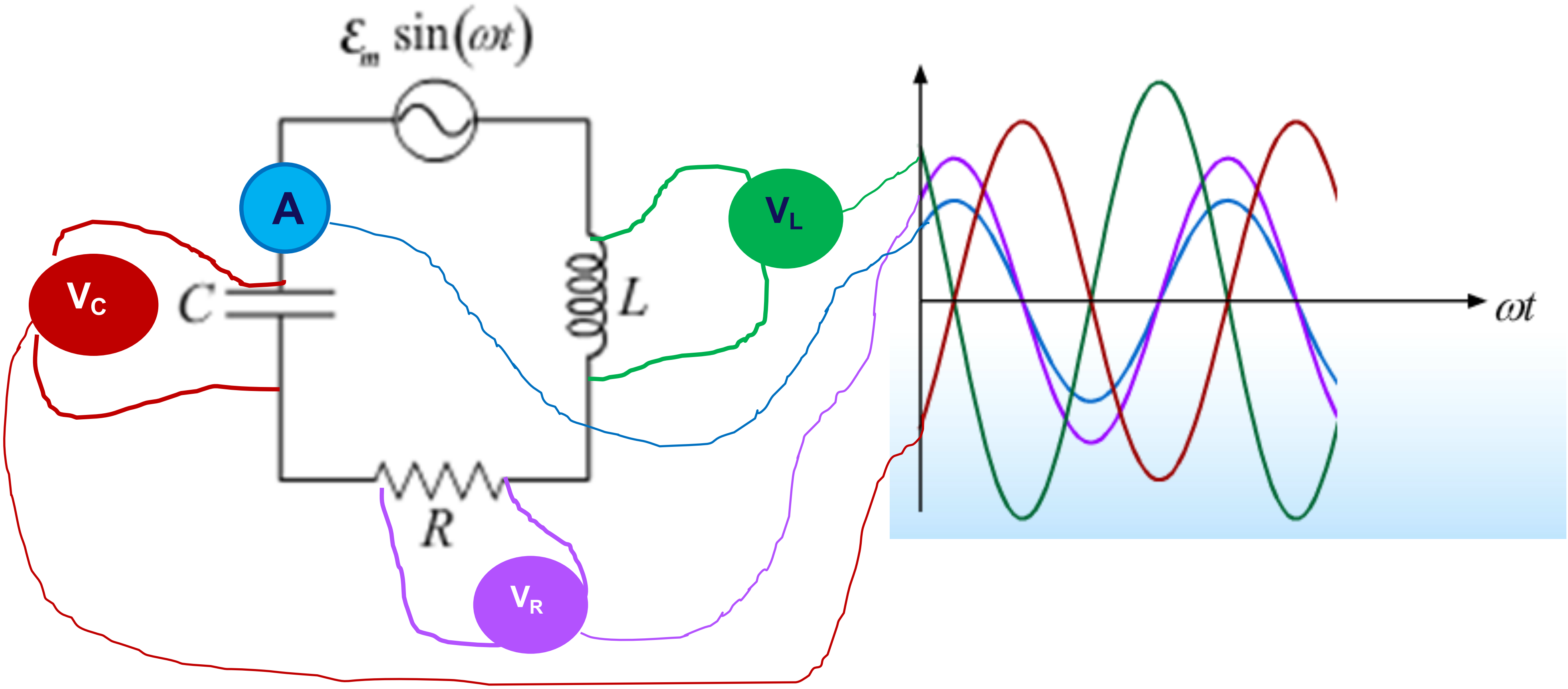
3 simple Phasor Diagrams

And now to all three components at once









Introducing...

ELI the ICE man

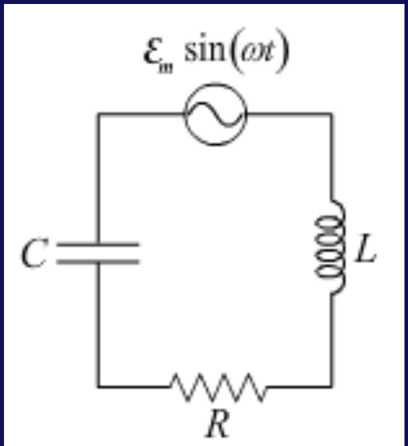
V_L leads I_L

V_C lags I_C

… we’ll see …

Now, to solve the series LCR circuit

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} + R\frac{dQ}{dt} = \varepsilon_m \sin \omega t$$



Solution form:

$$i = i_m \sin(\omega t - \phi)$$

We now must simply jump ahead to the answer

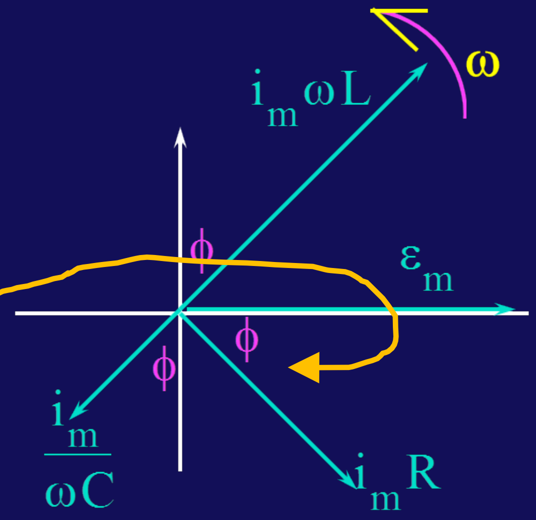
Picture is snapshot at t =0.

Projections are voltages at the given time.

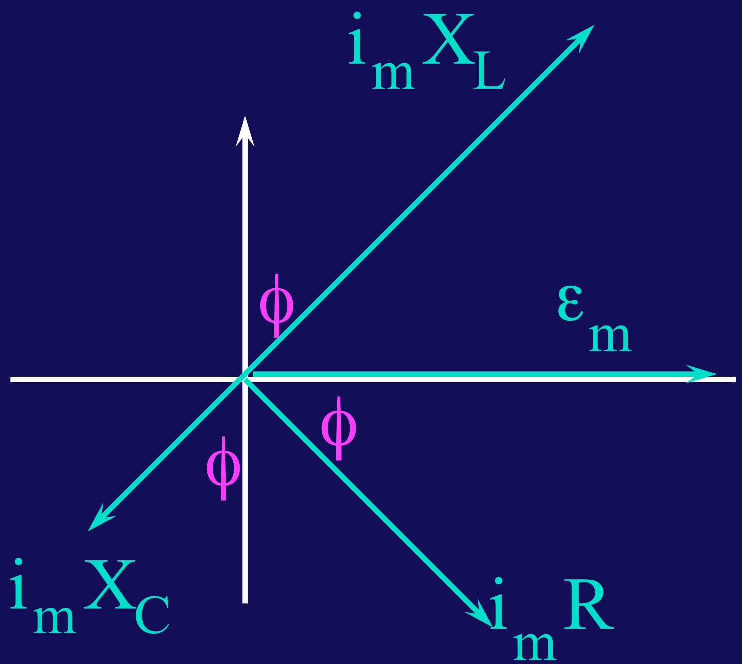
$$V_R = Ri = Ri_m \sin(\omega t - \phi)$$

$$V_C = \frac{Q}{C} = -\frac{1}{\omega C} i_m \cos(\omega t - \phi)$$

$$V_L = L\frac{di}{dt} = \omega L i_m \cos(\omega t - \phi)$$



Phasors, Reactance, Impedance



$$\tan\phi = \frac{X_L - X_C}{R}$$

Þ

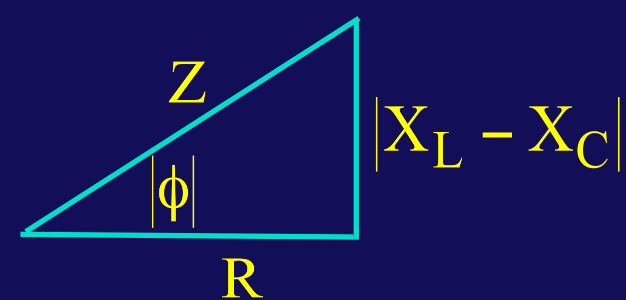
$$i_{max} = \frac{\varepsilon_{max}}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_{max}}{Z}$$

$X_L \equiv \omega L$ **Inductive Reactance**

$X_C \equiv \dfrac{1}{\omega C}$ **Capacitive Reactance**

$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$ **Impedance**

Unit: Ohms !



" Impedance Triangle"

Resonance

- **For driven RLC circuits, i_m will be a maximum at the resonant frequency ω_0 which makes the impedance Z purely resistive.**

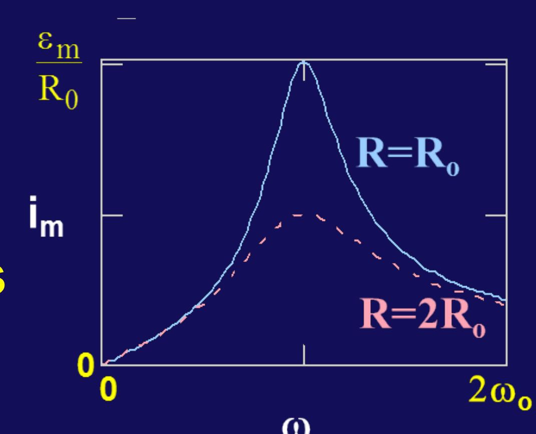
ie:

$$i_m = \frac{\varepsilon_m}{Z} = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

when: $X_L = X_C \qquad \omega_o L = \frac{1}{\omega_o C}$ þ $\boxed{\omega_o = \frac{1}{\sqrt{LC}}}$

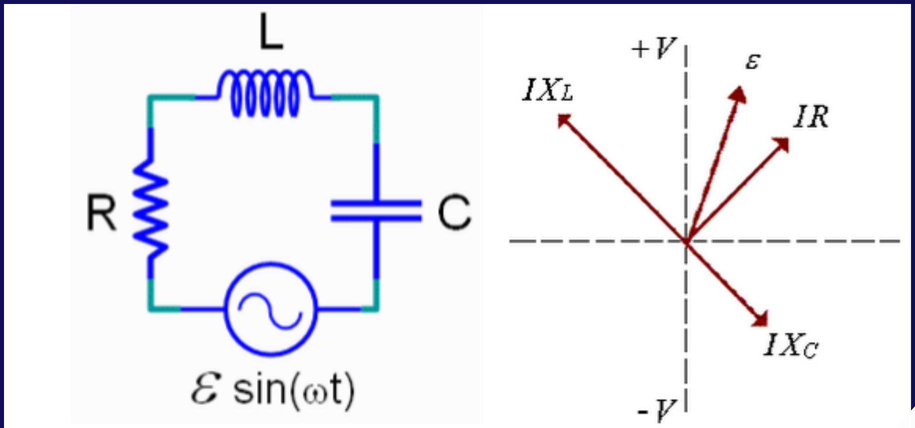
- **Note: identical to the natural frequency of the LC circuit**
- **And, at this frequency, the current and the driving voltage are in phase!**

$$\tan\phi = \frac{X_L - X_C}{R} = 0$$

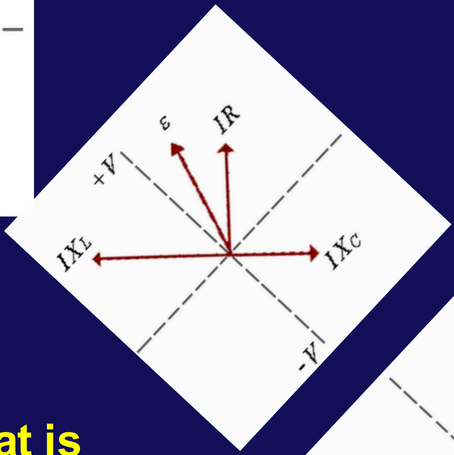


The width is related to the "Q" of the circuit which is very important for tuning. (we have to gloss over this, sorry)

Checkpoints

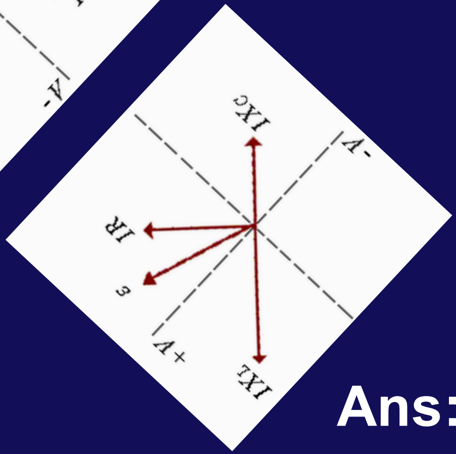


The vertical axis of the phasor diagram represents voltage. When the current through the circuit is maximum, what is the potential difference across the inductor?



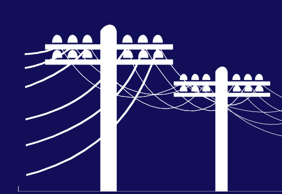
Ans: $V_L = 0$

When the capacitor is fully charged, what is the magnitude of the voltage across the inductor?



Ans: $V_L = V_{Lmax}$

Power Transmission



- **Why use high-voltage to transport electricity ?**
  - **At home: 120V AC at 60Hz.**
  - **Transmission typically ~500 kV**
  - **Transformers raise voltage for transmission; lower for use**
- **Why do we do that?**
  - **Calculate ohmic losses in the <u>transmission lines</u>:**
  - **Define *efficiency* of transmission:**

Resistance of long wires

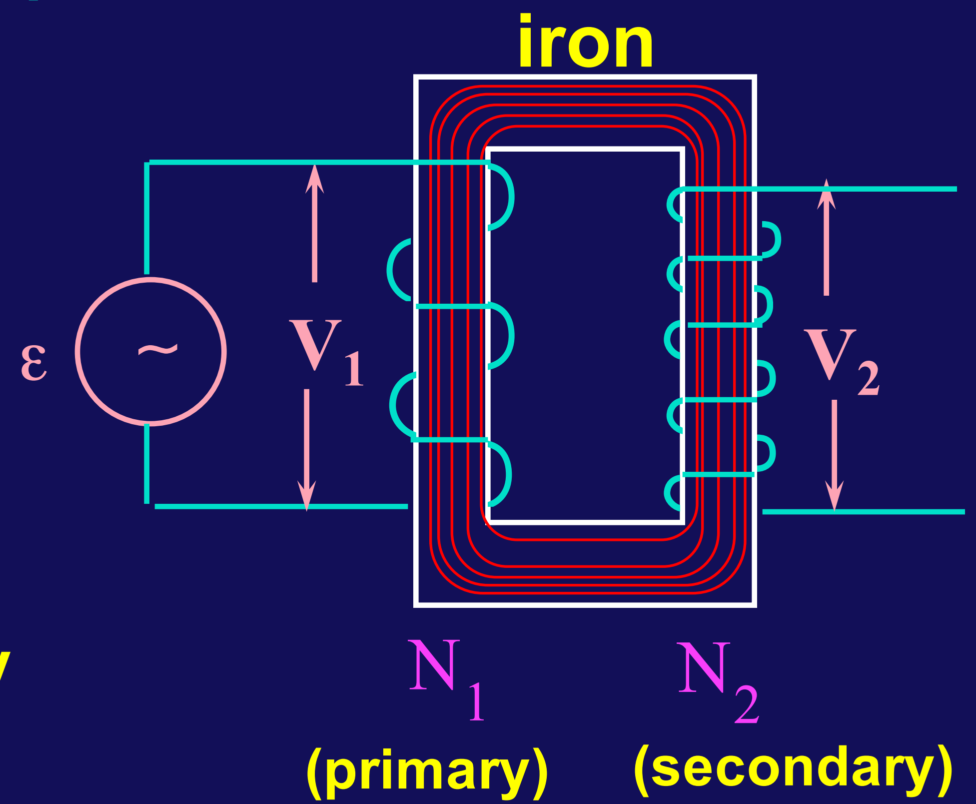
$$\varepsilon ff \equiv \frac{P_{out}}{P_{in}} = \frac{iV_{in} - i^2 R}{iV_{in}} = 1 - \frac{iR}{V_{in}}\left(\frac{V_{in}}{V_{in}}\right) = 1 - \frac{P_{in} R}{V^2{}_{in}}$$

Keep R small

Make V_{in} big

Transformers

- AC voltages can be *stepped up* or *stepped down*



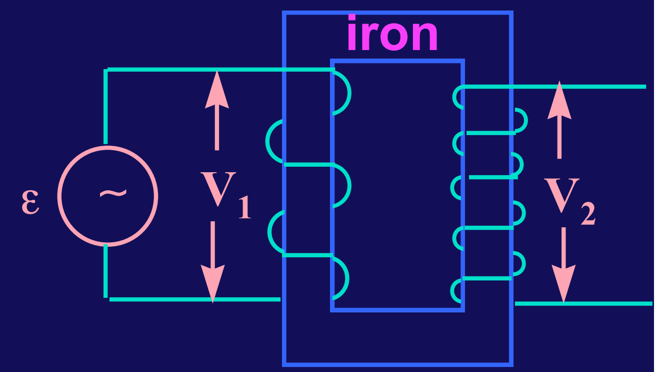
- The AC current in the primary circuit creates a time-varying magnetic field in the iron
- Induced *emf* on the secondary windings
- The iron maximizes mutual inductance. The entire flux produced by each turn of the primary is trapped in the iron for "ideal" transformer

Ideal Transformers (no load)

No resistance losses — All flux contained in iron — Nothing connected on secondary

- The flux in primary for each turn

$$\frac{d\phi_{turn}}{dt} = \frac{V_1}{N_1}$$



- Flux change same in secondary for each turn
- Induced voltage appears across the secondary coil

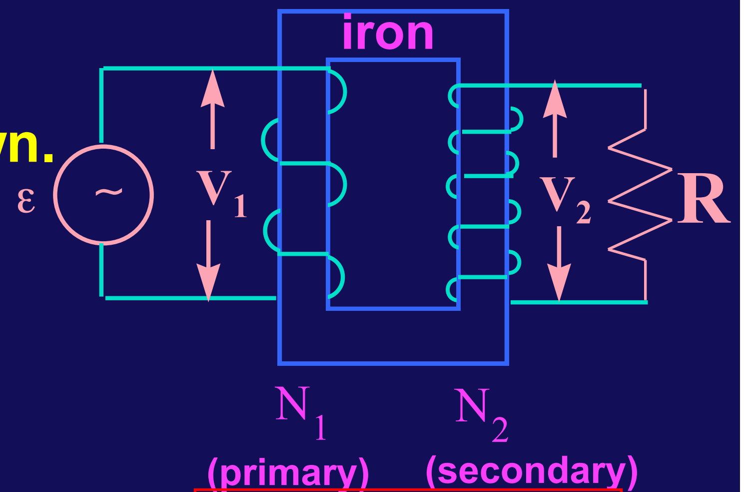
$$V_2 = N_2 \frac{d\phi_{turn}}{dt} = \frac{N_2}{N_1} V_1$$

$N_2 > N_1$ Þ secondary V_2 is larger than primary V_1 (step-up)

$N_1 > N_2$ Þ secondary V_2 is smaller than primary V_1 (step-down)

Clicker

- The primary coil of an ideal transformer is connected to an AC voltage source as shown. There are 50 turns in the primary and 200 turns in the secondary.
  - If V_1 = 120 V, what is the potential across the resistor R ?



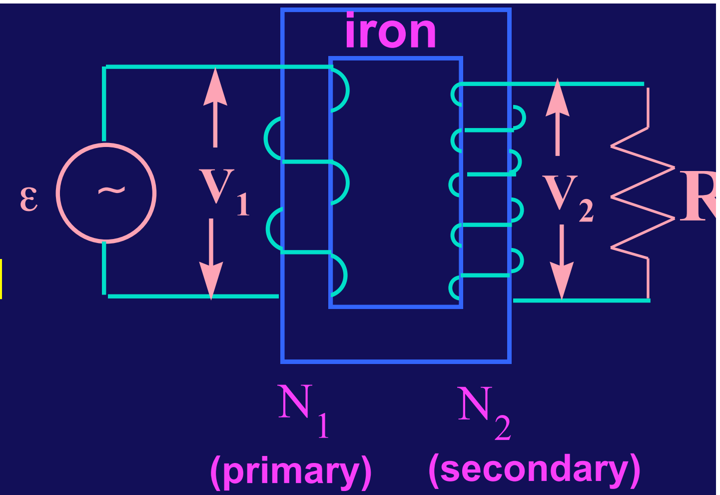
(a) 30 V (b) 120 V (c) 480 V

The ratio of turns, $(N_2/N_1) = (200/50) = 4$

The ratio of secondary voltage to primary voltage is equal to the ratio of turns, $(V_2/V_1) = (N_2/N_1)$

Therefore, (V_2/V_1) = 480 V

Clicker



- The primary coil of an ideal transformer is connected to a 120 AC voltage source as shown. There are 50 turns in the primary and 200 turns in the secondary.
  - If 960 W are dissipated in the resistor R, what is the current in the primary ?

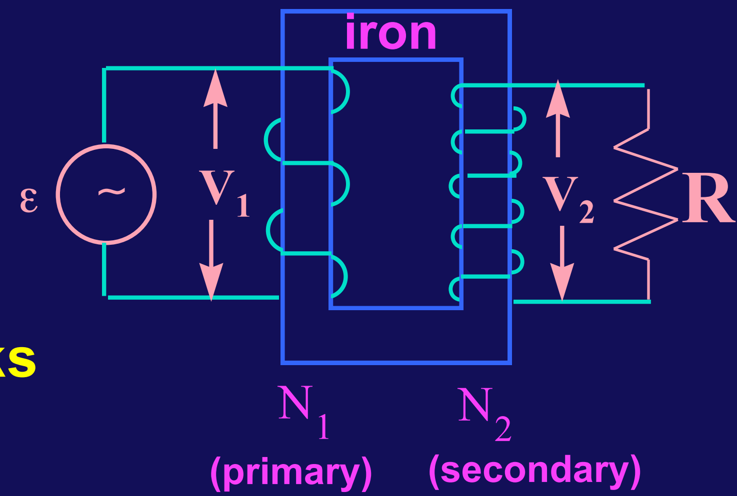
(a) 8 A (b) 16 A (c) 32 A

Let's assume energy is conserved

Therefore, 960 W should be produced in the primary

$P_1 = V_1 I_1$ implies that 960W/120V = 8 A

Transformers with a Load



To get that last Clicker, you had to use a general philosophy -- energy conservation.

An expression for the RMS power flow looks like this:

$$P_{rms} = V_{1rms} i_{1rms} = \frac{N_1}{N_2} V_{2rms} \frac{N_2}{N_1} i_{2rms} = V_{2rms} i_{2rms}$$

Note: This equation simply says that all power delivered by the generator is dissipated in the resistor ! Energy conservation!!