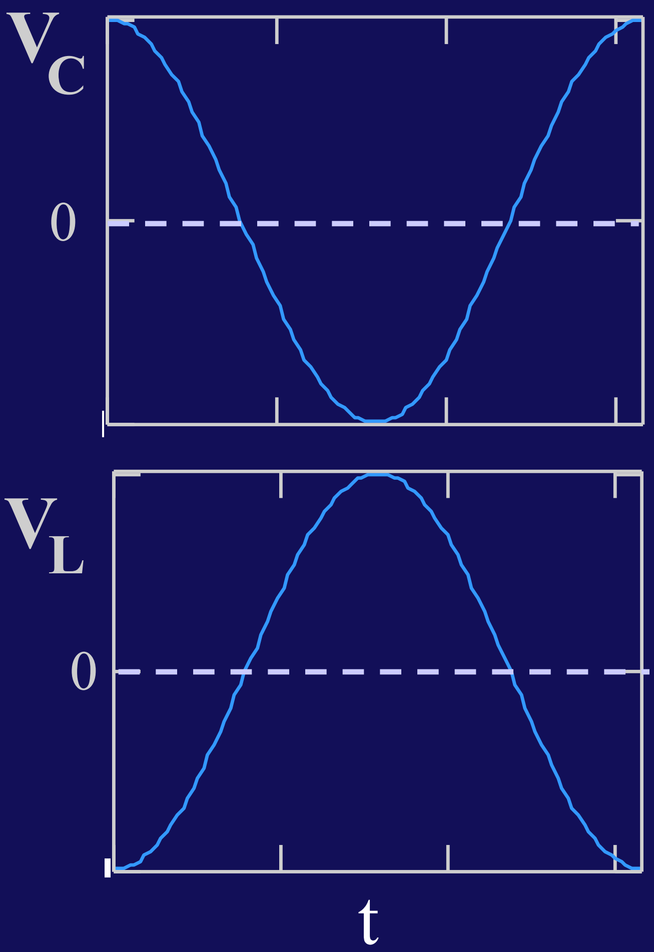
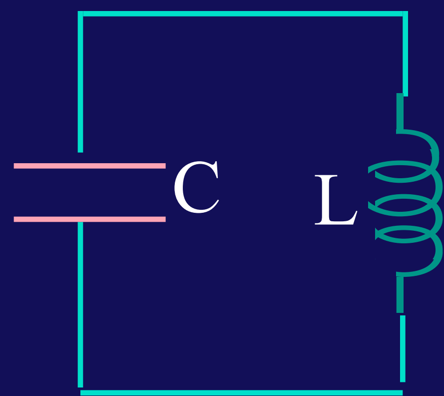
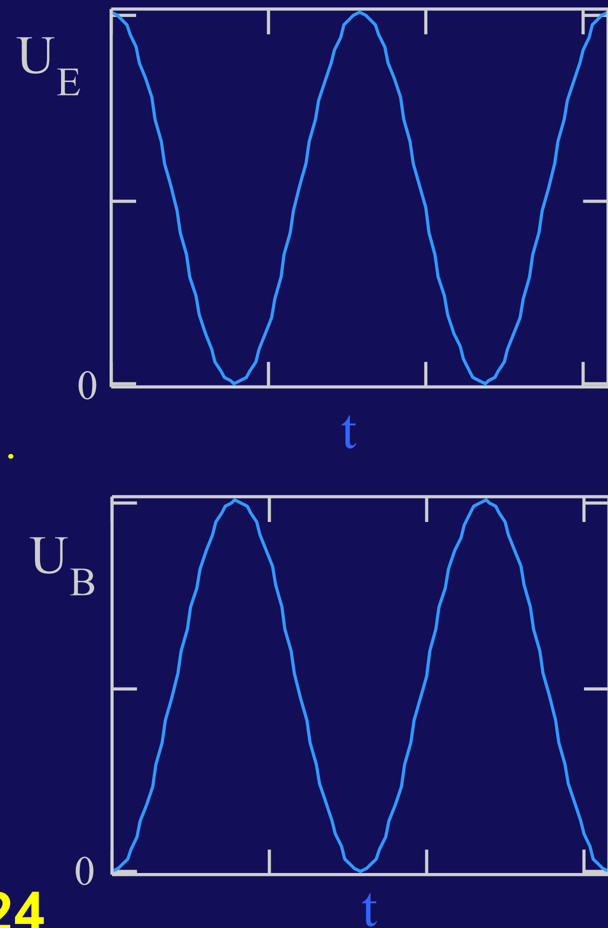
Oscillations





LC Circuits



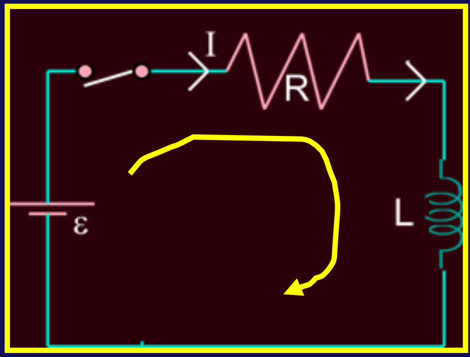
Physics 122 Lecture 24
G. Rybka

Exam 3

- Thursday
- Covers RC circuits through Faraday's law

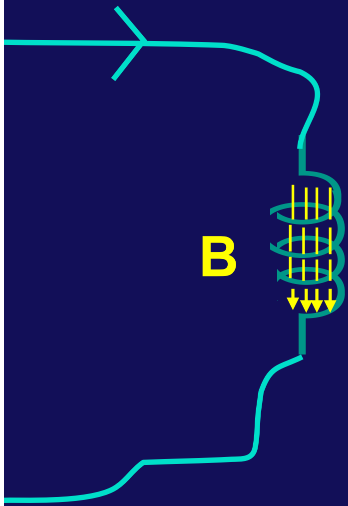
Energy in the *Magnetic* Field

"Power" accounting in a LR circuit...



$$\varepsilon I = I^2 R + LI\frac{dI}{dt}$$ Loop rule x I …

$$P_L = \frac{dU}{dt} = LI\frac{dI}{dt}$$ Rate of energy flow into L



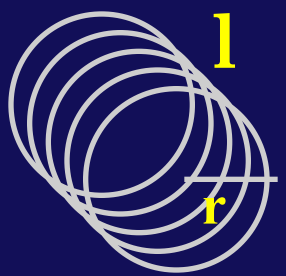
$$U = \int_0^U dU = \int_0^I LI\,dI$$ Total energy flow

$$U = \frac{1}{2}LI^2$$ … energy stored

But where is it "Stored"?

- Claim: energy is stored *in the Magnetic field itself*
- Consider the uniform field inside a long solenoid:

$$B = \mu_0 \frac{N}{l} I$$



- The inductance L is: $$L = \mu_0 \frac{N^2}{l} A$$

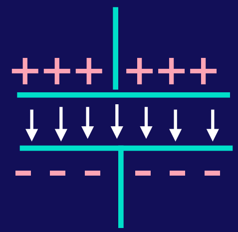
- Stored Energy U: $$U = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 \frac{N^2}{l} A\right) I^2 = \frac{1}{2} A l \frac{B^2}{\mu_0}$$

- Get energy density by dividing by the volume containing the field:

$$u_M = \frac{U}{Al} = \frac{U}{vol} = \frac{1}{2} \frac{B^2}{\mu_0}$$

Recall & Compare

Energy in the *Electric* Field



Work needed to add charge to capacitor...

$$dW = dq(V) = dq\left(\frac{q}{C}\right)$$

$$W = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$$

… total work …

Recall: $C = \varepsilon_0 A/d$ & $V = Ed$

$$u = \frac{W}{\text{volume}} = \frac{1}{2}\varepsilon_0 E^2$$

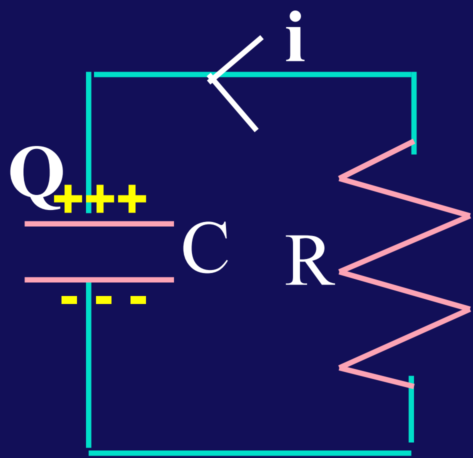
… energy density

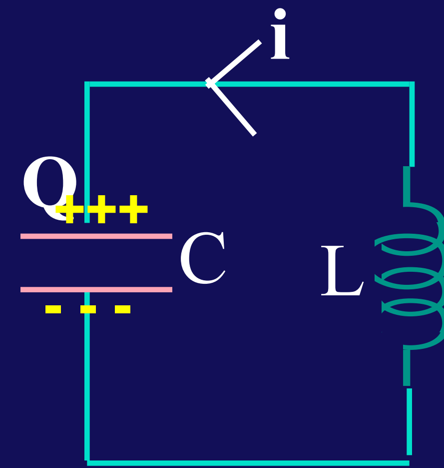
Energy Density:

$$u_{\text{electric}} = \frac{1}{2}\varepsilon_0 E^2$$

$$u_{\text{magnetic}} = \frac{1}{2}\frac{B^2}{\mu_0}$$

RC/LC Circuits



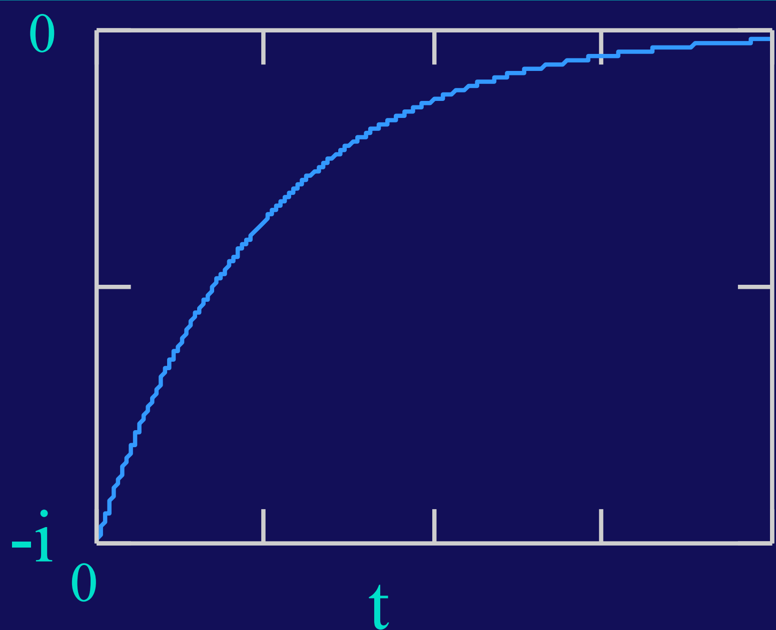


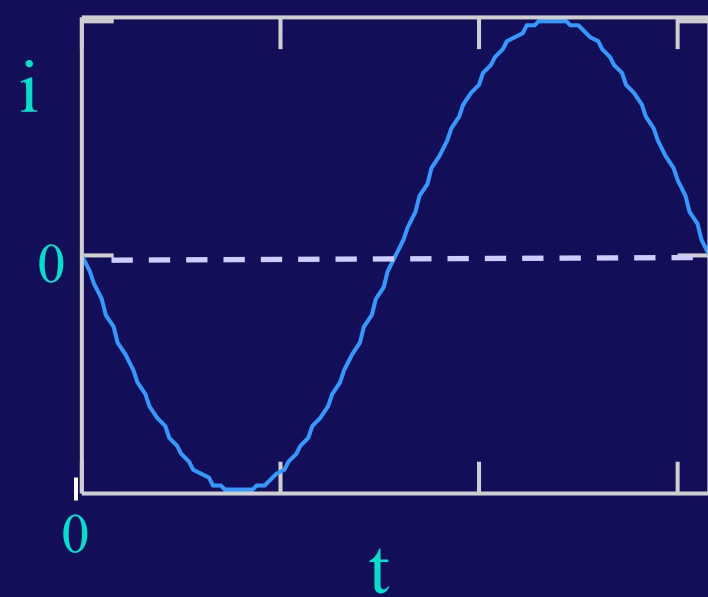
RC or LR:

current decays exponentially

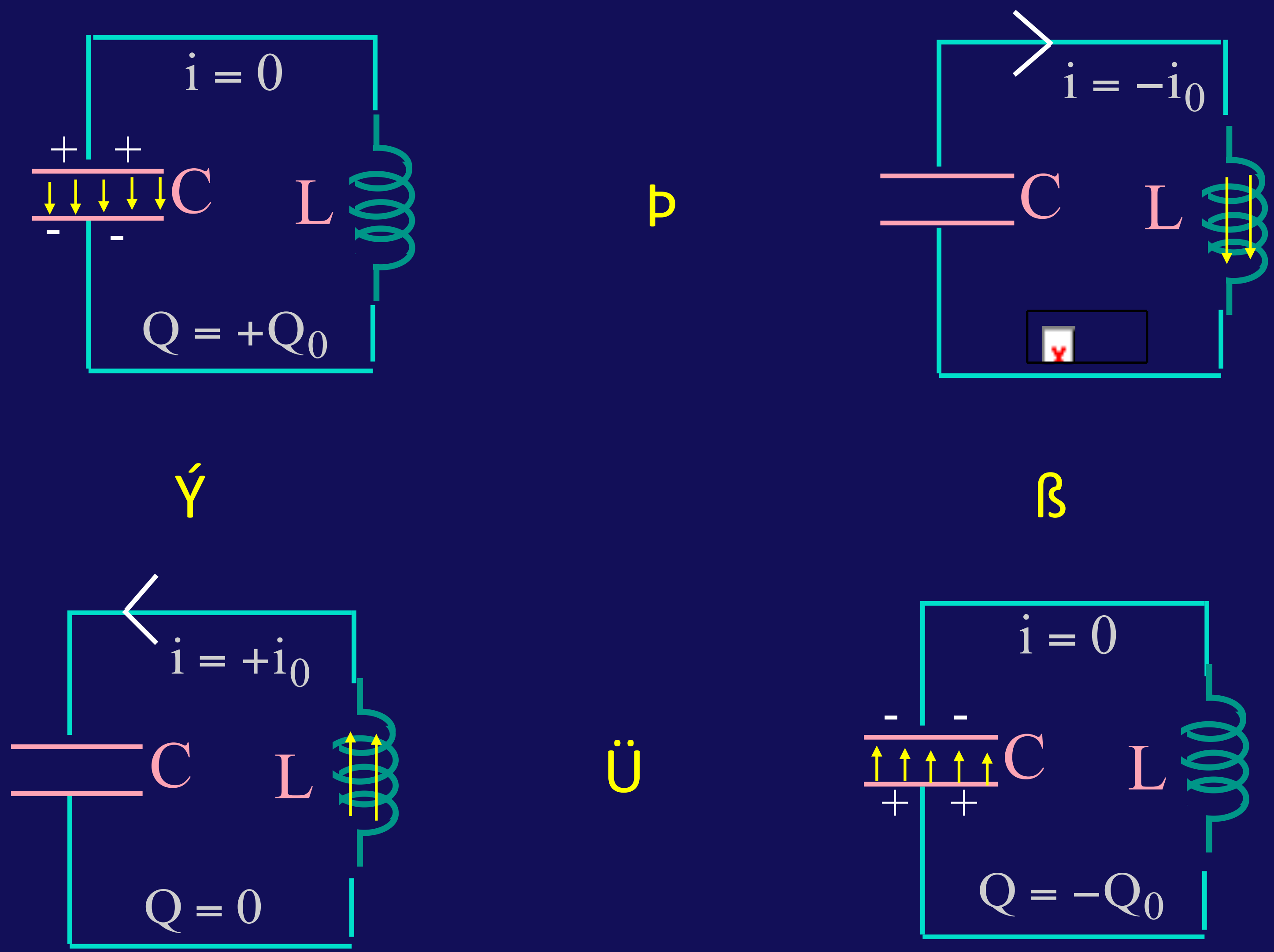
LC:

current oscillates

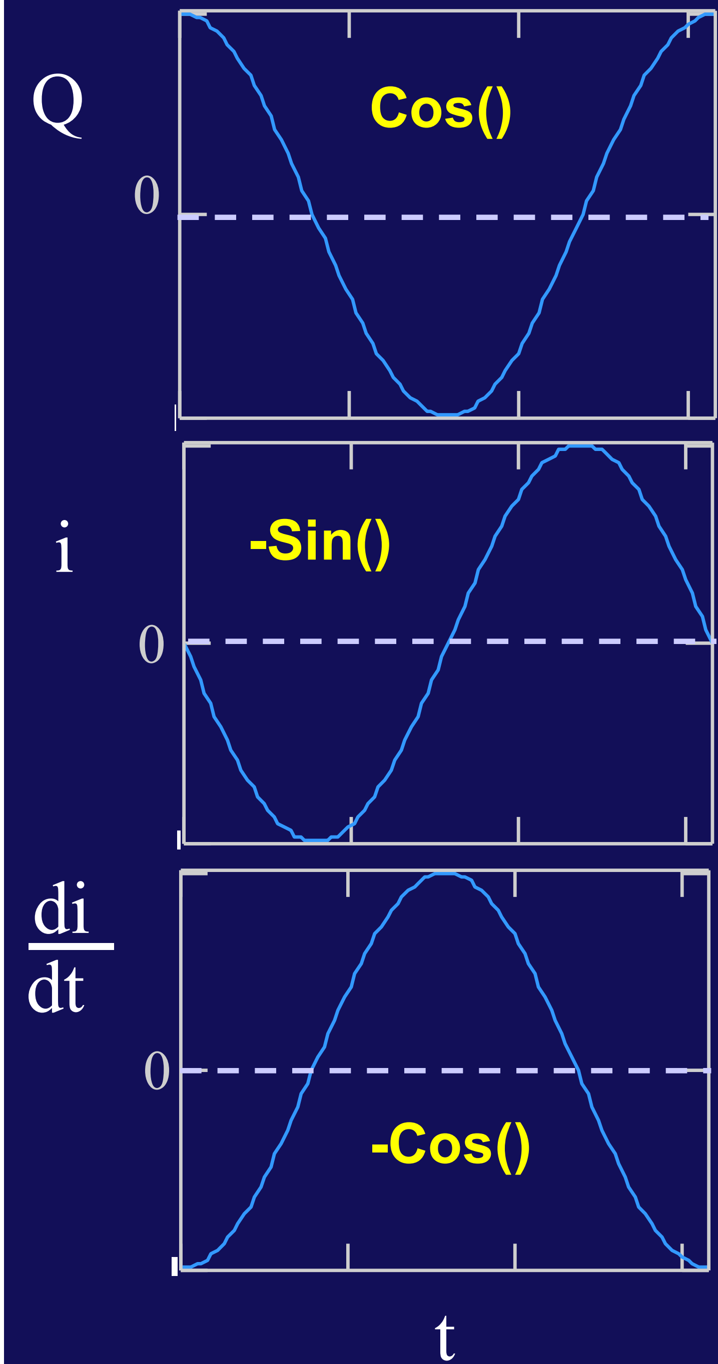


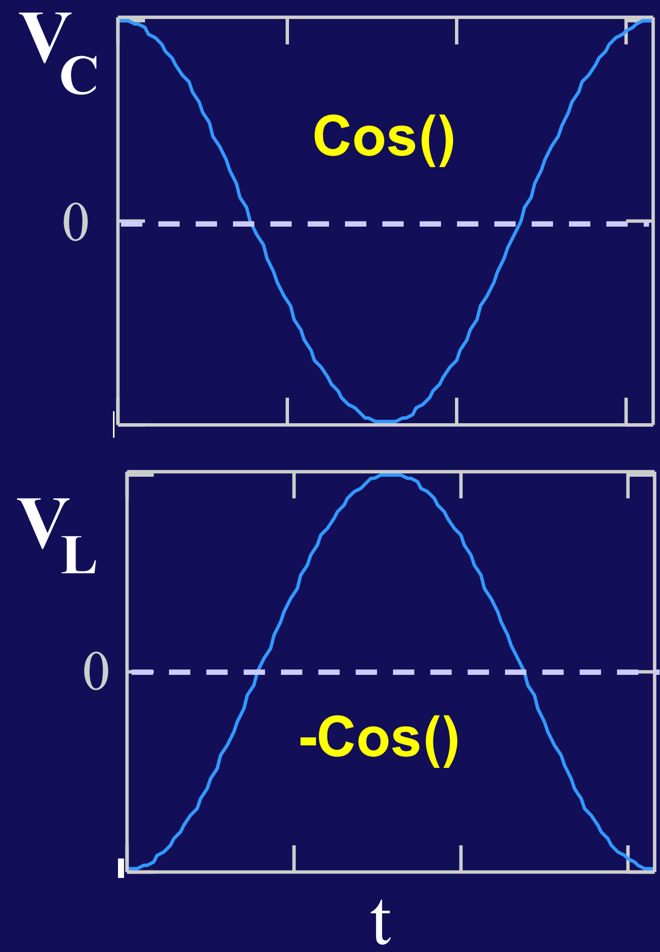


LC Oscillations (qualitative)



LC Oscillations
(qualitative)



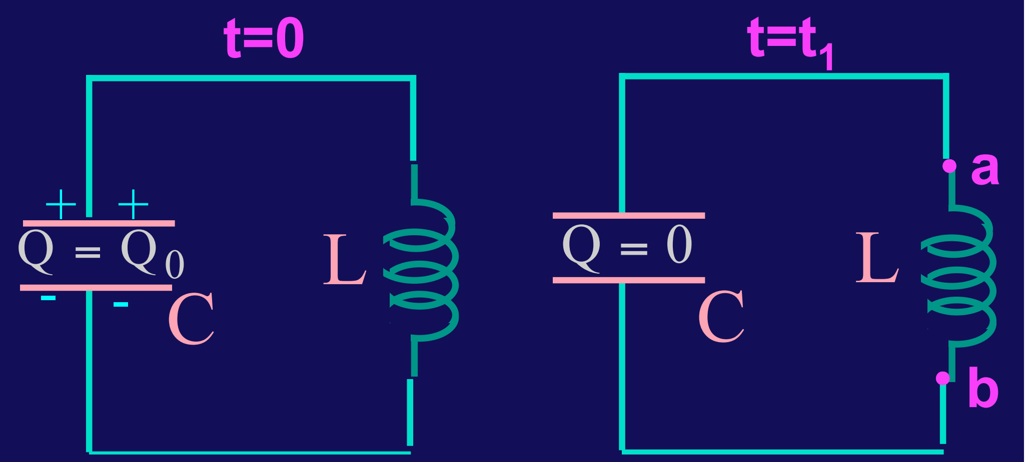


How do these change if L has a finite R?

Clicker

At t=0, the capacitor in the LC circuit shown has a total charge Q_0. At t = t_1, the capacitor is uncharged.

- What is the value of V_{ab}, the voltage across the inductor at time t_1?

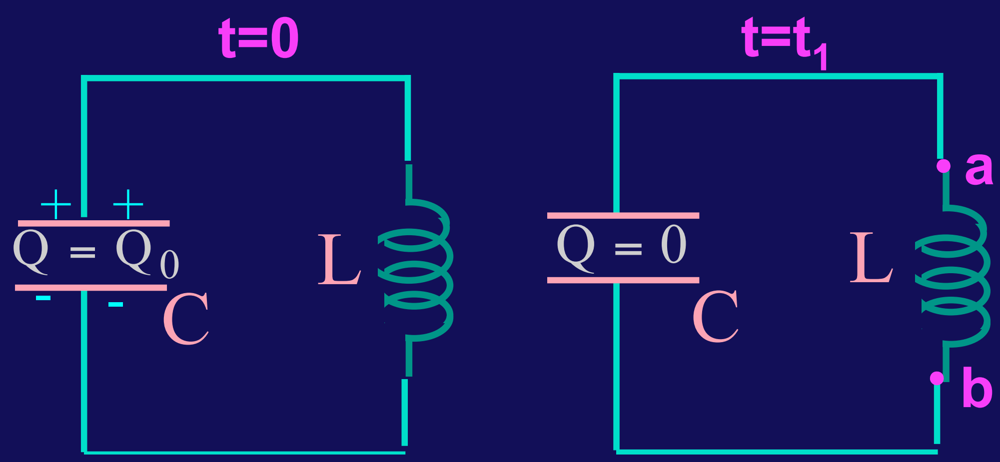


(a) $V_{ab} < 0$ | **(b) $V_{ab} = 0$** | (c) $V_{ab} > 0$

- V_{ab} is the voltage across the inductor, but it is also the voltage across the capacitor!
- Charge on the capacitor is zero, → $V_C = 0$
- When Q = 0 on capacitor, I is maximum through inductor
  - and dI/dt is zero then so $V_L = 0$ makes sense

Clicker

At t = 0, the capacitor in the LC circuit shown has a total charge Q_0. At $t = t_1$, the capacitor is uncharged.



What is the relation between U_{L1}, the energy stored in the inductor at $t = t_1$, and U_{C1} , the energy stored in the capacitor at $t = t_1$?

(a) $U_{L1} < U_{C1}$

(b) $U_{L1} = U_{C1}$

(c) $U_{L1} > U_{C1}$

At $t = t_1$, the charge on the capacitor is zero.

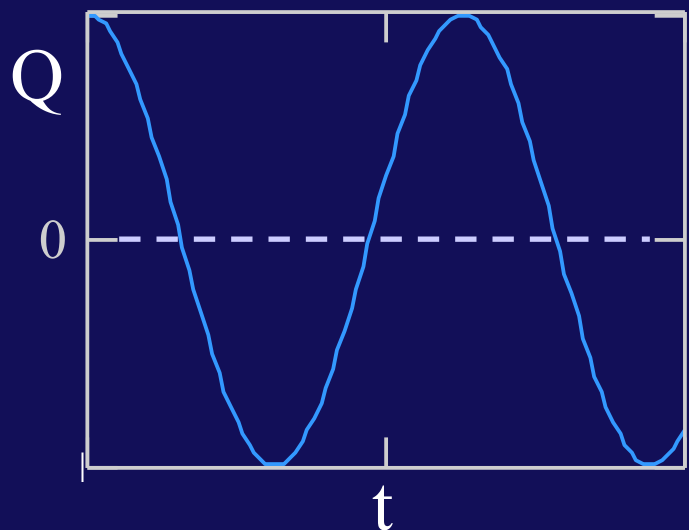
$$U_{C1} = \frac{Q_1^2}{2C} = 0$$

At $t = t_1$, the current is a maximum.

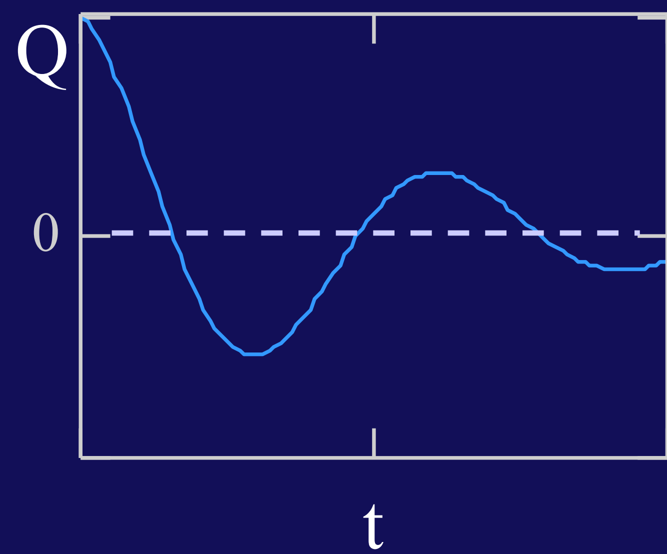
$$U_{L1} = \frac{1}{2} L I_1^2 = \frac{Q_0^2}{2C} > 0$$

LC Oscillations
(L with finite R)

- **If L has finite Resistance, then**
  - **energy will be dissipated in R and**
  - **the oscillations will become damped.**



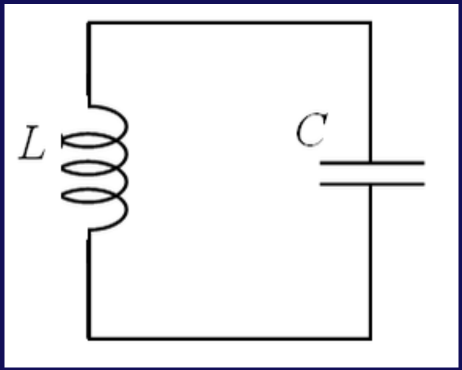
R = 0



R ¹ 0

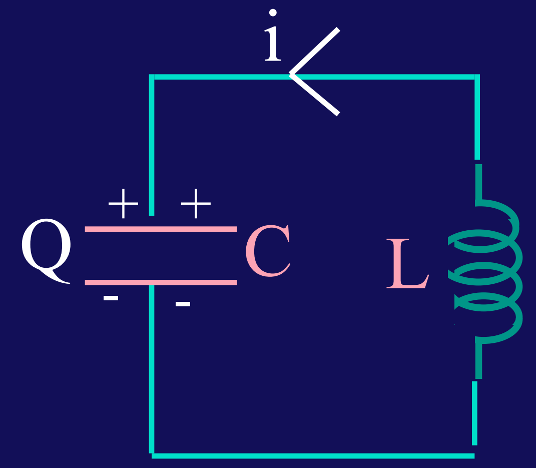
Quick checkpoint review

At time t = 0 the capacitor in the circuit below is fully charged with Q_{max}, and the current through the circuit is 0.



- What is the potential difference across the inductor at t = 0?
  - Ans: $V_L = Q_{max} / C$ (same as capacitor)
- What is the potential difference across the inductor when current is maximum?
  - Ans: 0
- How much energy is stored in C when I is max?
  - Ans: U = 0 (it's all in the inductor)

LC Oscillations
(quantitative)



- **Begin with the loop rule:**

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0$$

- **Guess solution: (harmonic oscillator)**

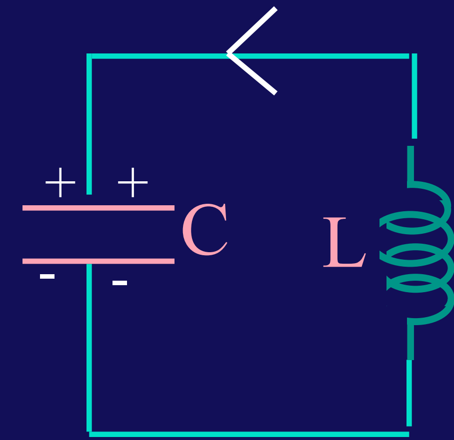
$$Q = Q_0\cos(\omega_0 t + \phi)$$

In mechanics → $-kx = m\frac{d^2x}{dt^2}$

where:
- ω_0 **determined from equation**
- ϕ, Q_0 **determined from initial conditions**

- **Procedure: differentiate above form for** Q **and substitute into loop equation to find** ω_0

LC Oscillations
(quantitative)



- **General solution:**

$$Q = Q_0 \cos(\omega_0 t + \phi)$$

- **Differentiate twice:**

$$\frac{dQ}{dt} = -\omega_0 Q_0 \sin(\omega_0 t + \phi)$$

$$\frac{d^2Q}{dt^2} = -\omega_0^2 Q_0 \cos(\omega_0 t + \phi)$$

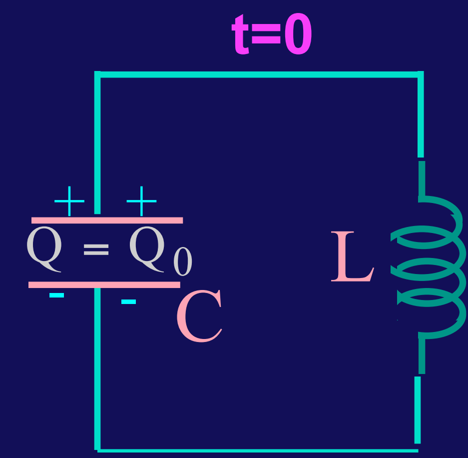
$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

- **Substitute into loop eqn:**

$$L\left(-\omega_0^2 Q_0 \cos(\omega_0 t + \phi)\right) + \frac{1}{C}\left(Q_0 \cos(\omega_0 t + \phi)\right) = 0$$ Þ $$-\omega_0^2 L + \frac{1}{C} = 0$$

$$\therefore \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Clicker



At t = 0 the capacitor has charge Q_0; the resulting oscillations have frequency ω_0. The maximum current in the circuit during these oscillations has value I_0 .

- What is the relation between ω_0 and ω_2 , the frequency of oscillations when the initial charge = $2Q_0$?

(a) $\omega_2 = 1/2\ \omega_0$

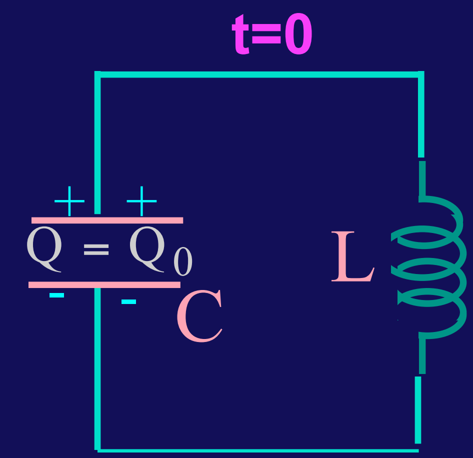
(b) $\omega_2 = \omega_0$

(c) $\omega_2 = 2\ \omega_0$

- Q_0 determines the amplitude of the oscillations (initial condition)
- The frequency is determined by the circuit parameters (L,C) only

Clicker

- At $t = 0$ the capacitor has charge Q_0; the resulting oscillations have frequency ω_0. The maximum current in the circuit during these oscillations has value I_0 .



What is the relation between I_0 and I_2 , the maximum current in the circuit when the initial charge = $2Q_0$?

(a) $I_2 = I_0$ (b) $I_2 = 2\ I_0$ (c) $I_2 = 4\ I_0$

- The initial charge determines the total energy : $U_0 = Q_0^2/2C$
- Maximum current occurs when $Q = 0$
- When all the energy is in the inductor: $U = 1/2\ LI_o^2$
- Doubling initial charge quadruples total energy.
  - Implies maximum current must double

LC Oscillations

Does solution conserve energy? YES !!

Energy in E field:

$$U_E(t) = \frac{1}{2}\frac{Q^2(t)}{C} = \frac{1}{2C}Q_0^2\cos^2(\omega_0 t + \phi)$$

Energy in B field:

$$U_B(t) = \frac{1}{2}Li^2(t) = \frac{1}{2}L\omega_0^2 Q_0^2\sin^2(\omega_0 t + \phi)$$

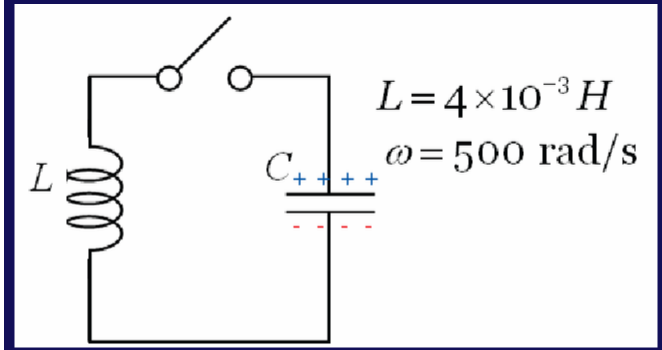
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$U_B(t) = \frac{1}{2C}Q_0^2\sin^2(\omega_0 t + \phi)$$

$$U_E(t) + U_B(t) = \frac{Q_0^2}{2C}$$

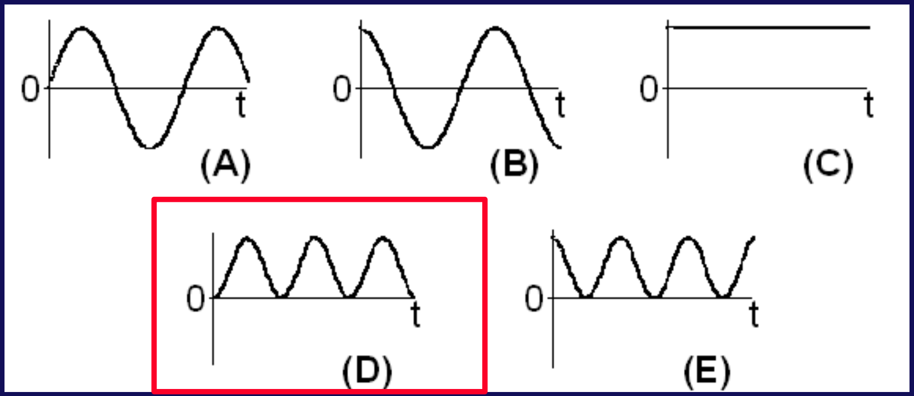
Quick checkpoint review

The capacitor charged such that the top plate has a charge $+Q_0$ and the bottom plate $-Q_0$. At time t=0, the switch is closed …



- What is the value of the capacitor C?
  - Ans: $(500)^2 \times L = 1 / C$
  - $C = 10^{-3}$ F

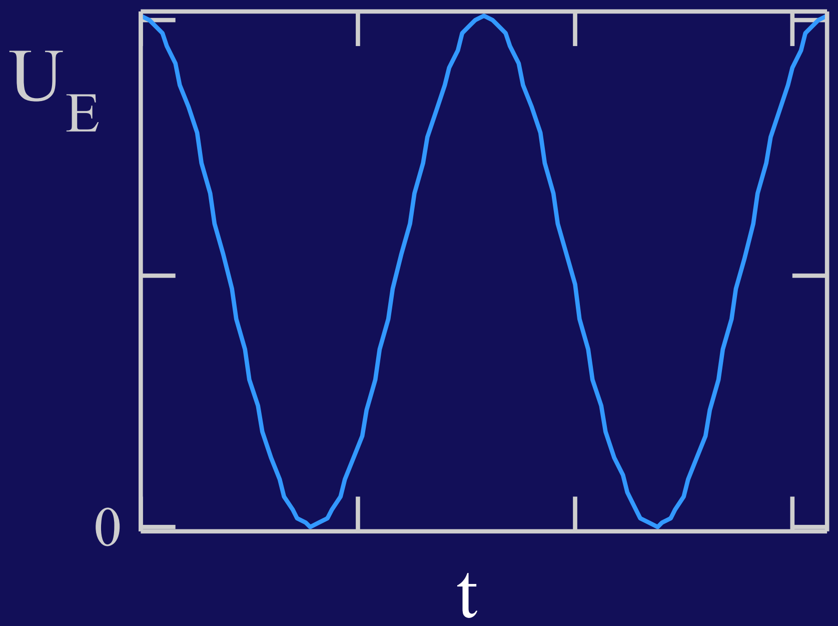
- Which of the following plots best represents the energy in the inductor as a function of time starting just after the switch is closed?

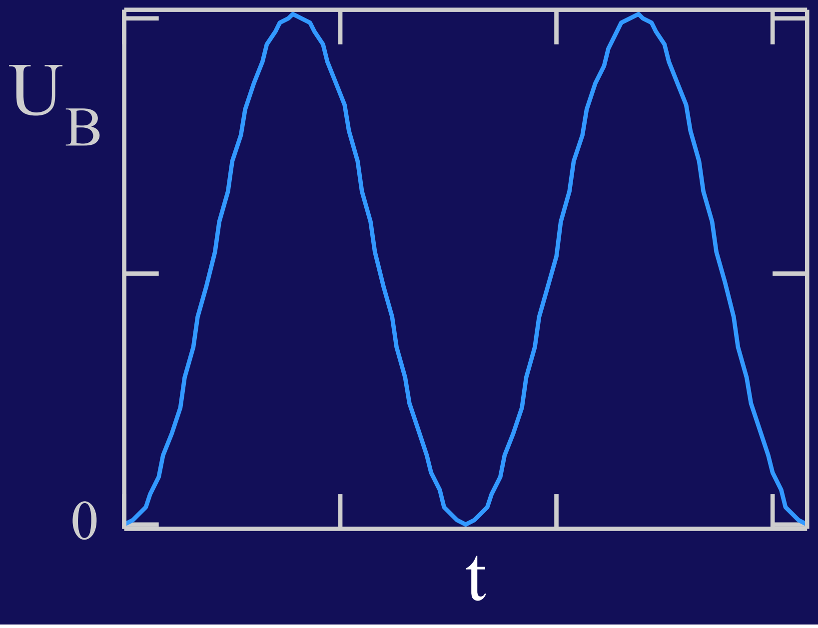


Energy is always POSITIVE (proportional to Square of current)

Energy Plotted vs Time

$$U_E(t) + U_B(t) = \frac{Q_0^2}{2C}$$





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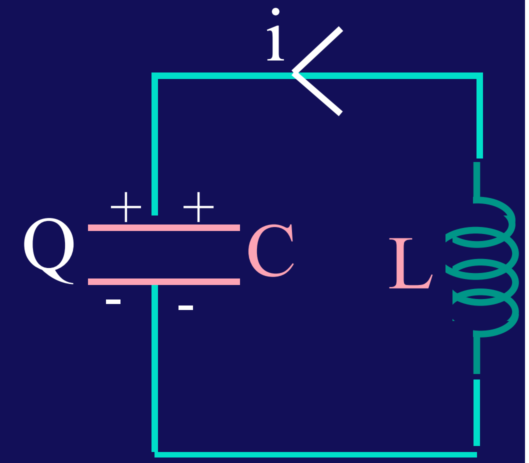


At t = 0 the current flowing through the circuit is 1/2 of its maximum value.

– Which of the following is a possible value for the phase ϕ, when the charge on the capacitor is described by: $Q(t) = Q_0\cos(\omega t + \phi)$.

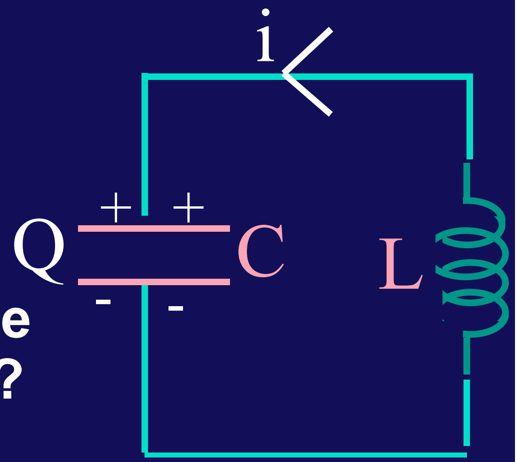
(a) $\phi = 30°$ **(b)** $\phi = 45°$ **(c)** $\phi = 60°$

- We are given a form for the charge on the capacitor as a function of time, but we need to know the current as a function of time.

$$I(t) = \frac{dQ}{dt} = -\omega_0 Q_0 \sin(\omega_0 t + \varphi)$$

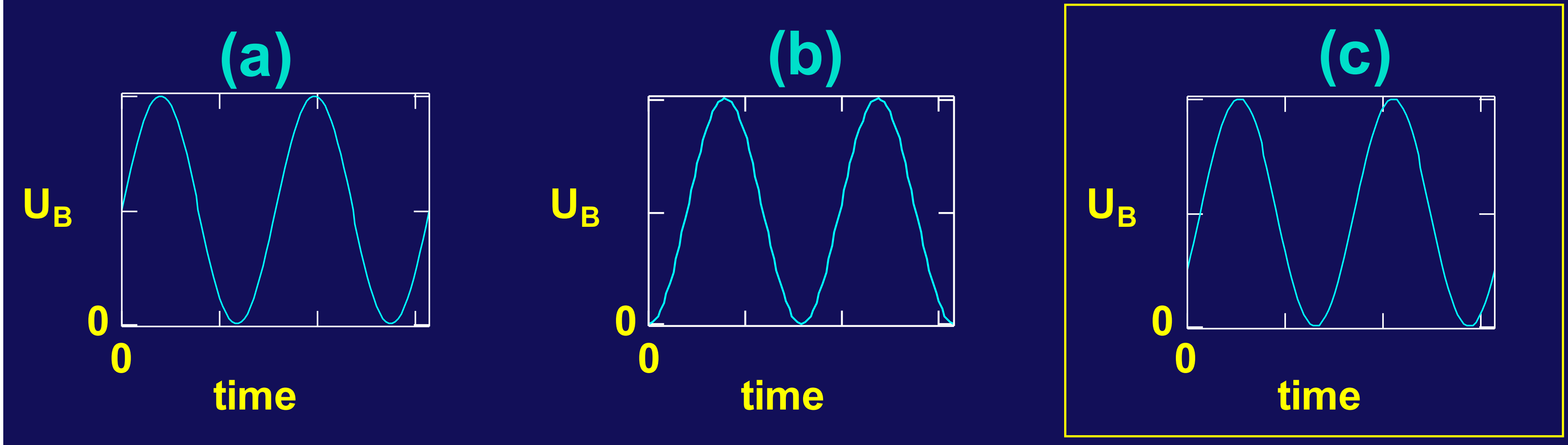
- At t = 0, the current is given by: $I(0) = -\omega_0 Q_0 \sin\varphi$
- The maximum value of the current is: $I_{max} = \omega_0 Q_0$
- Therefore, the phase angle must be given by: $\sin\varphi = \pm\frac{1}{2} \quad Þ \quad \varphi = \pm 30°$

Clicker



At t = 0 the current flowing through the circuit is 1/2 of its maximum value.

Which of the following plots best represents U_B, the energy stored in the inductor as a function of time?

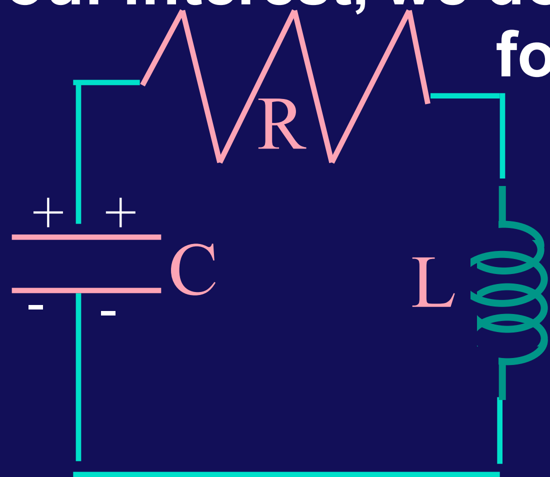


Energy stored in the inductor proportional to the CURRENT SQUARED.

If the current at t = 0 is 1/2 its maximum value, the energy stored in the inductor will be 1/4 of its maximum value!!

LCR Damping

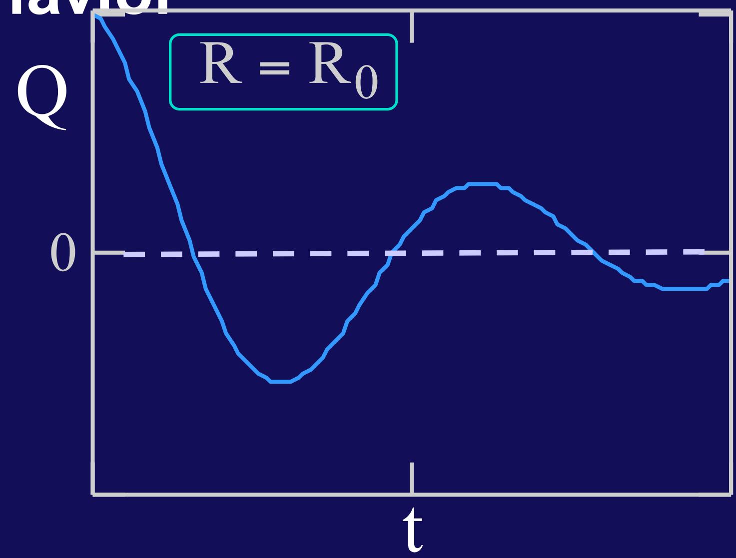
For your interest, we do not derive here, but only illustrate the following behavior



$$Q = Q_0 e^{-\beta t} \cos(\omega'_o t + \phi)$$

$$\beta = \frac{R}{2L}$$

$$\omega'_o = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$



In a LRC circuit, ω depends also on R !