Oscillations

LC Circuits

Physics 122  Lecture 24
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Exam 3

- Thursday
- Covers RC circuits through Faraday’s law
Energy in the *Magnetic* Field

“Power” accounting in a LR circuit...

\[ \varepsilon I = I^2 R + LI \frac{dI}{dt} \]

Loop rule $\times I$ ...

\[ P_L = \frac{dU}{dt} = LI \frac{dI}{dt} \]

Rate of energy flow into L

\[ U = \int dU = \int LI dI \]

Total energy flow

\[ U = \frac{1}{2} LI^2 \]

... energy stored
But where is it “Stored”?

- Claim: energy is stored *in the Magnetic field itself*
- Consider the uniform field inside a long solenoid:
  
  $$B = \mu_0 \frac{N}{l} I$$

- The inductance $L$ is:
  
  $$L = \mu_0 \frac{N^2}{l} A$$

- Stored Energy $U$:
  
  $$U = \frac{1}{2} LI^2 = \frac{1}{2} \left( \mu_0 \frac{N^2}{l} A \right) I^2 = \frac{1}{2} Al \frac{B^2}{\mu_0}$$

- Get energy density by dividing by the volume containing the field:
  
  $$u_M = \frac{U}{Al} = \frac{U}{vol} = \frac{1}{2} \frac{B^2}{\mu_0}$$
Energy in the *Electric* Field

Work needed to add charge to capacitor...

$$dW = dq(V) = dq\left(\frac{q}{C}\right)$$

$$W = \frac{1}{C_0} \int q dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2$$

... total work ...

Recall: $C = \varepsilon_0 A/d$ & $V = Ed$

$$u = \frac{W}{\text{volume}} = \frac{1}{2} \varepsilon_0 E^2$$

... energy density

**Energy Density:**

$$u_{\text{electric}} = \frac{1}{2} \varepsilon_0 E^2$$

$$u_{\text{magnetic}} = \frac{1}{2} \frac{B^2}{\mu_0}$$
RC/LC Circuits

- **RC or LR:**
  - Current decays exponentially

- **LC:**
  - Current oscillates

Graphs showing the decay of current in an RC circuit and the oscillations in an LC circuit.
How do these change if $L$ has a finite $R$?
At $t=0$, the capacitor in the LC circuit shown has a total charge $Q_0$. At $t = t_1$, the capacitor is uncharged.

- What is the value of $V_{ab}$, the voltage across the inductor at time $t_1$?

(a) $V_{ab} < 0$  
(b) $V_{ab} = 0$  
(c) $V_{ab} > 0$

- $V_{ab}$ is the voltage across the inductor, but it is also the voltage across the capacitor!

- Charge on the capacitor is zero, $\rightarrow V_C = 0$

- When $Q = 0$ on capacitor, $I$ is maximum through inductor
  - and $dI/dt$ is zero then so $V_L = 0$ makes sense
At \( t = 0 \), the capacitor in the LC circuit shown has a total charge \( Q_0 \). At \( t = t_1 \), the capacitor is uncharged.

What is the relation between \( U_{L1} \), the energy stored in the inductor at \( t = t_1 \), and \( U_{C1} \), the energy stored in the capacitor at \( t = t_1 \)?

(a) \( U_{L1} < U_{C1} \)  
(b) \( U_{L1} = U_{C1} \)  
(c) \( U_{L1} > U_{C1} \)

At \( t = t_1 \), the charge on the capacitor is zero.

\[
U_{C1} = \frac{Q_1^2}{2C} = 0
\]

At \( t = t_1 \), the current is a maximum.

\[
U_{L1} = \frac{1}{2} LI_1^2 = \frac{Q_0^2}{2C} > 0
\]
LC Oscillations
(L with finite R)

- If L has finite Resistance, then
  - energy will be dissipated in R and
  - the oscillations will become damped.
Quick checkpoint review

At time $t = 0$ the capacitor in the circuit below is fully charged with $Q_{\text{max}}$, and the current through the circuit is 0.

- What is the potential difference across the inductor at $t = 0$?
  - Ans: $V_L = \frac{Q_{\text{max}}}{C}$ (same as capacitor)

- What is the potential difference across the inductor when current is maximum?
  - Ans: 0

- How much energy is stored in C when I is max?
  - Ans: $U = 0$ (it’s all in the inductor)
LC Oscillations (quantitative)

- Begin with the loop rule:
  \[ L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = 0 \]

- Guess solution: (harmonic oscillator)
  \[ Q = Q_0 \cos(\omega_0 t + \phi) \]

  where:
  - \( \omega_0 \) determined from equation
  - \( \phi, Q_0 \) determined from initial conditions

- Procedure: differentiate above form for \( Q \) and substitute into loop equation to find \( \omega_0 \)

In mechanics:

\[ -kx = m \frac{d^2 x}{dt^2} \]
LC Oscillations
(quantitative)

• **General solution:**
  \[ Q = Q_0 \cos(\omega_0 t + \phi) \]

• **Differentiate twice:**
  \[
  \frac{dQ}{dt} = -\omega_0 Q_0 \sin(\omega_0 t + \phi)
  \]
  \[
  \frac{d^2Q}{dt^2} = -\omega_0^2 Q_0 \cos(\omega_0 t + \phi)
  \]

• **Substitute into loop eqn:**
\[
L \left( -\omega_0^2 Q_0 \cos(\omega_0 t + \phi) \right) + \frac{1}{C} \left( Q_0 \cos(\omega_0 t + \phi) \right) = 0
\]
\[\therefore -\omega_0^2 L + \frac{1}{C} = 0\]
\[\therefore \omega_0 = \frac{1}{\sqrt{LC}}\]
At $t = 0$ the capacitor has charge $Q_0$; the resulting oscillations have frequency $\omega_0$. The maximum current in the circuit during these oscillations has value $I_0$.

- What is the relation between $\omega_0$ and $\omega_2$, the frequency of oscillations when the initial charge = $2Q_0$?

(a) $\omega_2 = \frac{1}{2} \omega_0$  
(b) $\omega_2 = \omega_0$  
(c) $\omega_2 = 2 \omega_0$

- $Q_0$ determines the amplitude of the oscillations (initial condition)

- The frequency is determined by the circuit parameters (L,C) only
At \( t = 0 \) the capacitor has charge \( Q_0 \); the resulting oscillations have frequency \( \omega_0 \). The maximum current in the circuit during these oscillations has value \( I_0 \).

What is the relation between \( I_0 \) and \( I_2 \), the maximum current in the circuit when the initial charge = \( 2Q_0 \)?

- (a) \( I_2 = I_0 \)
- (b) \( I_2 = 2I_0 \)
- (c) \( I_2 = 4I_0 \)

The initial charge determines the total energy: \( U_0 = \frac{Q_0^2}{2C} \)

Maximum current occurs when \( Q = 0 \)

When all the energy is in the inductor: \( U = \frac{1}{2} LI_o^2 \)

Doubling initial charge quadruples total energy.
  - Implies maximum current must double
LC Oscillations

Does solution conserve energy?  YES !!

Energy in E field:

\[ U_E(t) = \frac{1}{2} \frac{Q^2(t)}{C} = \frac{1}{2C} Q_0^2 \cos^2(\omega_0 t + \phi) \]

Energy in B field:

\[ U_B(t) = \frac{1}{2} Li^2(t) = \frac{1}{2} L\omega_0^2 Q_0^2 \sin^2(\omega_0 t + \phi) \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ U_B(t) = \frac{1}{2C} Q_0^2 \sin^2(\omega_0 t + \phi) \]

\[ U_E(t) + U_B(t) = \frac{Q_0^2}{2C} \]
Quick checkpoint review

The capacitor charged such that the top plate has a charge $+Q_0$ and the bottom plate $-Q_0$. At time $t=0$, the switch is closed …

- What is the value of the capacitor $C$?
  - Ans: $(500)^2 \times L = 1 / C$
  - $C = 10^{-3} \text{ F}$

- Which of the following plots best represents the energy in the inductor as a function of time starting just after the switch is closed?

  ![Plots](image)

Energy is always POSITIVE (proportional to Square of current)
Energy Plotted vs Time

\[ U_E(t) + U_B(t) = \frac{Q_0^2}{2C} \]
At $t = 0$ the current flowing through the circuit is $1/2$ of its maximum value.

- Which of the following is a possible value for the phase $\phi$, when the charge on the capacitor is described by: $Q(t) = Q_0\cos(\omega t + \phi)$.

(a) $\phi = 30^\circ$  
(b) $\phi = 45^\circ$  
(c) $\phi = 60^\circ$

- We are given a form for the charge on the capacitor as a function of time, but we need to know the current as a function of time.

$$I(t) = \frac{dQ}{dt} = -\omega_0 Q_0 \sin(\omega_0 t + \phi)$$

- At $t = 0$, the current is given by: $I(0) = -\omega_0 Q_0 \sin \phi$

- The maximum value of the current is: $I_{\text{max}} = \omega_0 Q_0$

- Therefore, the phase angle must be given by: $\sin \phi = \pm \frac{1}{2} \implies \phi = \pm 30^\circ$
At $t = 0$ the current flowing through the circuit is $1/2$ of its maximum value.

Which of the following plots best represents $U_B$, the energy stored in the inductor as a function of time?

Energy stored in the inductor proportional to the **CURRENT Squared**.

If the current at $t = 0$ is $1/2$ its maximum value, the energy stored in the inductor will be $1/4$ of its maximum value!!
LCR Damping

For your interest, we do not derive here, but only illustrate the following behavior:

\[ Q = Q_0 e^{-\beta t} \cos(\omega'_0 t + \phi) \]

\[ \beta = \frac{R}{2L} \]

\[ \omega'_0 = \sqrt{\left( \frac{1}{LC} - \frac{R^2}{4L^2} \right)} \]

In a LRC circuit, \( \omega \) depends also on \( R \)!