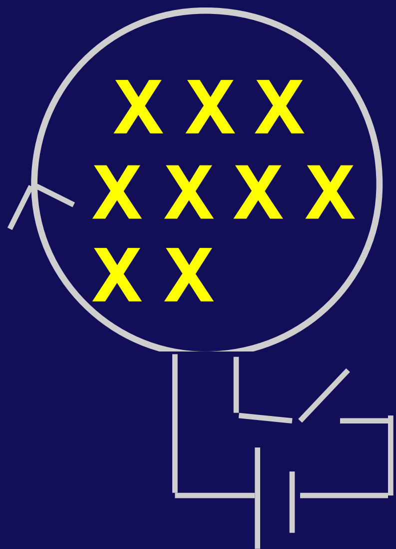
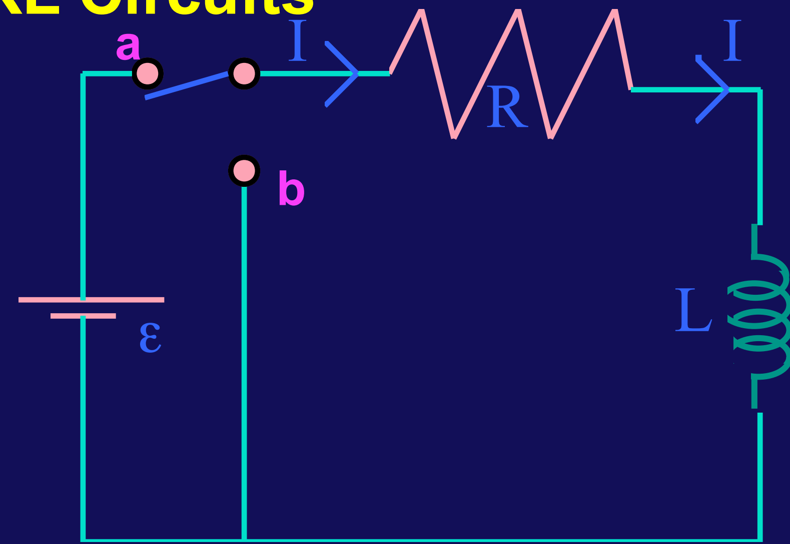
Finish Faraday
Introduce: Induction

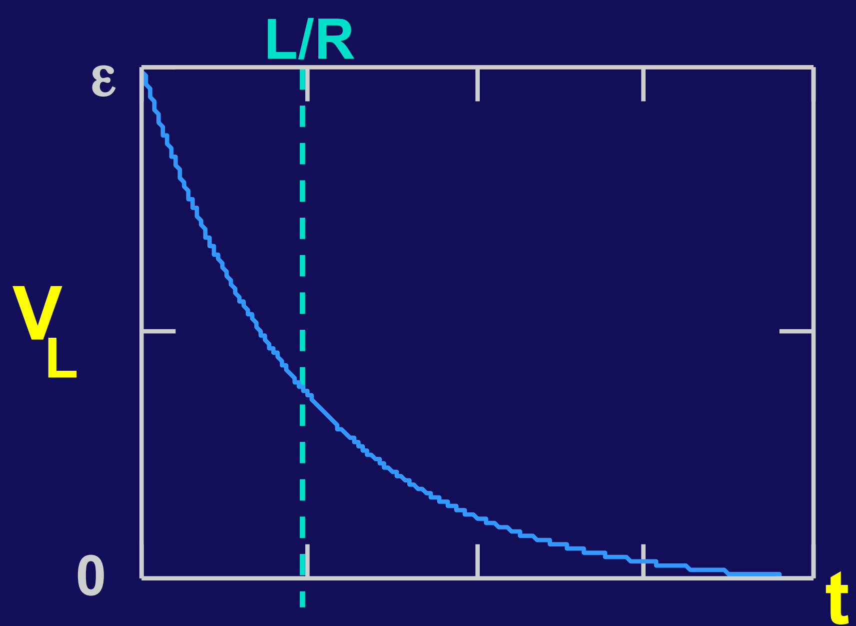
Self-Inductance, RL Circuits





$$L \equiv \frac{\Phi_B}{I}$$

$$L \equiv -\frac{\varepsilon}{(dI/dt)}$$



Physics 122 Lecture 23
G. Rybka

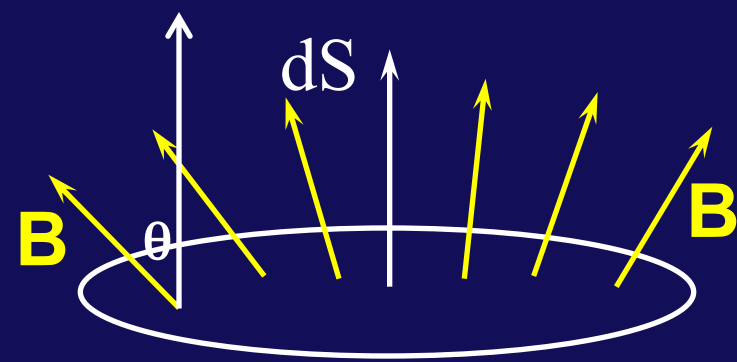
Business

- **Test Next Thursday – will cover through Faraday’s & Lenz’s law**
- **Practice Exam 3 to go up today, solutions next week**
- **Schedule Change: We’ll do LRC Circuits on Monday and exam review on Wednesday (NOTE THIS MEANS FLIPITPHYSICS HAS BEEN MOVED TO MONDAY)**

Last time: Faraday's Law

- **Define the flux of the magnetic field through a surface (closed or open) from:**

$$\Phi_B \equiv \int \vec{B} \bullet d\vec{S}$$



- **Faraday's Law:**

The emf induced in a circuit is determined by the time rate of change of the magnetic flux through that circuit.

$$emf = \oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

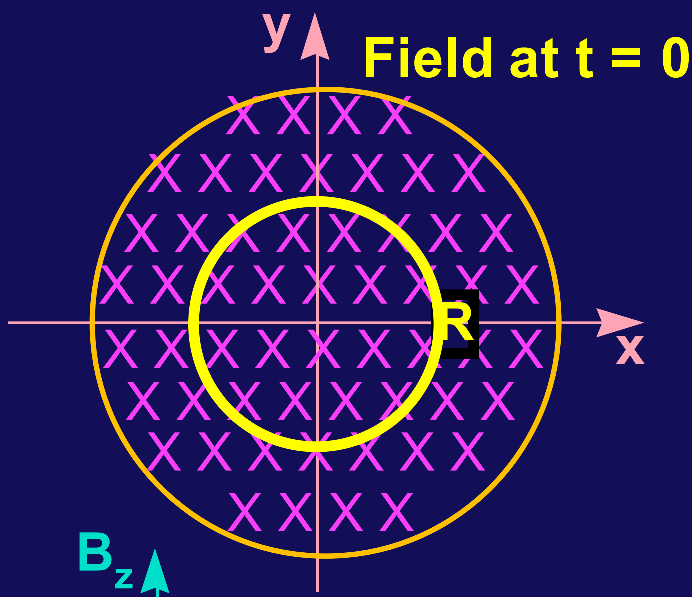
The minus sign indicates direction of induced current (given by Lenz's Law).

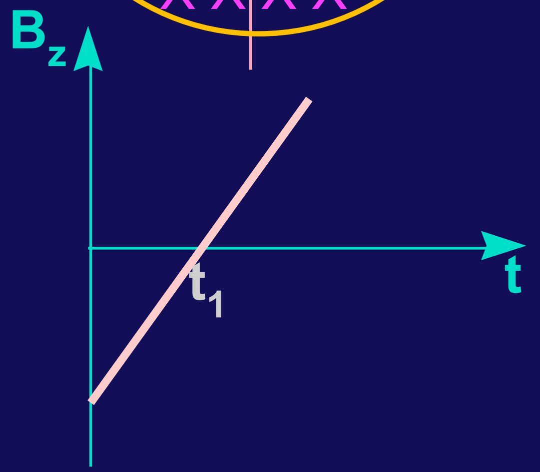
Clicker Part 1

The magnetic field in a region of space of radius 2R is aligned with the z-direction and changes in time as shown in the plot.

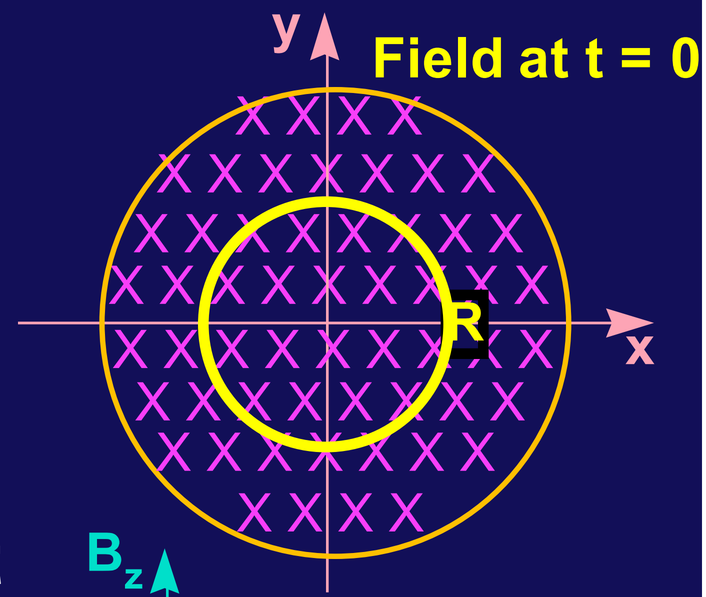
Which way would the induced current flow in yellow loop at time $t=t_1$?

(a) ccw (b) cw (c) No current





Clicker Part 1



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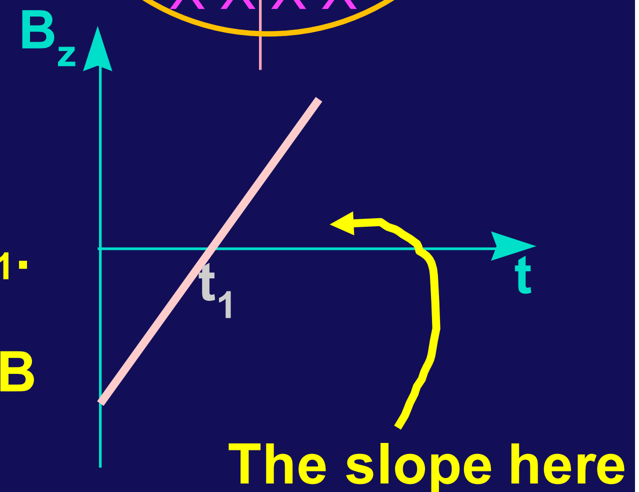


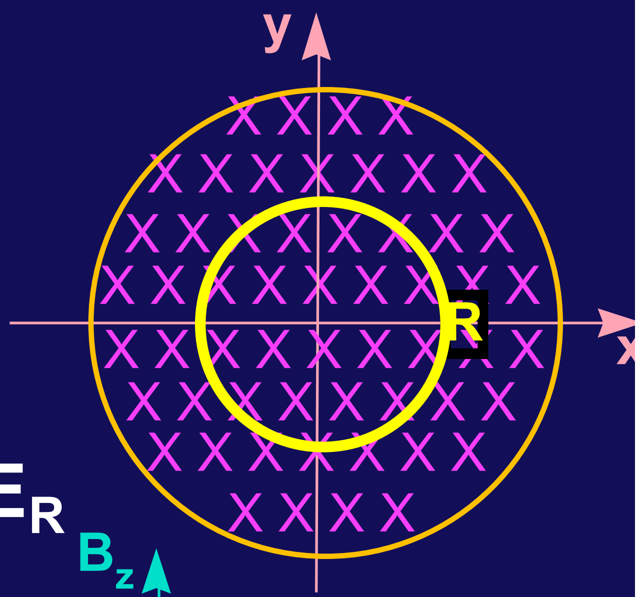
$d\Phi/dt > 0$ in +z direction at all times, even at $t = t_1$.

Induced current in direction to create a induced B field in negative z direction

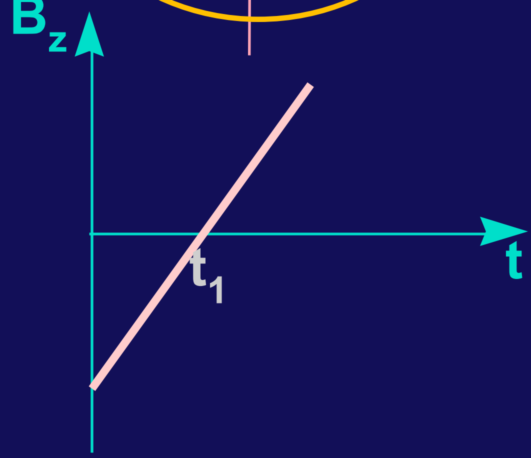
Into the page: → clockwise current

Clicker Part 2

What is the relation between the magnitudes of the induced electric fields E_R at radius R and E_{2R} at radius 2R ?



(a) $E_{2R} = E_R$ (b) $E_{2R} = 2E_R$ (c) $E_{2R} = 4E_R$



$$emf = \oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

The rate of change of the flux is proportional to the area:

$$\frac{d\Phi_B}{dt} = -\pi R^2 \frac{dB}{dt}$$

The path integral of the induced electric field is proportional to the radius.

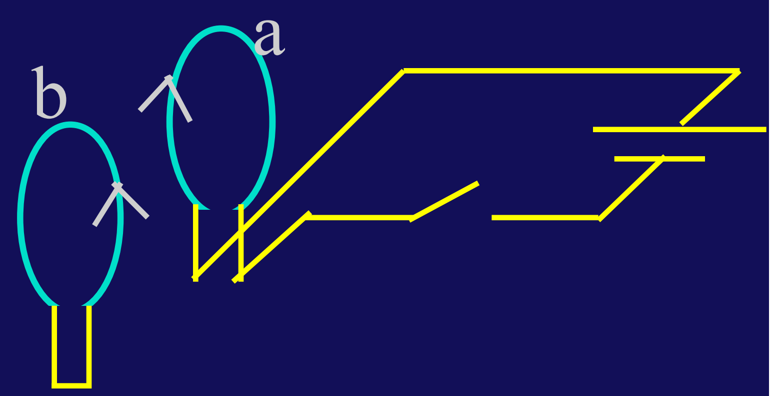
$$\oint \vec{E} \bullet d\vec{l} = E(2\pi R)$$

Therefore:

$$E \propto R$$

New: Mutual Inductance

- Demo: current is induced in coil b when the current is changed coil a
- Describe in terms of the mutual inductance: *M*
- Ratio of flux through the loop to current in opposite loop.



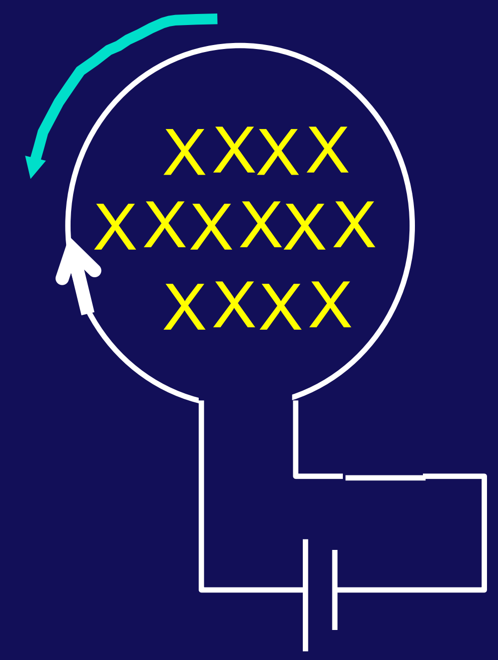
$$M \equiv \frac{\Phi_{ab}}{I_b} = \frac{\Phi_{ba}}{I_a}$$

- Note M has this symmetry (not obvious perhaps, but true)
- You can use M as you will use L, the <u>self inductance</u>, in the development which follows.

Meanwhile, what is L?

*Self**-Inductance

- **Consider the loop at the right.**



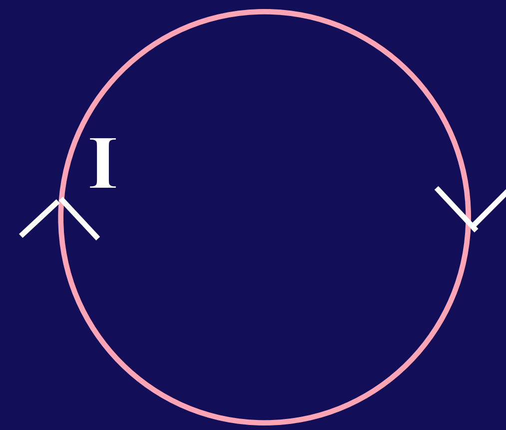
  - **Switch closed Þ current starts to flow in the loop.**
  - **Magnetic field produced in loop.**
  - **Flux through loop is changing**
  - **An emf is induced in loop in direction to oppose the initial emf from the battery**

- ***Self-Induction*: the act of a changing current through a loop inducing an opposing emf in that same loop.**

we will mostly drop the word 'self'

Defining Self-Inductance

- The magnetic field produced by the current in the loop shown is proportional to that current.



$$B \propto I$$

- Flux, is proportional to current.

$$\Phi_B \equiv \int \vec{B} \bullet d\vec{S} \propto I$$

- Define constant of proportionality between Φ and I to be the inductance L.

$$L \equiv \frac{\Phi_B}{I}$$

- Use Faraday's Law, in terms of the emf induced by a changing current,

$$\Phi_B = IL$$

$$\frac{d\Phi_B}{dt} = \frac{dI}{dt} L = -\varepsilon$$

$$\Rightarrow \quad \varepsilon = -L\frac{dI}{dt}$$

The inductance is a property of the device
(sound familiar? Recall resistors and capacitors)

- **Long Solenoid:**

N turns total, radius r, Length *l*

$$r << l \Rightarrow B = \mu_0 \frac{N}{l} I$$

For a single turn:

$$A = \pi r^2 \Rightarrow \phi = BA = \mu_0 \frac{N}{l} IA$$

The total flux through solenoid is given by:

$$\Phi_B = N\phi = \mu_0 \frac{N^2}{l} I\pi r^2 = \mu_0 n^2 lIA$$

Note: $n = N / l$

Inductance of solenoid can then be calculated as:

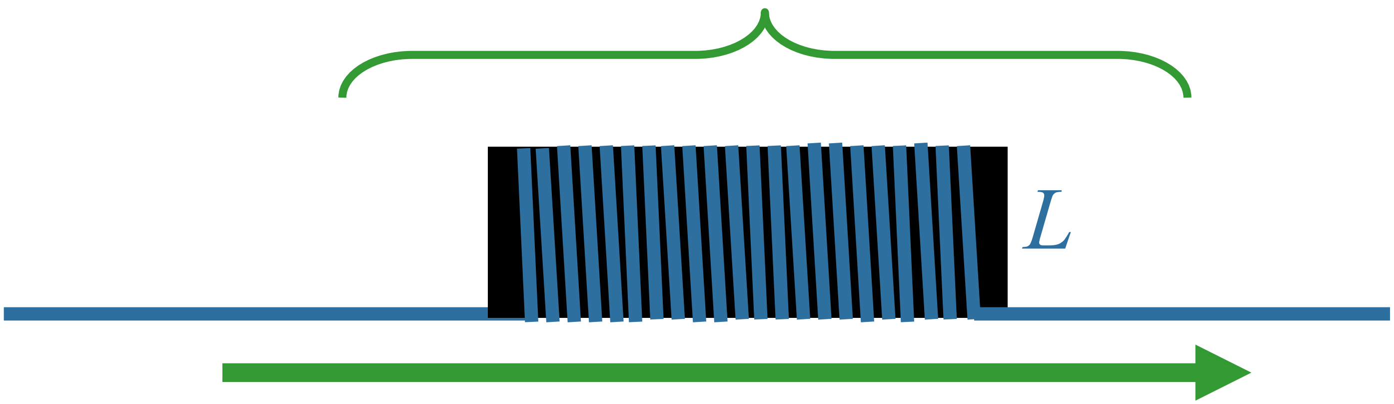
$$L \equiv \frac{\Phi_B}{I} = \mu_0 \frac{N^2}{l} A = \mu_0 n^2 Al$$

Note: if there are iron cores or other inserts, materials properties enter

What an inductor does ..

emf induced across ***L*** tries to keep ***I*** constant.

$$\varepsilon_L = -L\frac{dI}{dt}$$



current ***I*** through an inductor in a circuit

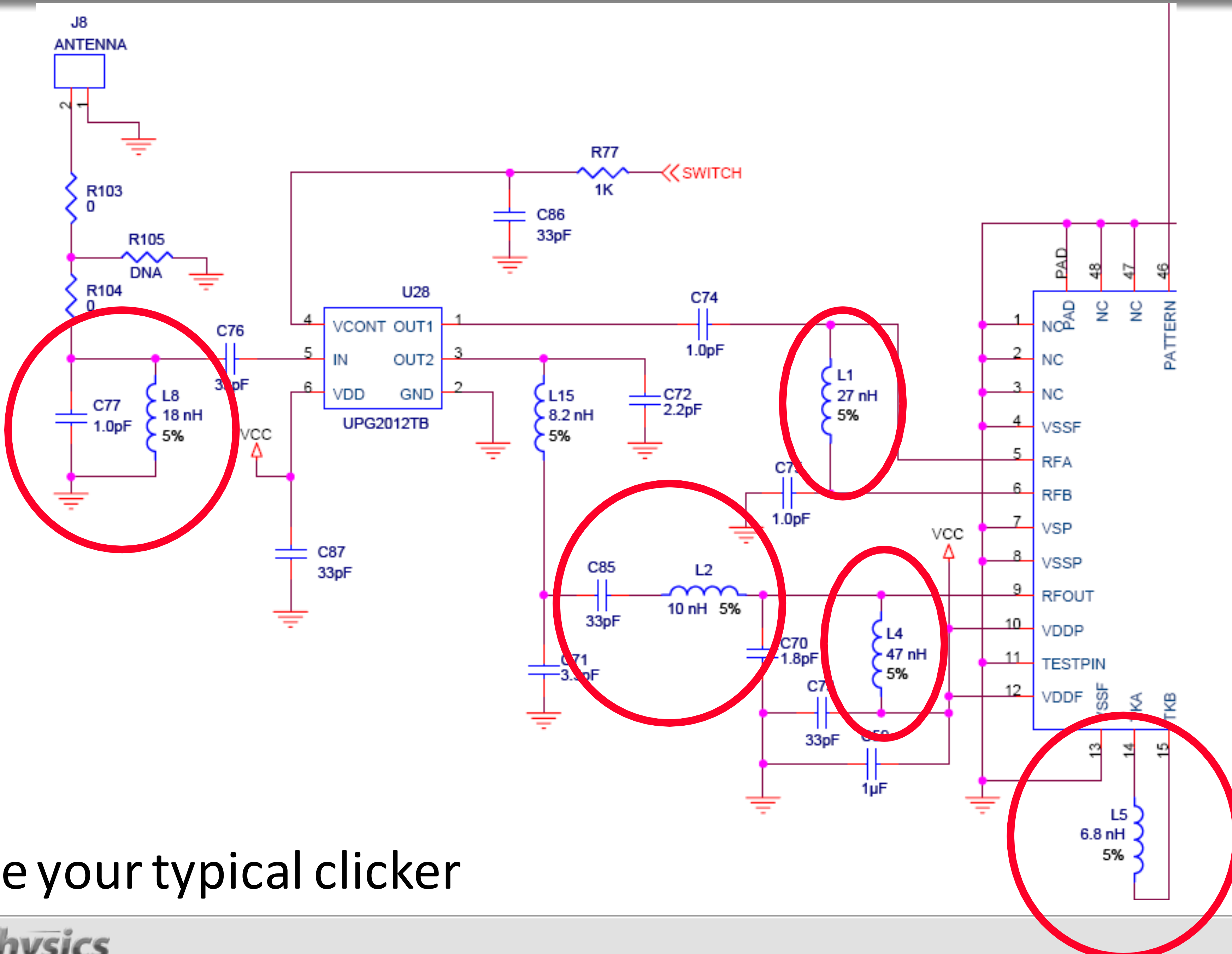
Inductors prevent discontinuous current changes!

It's like **inertia** for the current

If **no current is flowing**, you can't instantaneously start it

If **current is flowing**, you can't instantaneously stop it

What are Inductors and Capacitors Good For?

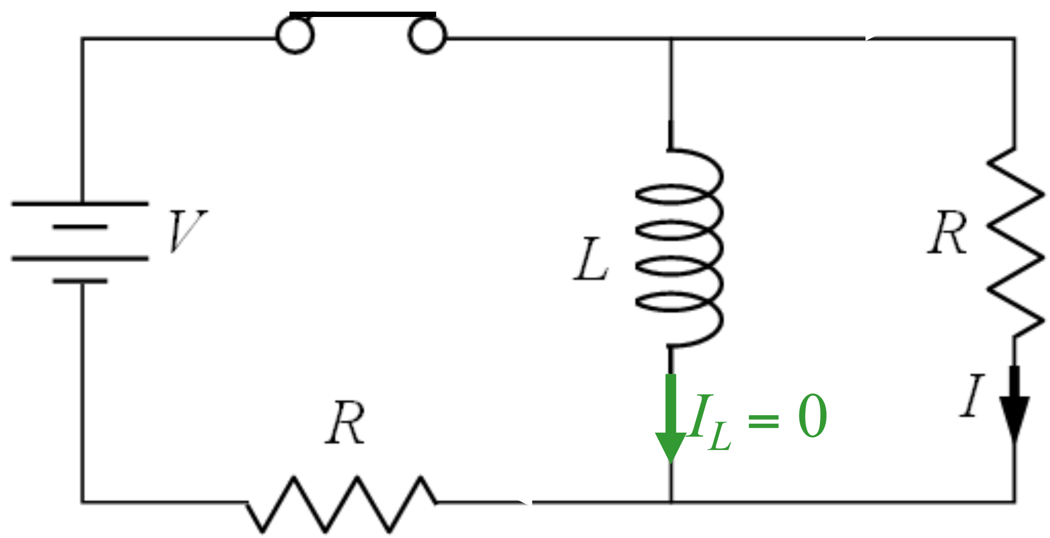


Inside your typical clicker

CheckPoint 4

In the circuit, the switch has been open for a long time, and the current is zero everywhere.

At time $t = 0$ the switch is closed.



What is the current I through the vertical resistor immediately after the switch is closed?

(+ is in the direction of the arrow)

A) $I = V/R$

B) $I = V/2R$

C) $I = 0$

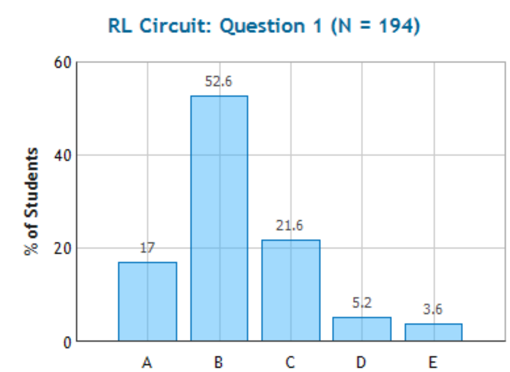
D) $I = -V/2R$

E) $I = -V/R$

Before: $I_L = 0$

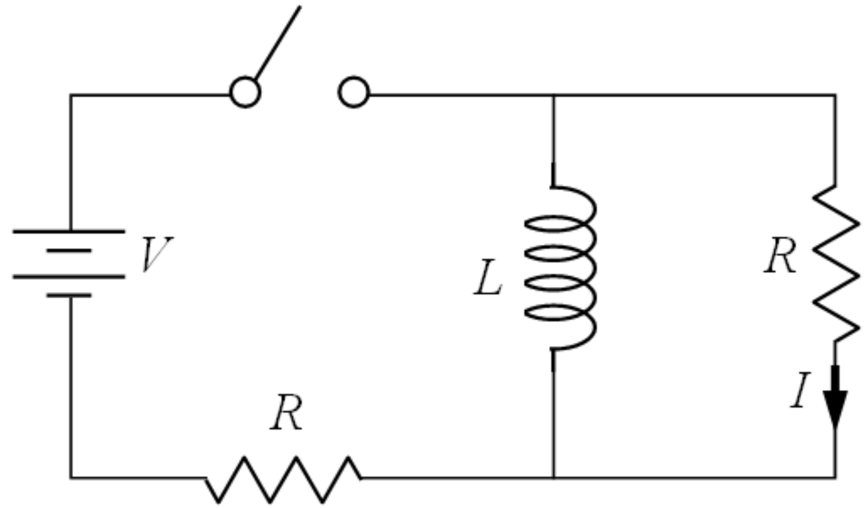
After: $I_L = 0$

$I = +V/2R$



CheckPoint 6

After a long time, the switch is opened, abruptly disconnecting the battery from the circuit.



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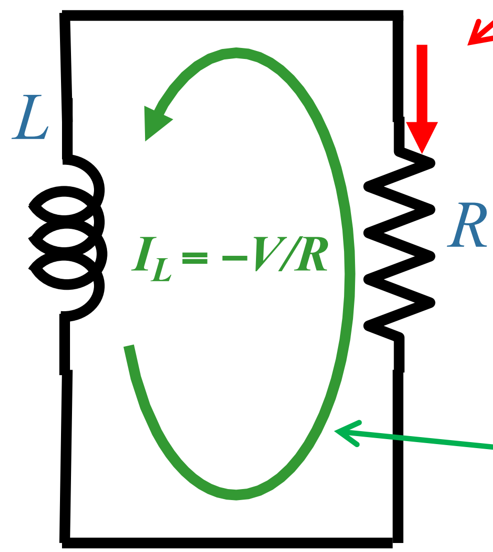
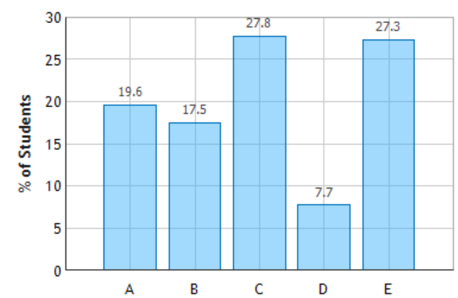
C) $I = 0$
D) $I = -V/2R$
E) $I = -V/R$

circuit when switch opened



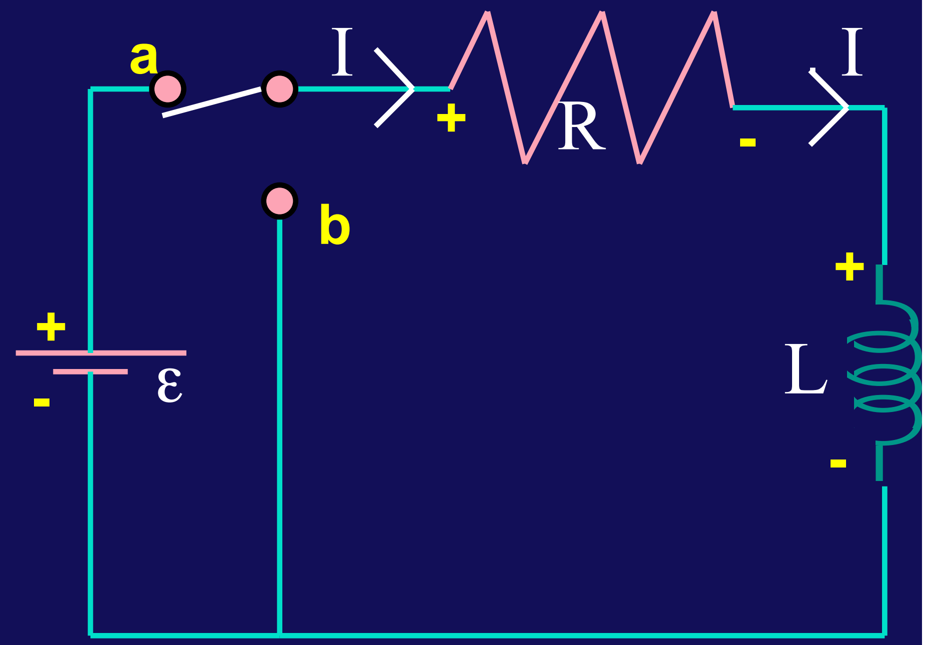
Note, their arrow for *I*



Current through inductor cannot change DISCONTINUOUSLY

Actual current flow in same direction as before switch opened

RL Circuits



- At t=0, the switch is closed and the current I starts to flow.
- Loop rule:

$$IR + L\frac{dI}{dt} - \varepsilon = 0$$

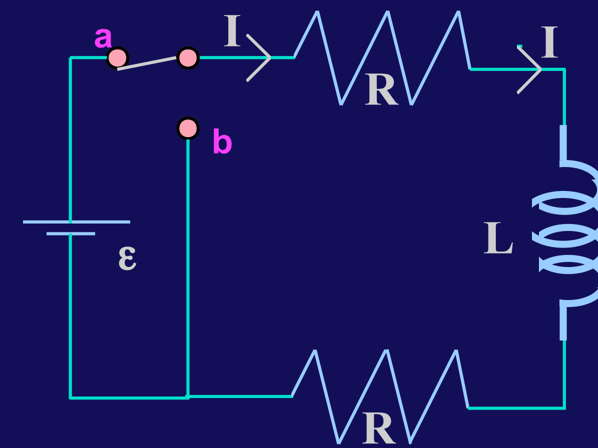
Equation is identical in form to that for the RC circuit with the following substitutions:

RC: $$\frac{q}{C} + R\frac{dq}{dt} - \varepsilon = 0$$

⇛ RC→RL: $R \to L$, $\frac{1}{C} \to R$, $Q \to I$

$\therefore \quad \tau_{RC} = RC$ ⇛ $$\tau_{RL} = \frac{L}{R}$$

Clicker

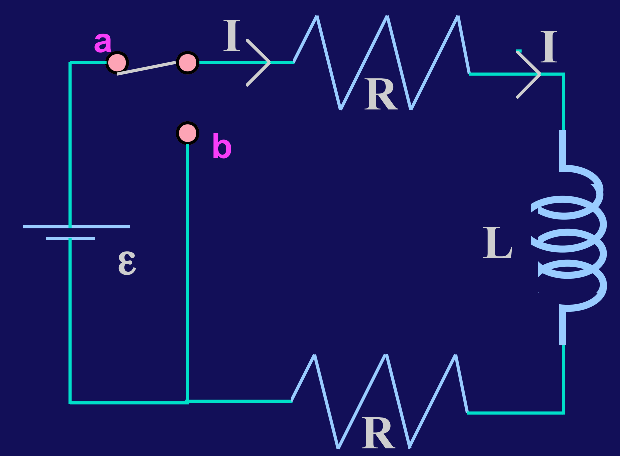


- At t = 0 the switch is thrown from position b to position a in the circuit shown:
  - What is the value of the current $I_{¥}$ a long time after the switch is thrown?

(a) $I_{¥} = 0$ (b) $I_{¥} = \varepsilon / 2R$ (c) $I_{¥} = 2\varepsilon / R$

- A long time after the switch is thrown, the current approaches an asymptotic value. ie as $t \rightarrow ¥$, $dI/dt \rightarrow 0$.
- As $dI/dt \rightarrow 0$, the voltage across the inductor $\rightarrow 0$.
- $\therefore I_{¥} = \varepsilon / 2R$.

Clicker



What is the value of the current I_0 immediately after the switch is thrown?

(a) $I_0 = 0$

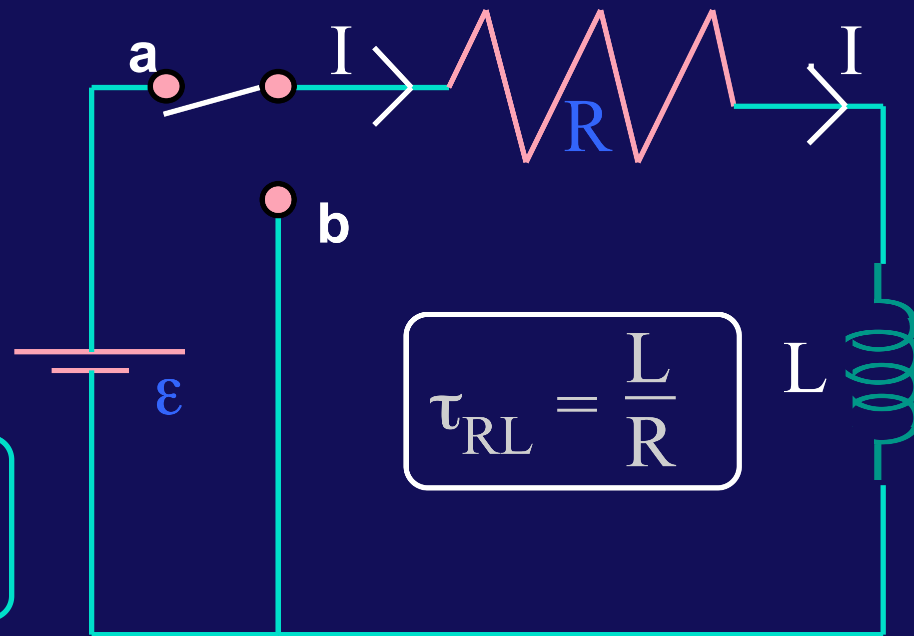
(b) $I_0 = \varepsilon / 2R$

(c) $I_0 = 2\varepsilon / R$

- **Immediately after the switch is thrown, the rate of change of current is as large as it can be**
  - ***(before the inductor was introduced , we assumed the rate of change was ¥ !)***
- **The inductor limits dI/dt to be initially equal to ε / L.**
  - **ie the voltage across the inductor = ε;**
  - **the current then must be 0**

RL Circuits

- **For $I(t)$ choose exponential solution that satisfies the boundary conditions:**



$$\tau_{RL} = \frac{L}{R}$$

$$\frac{dI}{dt}(t = \infty) = 0 \Rightarrow I(t = \infty) = \frac{\varepsilon}{R}$$

- **We therefore write:**

$$I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

- **The voltage drop across the inductor is given by:**

$$V_L = L\frac{dI}{dt} = \varepsilon e^{-Rt/L}$$

RL Circuit (switch ε on)

Current

$$I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

Max = ε/R

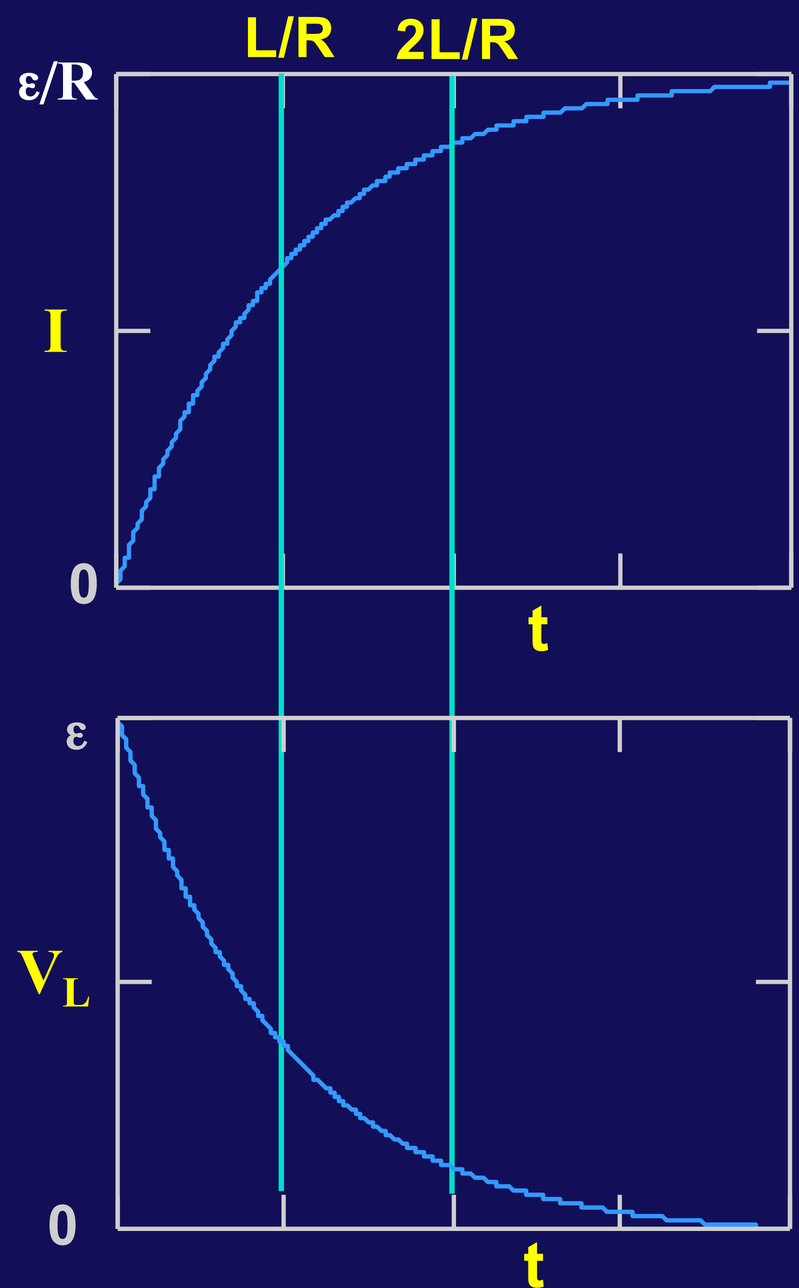
63% Max at t=L/R

Voltage on L

$$V_L = L\frac{dI}{dt} = \varepsilon e^{-Rt/L}$$

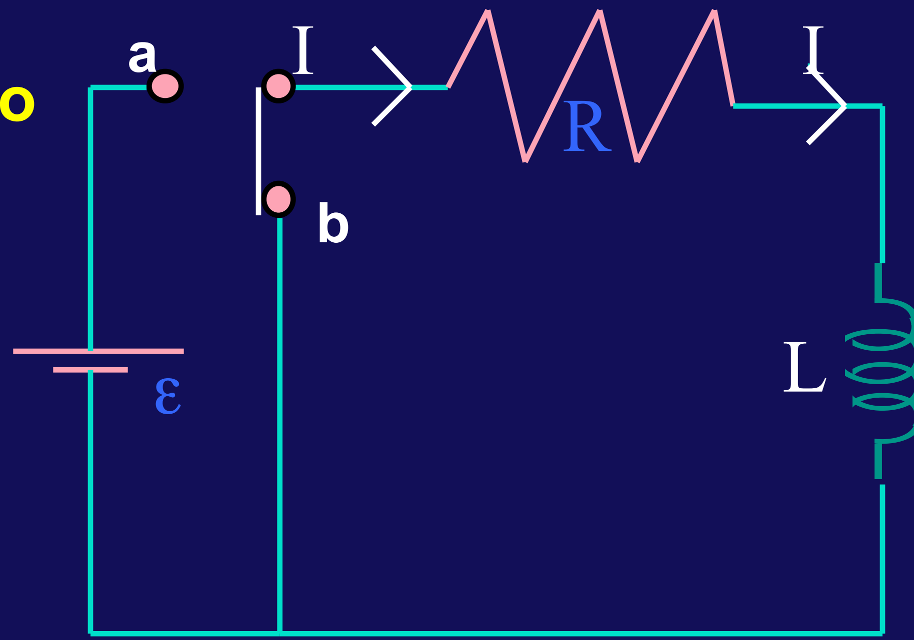
Max = ε

37% Max at t=L/R



RL Circuits

- After the switch has been in position a for a long time, redefined to be t=0, it is moved to position b.



- Loop rule:

$$IR + L\frac{dI}{dt} = 0$$

- The appropriate initial condition is:

$$I(t = 0) = \frac{\varepsilon}{R}$$

Warning: this confuses people

- The solution then must have the form:

$$I = \frac{\varepsilon}{R} e^{-Rt/L}$$

$$V_L = L\frac{dI}{dt} = -\varepsilon e^{-Rt/L}$$

RL Circuit (turn off ε)

Current

$$I = \frac{\varepsilon}{R} e^{-Rt/L}$$

Max = ε/R

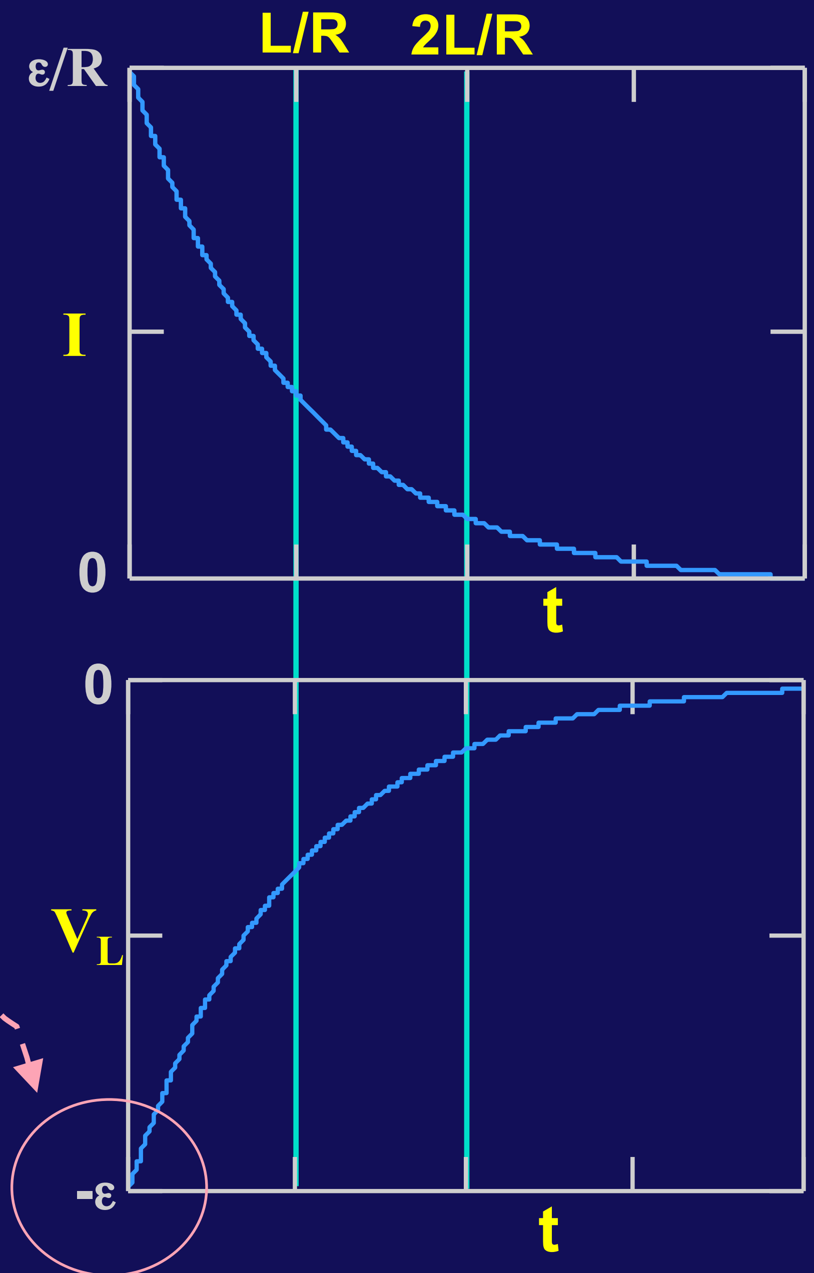
37% Max at t=L/R

Voltage on L

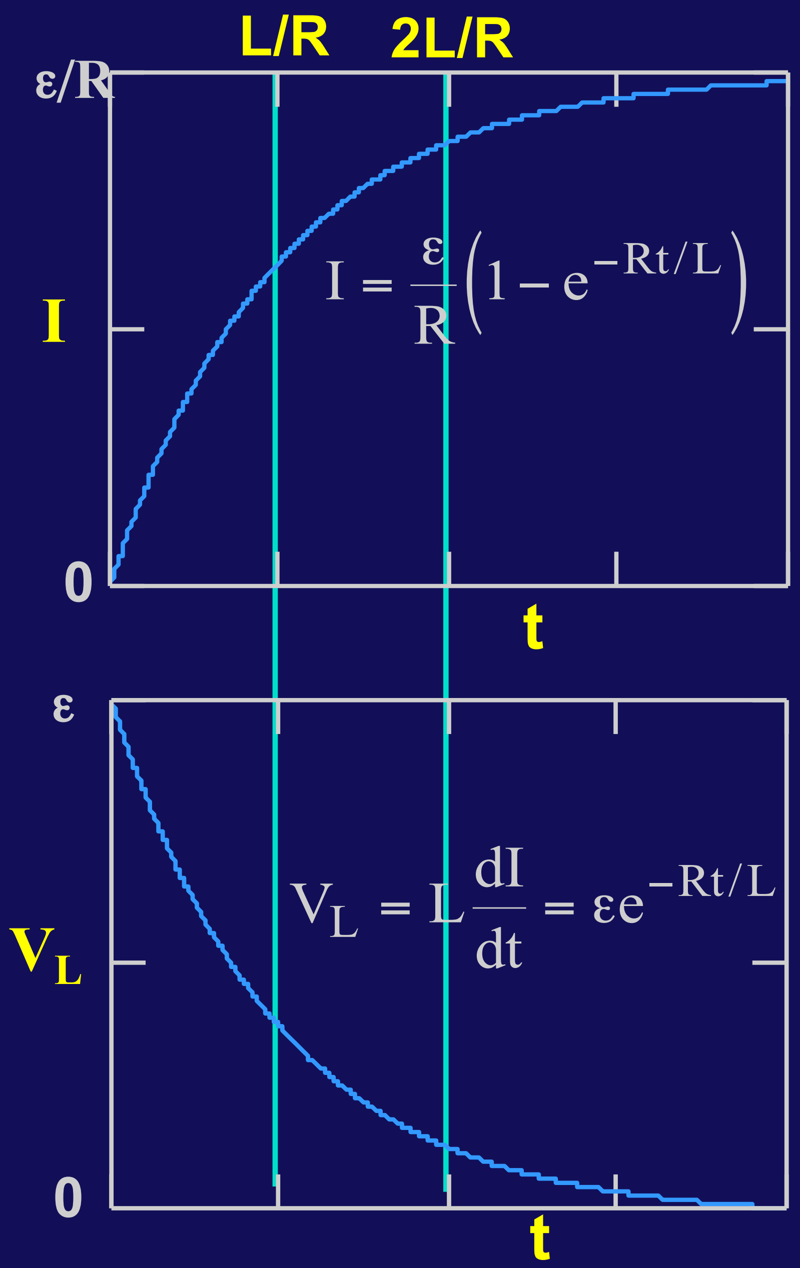
$$V_L = L\frac{dI}{dt} = -\varepsilon e^{-Rt/L}$$

Max = -ε

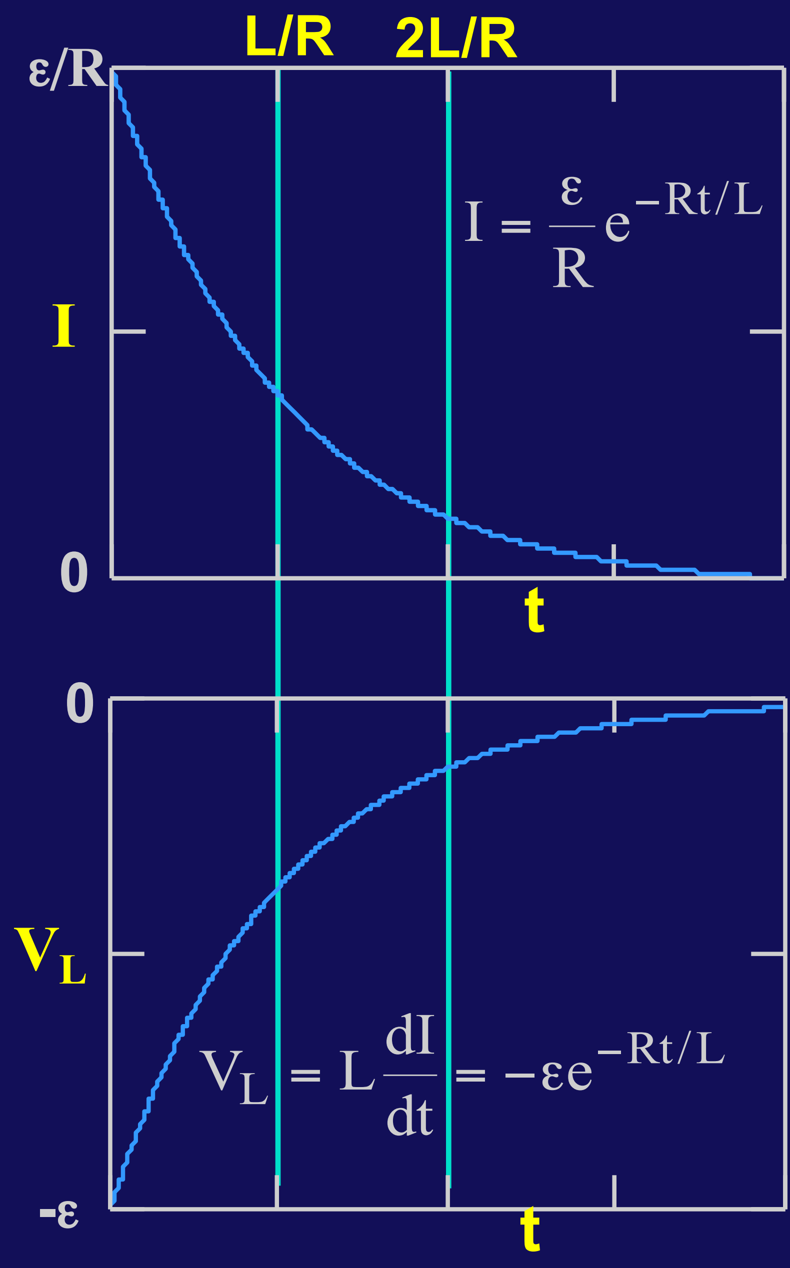
37% Max at t=L/R



ε on

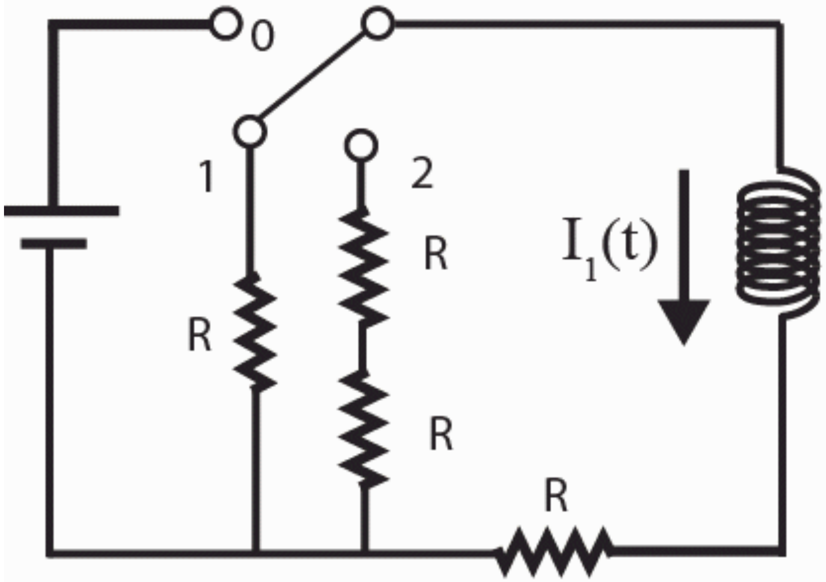

L/R
2L/R
ε/R
I
0
$I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$
t
ε
V_L
0
$V_L = L\frac{dI}{dt} = \varepsilon e^{-Rt/L}$
t

ε off


L/R
2L/R
ε/R
I
0
$I = \frac{\varepsilon}{R} e^{-Rt/L}$
t
0
V_L
-ε
$V_L = L\frac{dI}{dt} = -\varepsilon e^{-Rt/L}$
t

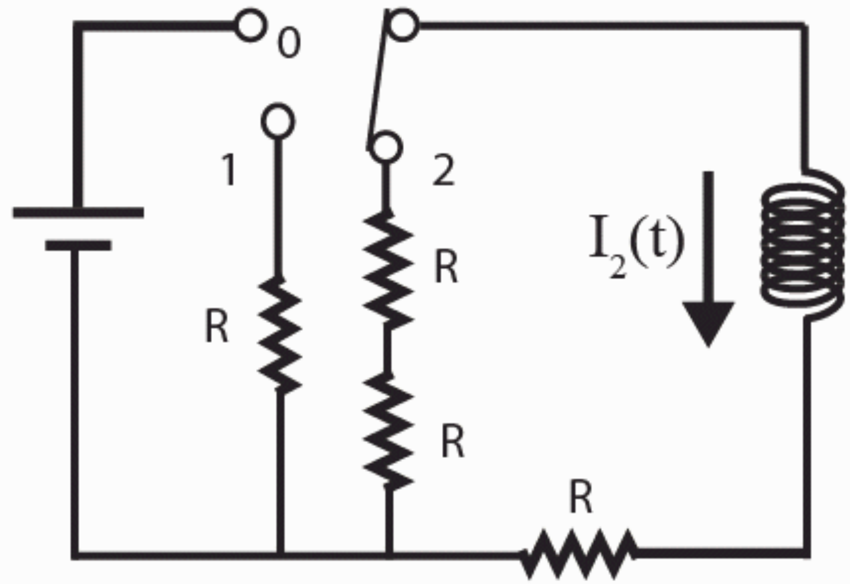
CheckPoint 8

After long time at 0, moved to 1



Case 1

After long time at 0, moved to 2



Case 2

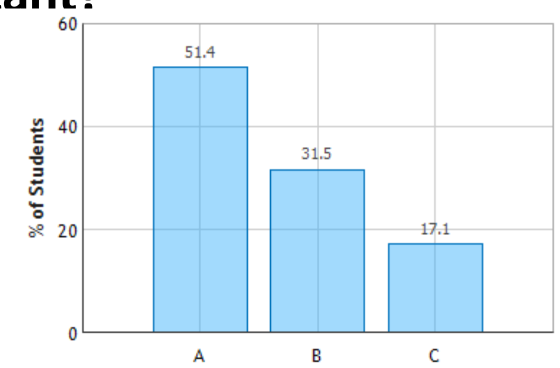
After switch moved, which case has larger time constant?

A) Case 1

B Case 2

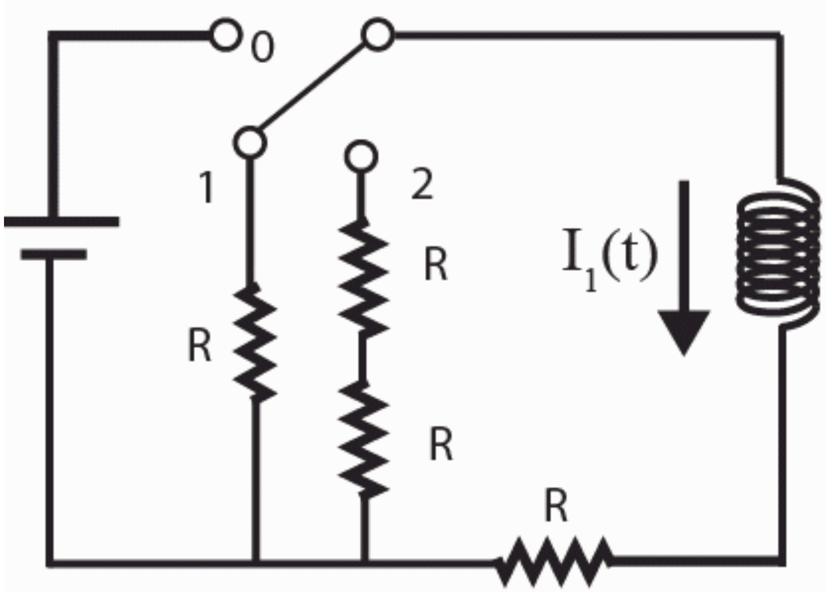
C) The same

$$\tau_1 = \frac{L}{2R} \qquad \tau_2 = \frac{L}{3R}$$



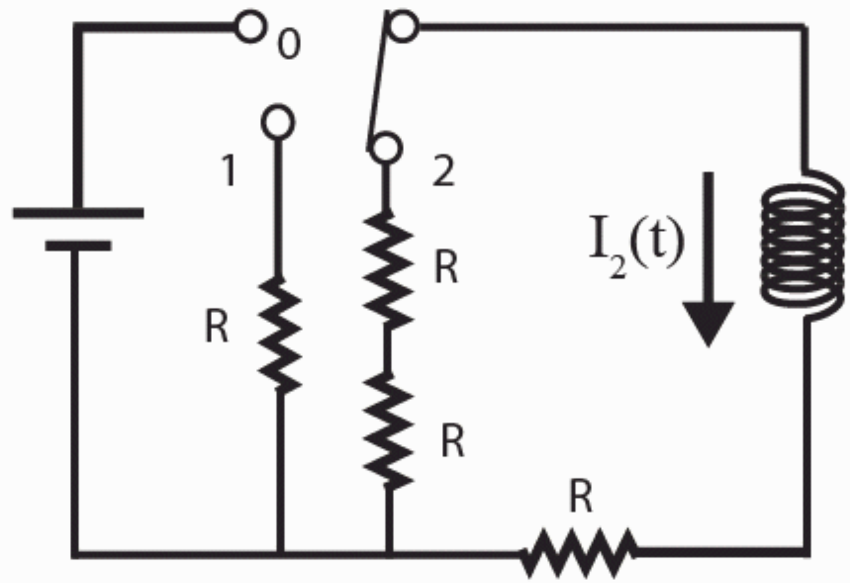
CheckPoint 10

After long time at 0, moved to 1



Case 1

After long time at 0, moved to 2



Case 2

Immediately after switch moved, in which case is the voltage across the inductor larger?

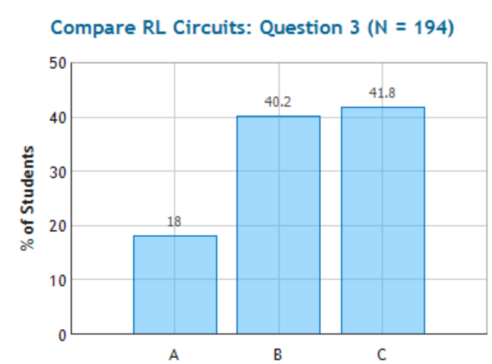
A) Case 1
B) Case 2
C) The same

Before switch moved: $I = \frac{V}{R}$

After switch moved:

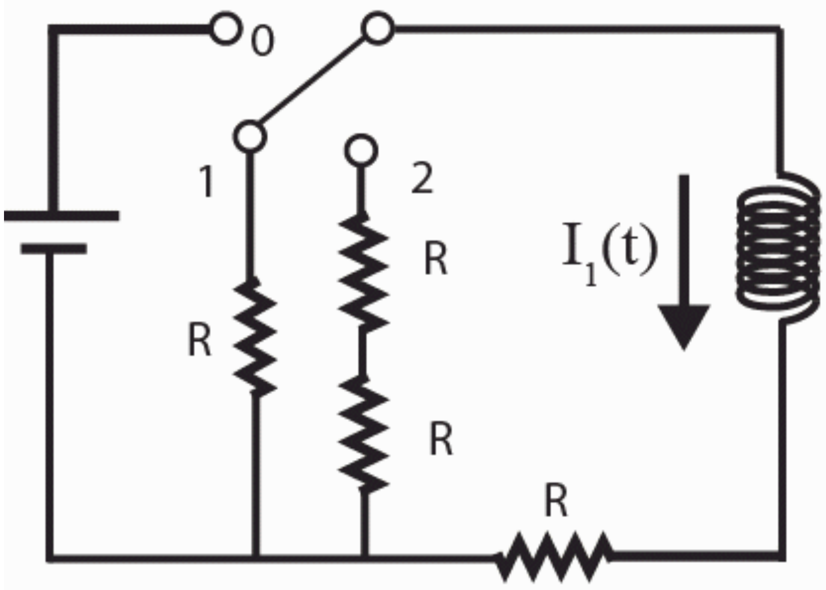
$$V_{L1} = \left(\frac{V}{R}\right)2R$$

$$V_{L2} = \left(\frac{V}{R}\right)3R$$



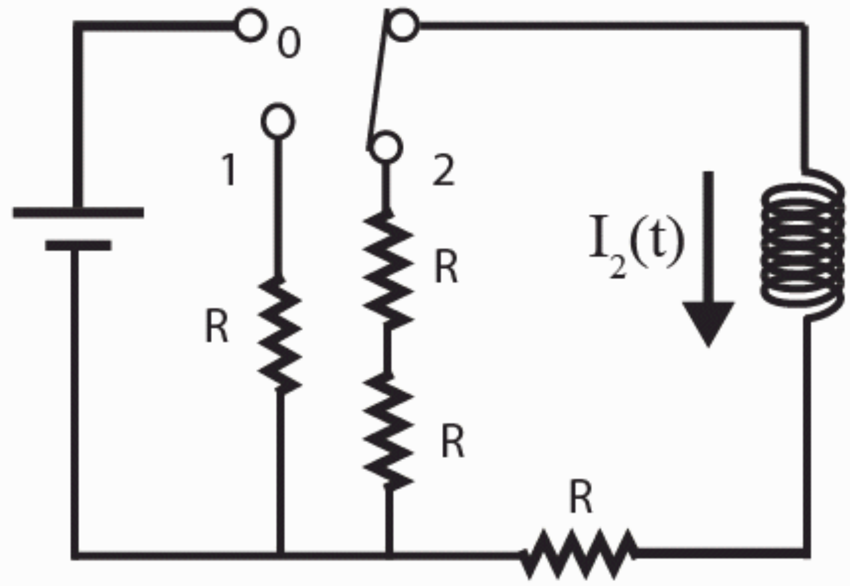
CheckPoint 12

After long time at 0, moved to 1



Case 1

After long time at 0, moved to 2



Case 2

After switch moved for finite time, in which case is the current through the inductor larger?

A) Case 1
B) Case 2
C) The same

After a while … but not forever

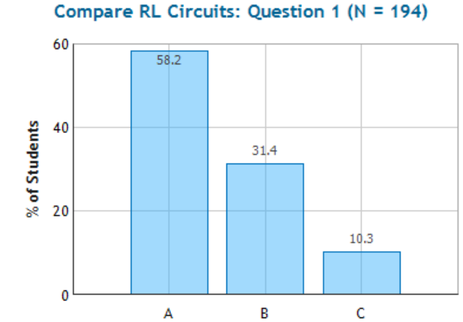
$$I_1 = Ie^{-t/\tau_1}$$

$$I_2 = Ie^{-t/\tau_2}$$

$$\tau_1 > \tau_2$$

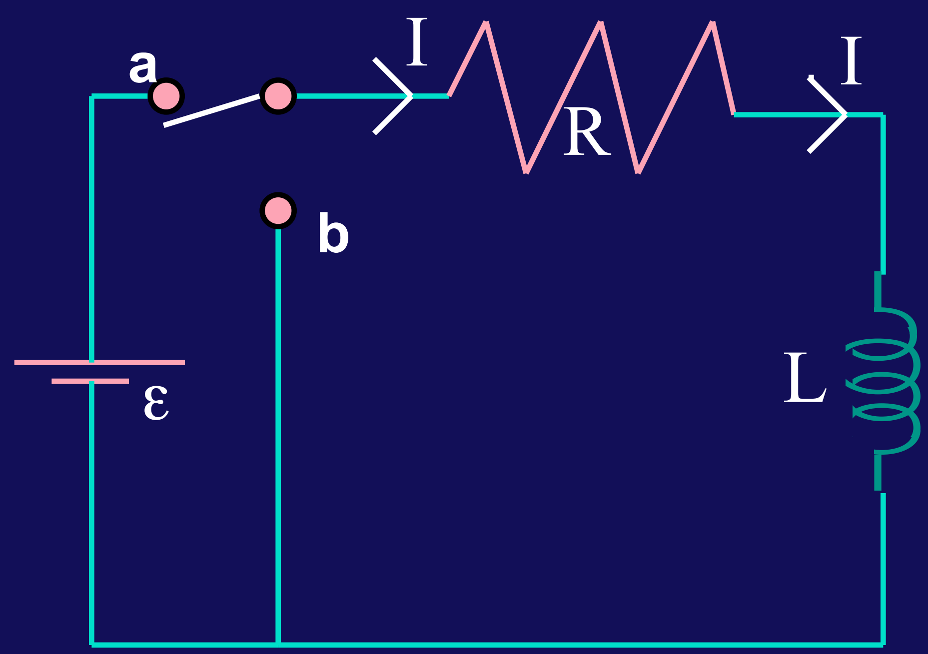
Immediately after: $I_1 = I_2$

Longer time constant to get to zero



Energy of an Inductor

- **How much energy is stored in an inductor when a current is flowing through it?**



- **Start with loop rule:**

$$\varepsilon = IR + L\frac{dI}{dt}$$

- **Multiply this equation by I:**

$$\varepsilon I = I^2R + LI\frac{dI}{dt}$$

- **From this equation, we can identify P_L, the rate at which energy is being stored in the inductor:** $P_L = \frac{dU}{dt} = LI\frac{dI}{dt}$

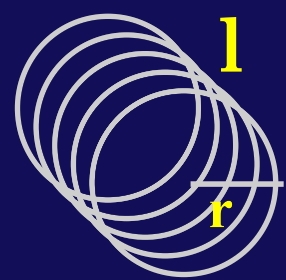
- **We can integrate this equation to find an expression for U, the energy stored in the inductor when the current = I:**

$$U = \int_0^U dU = \int_0^I LI\,dI \quad \Rightarrow \quad U = \frac{1}{2}LI^2$$

Where is the Energy Stored?

- Claim: (without proof) energy is stored ***in the Magnetic field itself*** (just as in the Capacitor / Electric field case).
- Consider the uniform field inside a long solenoid:

$$B = \mu_0 \frac{N}{l} I$$



N turns

- The inductance L i $L = \mu_0 \frac{N^2}{l} \pi r^2 = \mu_0 \frac{N^2}{l} A$

- Energy U: $U = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 \frac{N^2}{l} A\right) I^2 = \frac{1}{2} A l \frac{B^2}{\mu_0}$

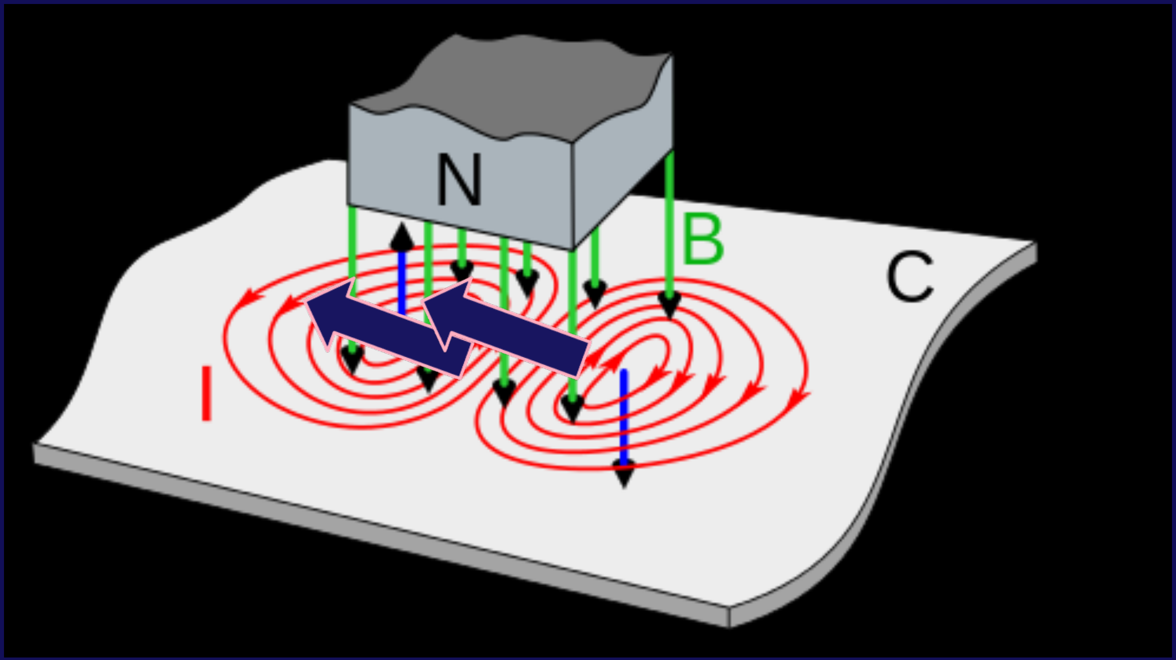
- We can turn this into an energy density by dividing by the volume containing the field:

$$u_M = \frac{U}{Al} = \frac{U}{vol} = \frac{1}{2}\frac{B^2}{\mu_0}$$

Recall for E: $u_E = \frac{1}{2}\varepsilon_0 E^2$

Bonus material

Some follow-up to spectacular demos



- When the magnet is moving past the conductors …
- Or when the conductors were moving past the magnets …
- “whirlpools” of current induced. Notice direction of force … it opposes the change … causes a “braking” force

Self-Inductance

- The inductance of an inductor (a set of coils in some geometry ..eg solenoid, toroid) then, like a capacitor, can be calculated from its geometry alone if the device is constructed from conductors and air.
- If extra material (eg iron core) is added, then we need to add some knowledge of materials as we did for capacitors (dielectrics) and resistors (resistivity)

$$L \equiv -\frac{\varepsilon}{(dI/dt)}$$

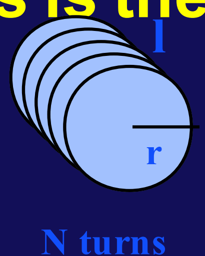
$$L \equiv \frac{\Phi_B}{I}$$

$$C \equiv \frac{Q}{V}$$

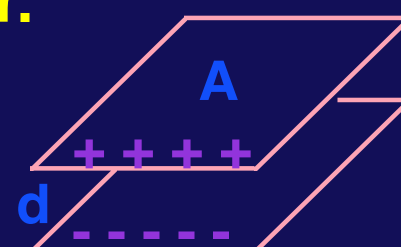
$$C \equiv \kappa C_0$$

$$R = \rho \frac{L}{A}$$

- Archetypal inductor is a long solenoid, just as a pair of parallel plates is the archetypal capacitor.



$$r << l$$



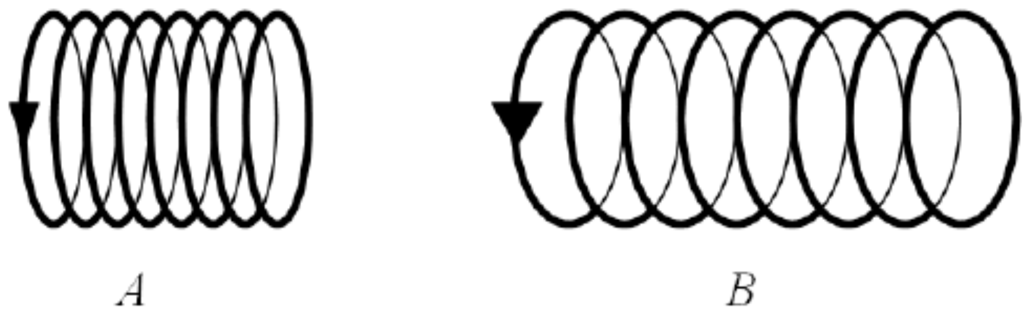
$$d << \sqrt{A}$$

A similar CheckPoint

Two solenoids are made with the same cross sectional area and total number of turns. Inductor ***B*** is twice as long as inductor ***A***

$$L_B = \mu_0 n^2 A z$$

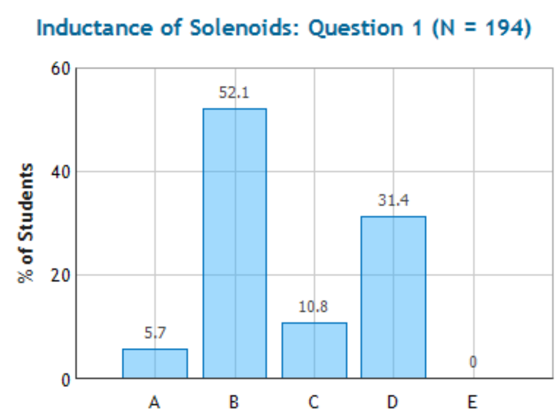
$(1/2)^2$ 2



$$\rightarrow \quad L_B = \frac{1}{2} L_A$$

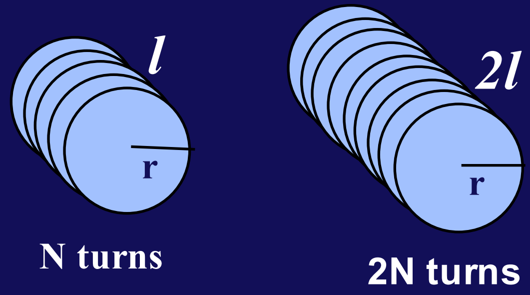
Compare the inductance of the two solenoids

A) $L_A = 4\, L_B$
B) $L_A = 2\, L_B$
C) $L_A = L_B$
D) $L_A = (1/2)\, L_B$
E) $L_A = (1/4)\, L_B$



Clicker (like the prelecture question)

- Consider the two inductors shown:
  - Inductor 1 has length l, N total turns and has inductance L_1.
  - Inductor 2 has length $2l$, 2N total turns and has inductance L_2.
  - What is the relation between L_1 and L_2?



(a) $L_2 < L_1$ (b) $L_2 = L_1$ (c) $L_2 > L_1$

$$L = \mu_0 n^2 A l$$

- N/l does not change → n is same for both
- l is twice as long for 2nd one
- $L_2 = 2L_1$

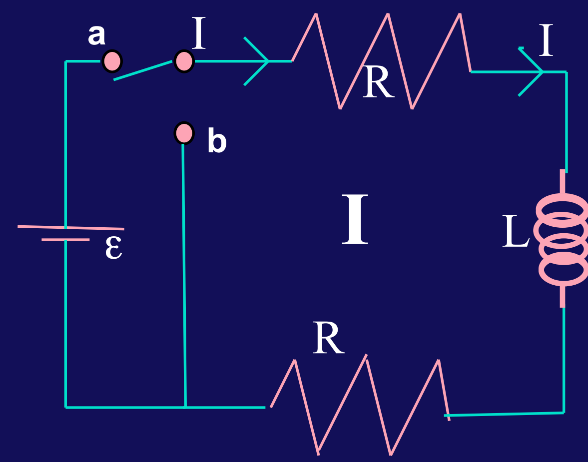
Clicker

At t = 0, the switch is thrown from position b to position a as shown:

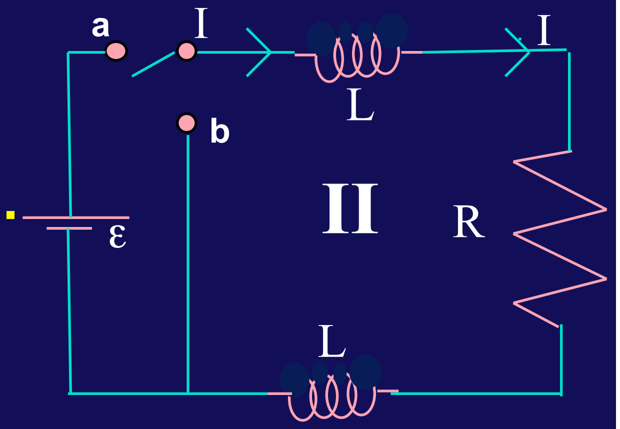
– Let t_I be the time for circuit I to reach 1/2 of its asymptotic current.

– Let t_{II} be the time for circuit II to reach 1/2 of its asymptotic current.

– What is the relation between t_I and t_{II}?



(a) $t_{II} < t_I$ (b) $t_{II} = t_I$ (c) $t_{II} > t_I$



- Determine the time constants of the two circuits.
- Write down the loop eqns:

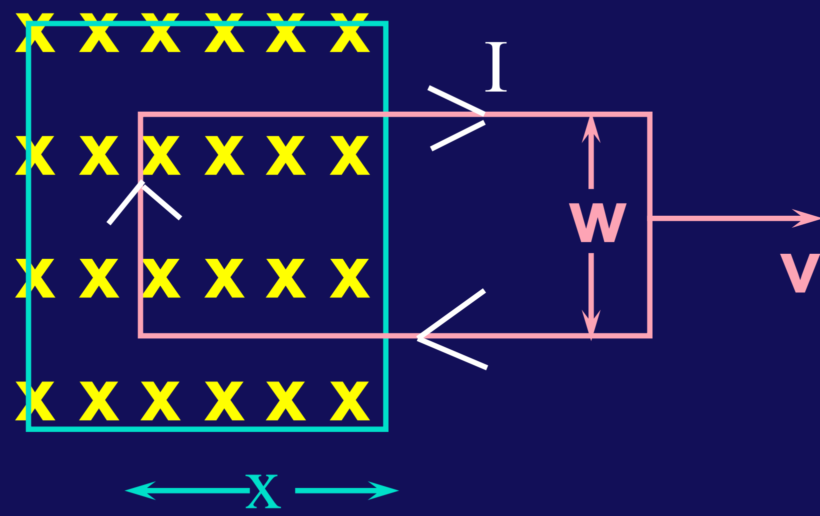
I: $$IR + L\frac{dI}{dt} + IR - \varepsilon = 0 \Rightarrow \tau_I = \frac{L}{2R}$$

II: $$L\frac{dI}{dt} + IR + L\frac{dI}{dt} - \varepsilon = 0 \Rightarrow \tau_{II} = \frac{2L}{R}$$

Can you determine from this Clicker the rule for adding inductors in series?

Classic Calculation

- **Suppose we pull with velocity v a coil of resistance R through a region of constant magnetic field B.**
  - **What will be the induced current?**
    - **What direction?**
      - **Lenz' Law þ clockwise!!**
  - **What is the magnitude?**
    - **Magnetic Flux:** $$\Phi_B = xwB$$
    - **Faraday's Law:** $$\varepsilon = -\frac{d\Phi_B}{dt}$$



$$\therefore \quad \frac{d\Phi_B}{dt} = wB\frac{dx}{dt} = wBv$$ þ $$I = \frac{\varepsilon}{R} = \frac{wBv}{R}$$