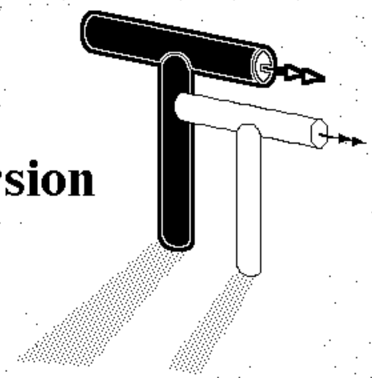
Torsion



An Introduction to Torsion in Circular Shafts

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How to solve the Torsion Problem

1) **Equilibrium** for equating forces and moments on free body diagrams,

2) **Kinematics** (or geometry of deformations) for postulating displacements and developing strain-displacement relationships,

3) **Constitutive Relations** for relating the stresses to

In this stack we will investigate **Torsion in Circular Shafts** using the three major components from the study of mechanics of materials. They are listed above.

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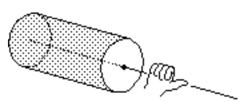
Sign Convention



In order to apply the equations we are about to develop, we must develop a sign convention for the rotation of an object about a given axis. To do this we will use the right-hand rule. Specifically, if the thumb is directed outward from the shaft → then the fingers will indicate the direction of *positive* rotation and torque.→

It is should be noted that the right hand rule may be applied to any cut.→

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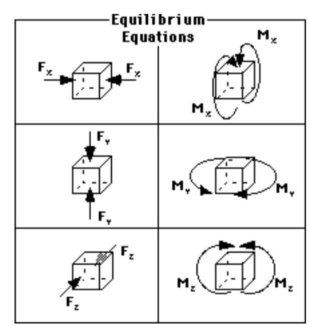
3-D Equilibrium Equations

$$\Sigma F_X = 0, \ \Sigma F_Y = 0, \ \Sigma F_Z = 0,$$
$$\Sigma M_X = 0, \ \Sigma M_Y = 0, \ \Sigma M_Z = 0.$$

Recall that in three dimensions there are six potential equations of equilibrium. They are the three force equilibrium equations and the three moment equilibrium equations.



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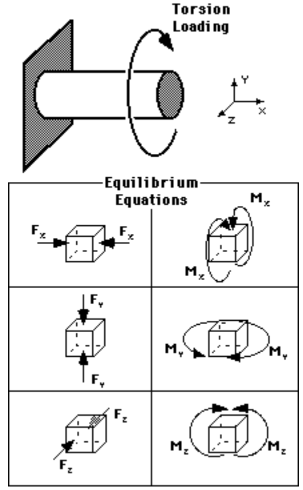

Although there are six potential equilibrium equations, the torsion problems we will be addressing (as is shown in the figure) will only require one of these equilibrium equations. Which equation do you think it is?

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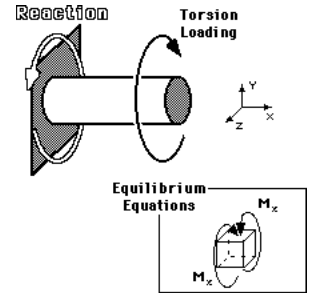

The fact that our torsion problems are governed by a single moment equilibrium equation implies that, for a statically determinate system, there can be *only one reaction*. It also follows that this reaction must be a moment about the X-axis.

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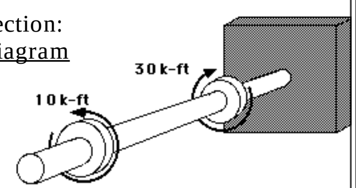
Torsion at a Section: The Torsion Diagram



Once we have determined the reaction, we must next calculate where in the shaft the internal torsion force is a maximum . This is done by using the *method of sections*..

Recall that to use the method of sections we must first solve for the reaction at the support. The reaction in a torsion problem is determined by applying moment equilibrium about the X-axis. → → →

$$\sum M_x = 0$$

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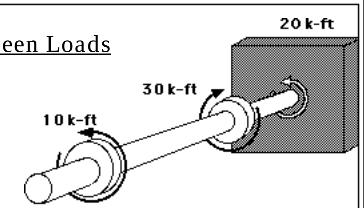
Take a Cut Between Loads



Once we have determined the reaction at the support we may now use the method of sections to determine the internal torque at any point along the member. To do this we first take a cut at any section →

$$\sum M_x = 0$$

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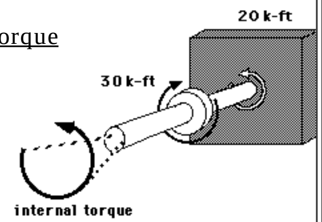
Calculate Internal Torque



$\sum M_x = 0$

.... and look at moment equilibrium for the portion of the member which is remaining.→ → →

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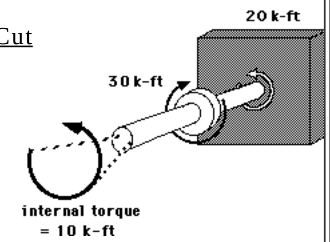
Take a Second Cut



A final cut will give us all the information we need to draw the torsion diagram for this element.→

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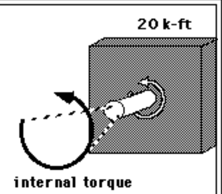
More Internal Torque



Using moment equilibrium once again, we find internal torque in the shaft between the 10 k-ft load and the wall.→

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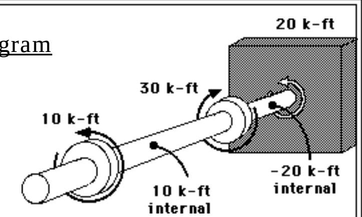
Torsion Diagram

Once we know the value of the internal moment in the shaft between each change in load, we are ready to plot the torsion diagram.

The torsion diagram plots just like the shear diagram for a beam.→ →



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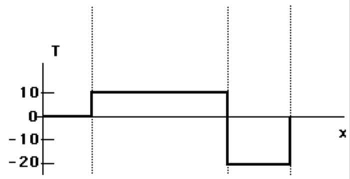
The first use of equilibrium in our examination of the torsion problem is to calculate the internal torque in the shaft given the external loadings. Once we have plotted the torsion diagram, it is then easy to read off the maximum internal torque experienced by the shaft.

1) **Equilibrium** for equating forces and moments on free body diagrams,

$$\sum M_x = 0 \quad \Rightarrow$$



Torsion Diagram

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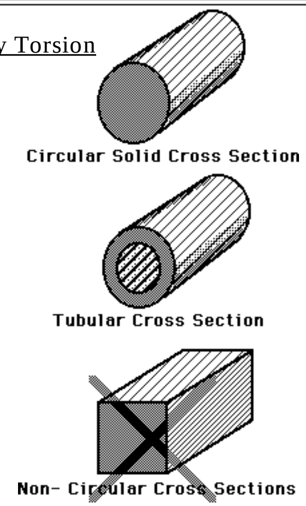
Part II: Stresses Induced by Torsion

This section examines the stresses induced by torsion loading in members having circular solid and tubular cross sections. Shafts with non-circular sections are not considered in this course.

To relate the stresses to internal torque we will first make several assumptions about the deflected shape of a torsion member.

Using these kinematic assumptions, along with Hooke's Law for linear elastic materials, we will them derive a formula which can be used to calculate shear stresses for known sections and internal torques.



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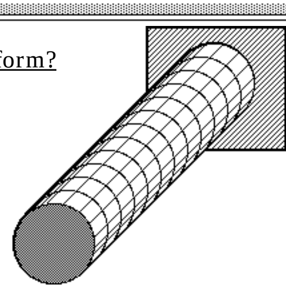
How Does a Shaft Deform?

As stated previously, it is necessary to make two assumptions about the deflected shape of the torsion element in order to establish the relationship between internal torque and the stresses resulting from the torque. In this section we will discuss the type of deformation that occurs when a torque is applied to a circular shaft made of a homogeneous material.

We can illustrate physically what happens when a torque is applied to a circular shaft by considering a shaft made of a highly deformable material, such as rubber. If we create a grid on the shaft, then when a torque is applied, the circles and longitudinal grid lines will distort.→



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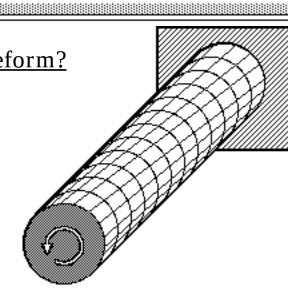

What conclusions about the deflected shape of the shaft can we make by looking at the deformed grid above?

Do the circles appear to have changed shape? Are the longitudinal

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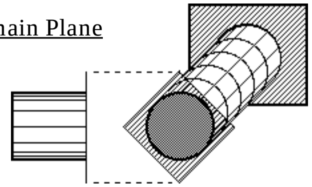
(1) Plane Sections Remain Plane



If you were looking closely, you should have seen that the circles drawn on the undeformed torsion bar remained circles after loading.

Based on this observation we will assume a plane section of material perpendicular to the axis of a circular member→ remains plane after the torques are applied,→ → and that no warping of the section occurs.→

A plane section before loading remains plane after loading,

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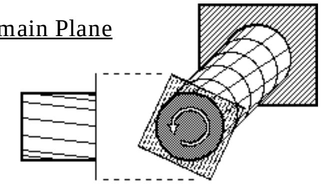

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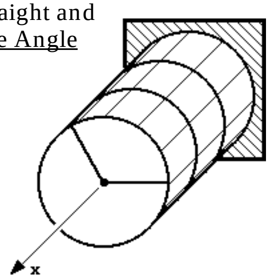
(2) Radial Lines Remain Straight and Rotate Through the Same Angle



The second assumption we make about the deflected shape of the shaft is that any line drawn from the neutral axis to the outside edge of the shaft → will remain straight after the torques are applied → →, and all such lines at a given section will rotate through the same angle, ϕ. →

A radial line on the undeformed section....

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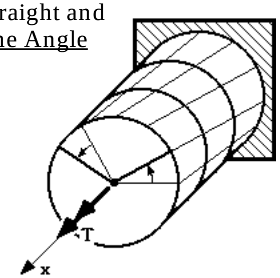

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A radial line on the undeformed section will remain straight after deformations. and all

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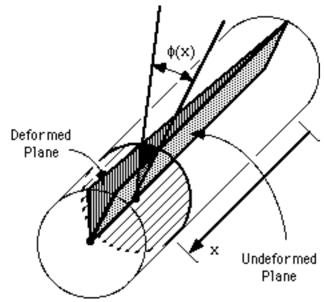
Angle of Twist, ϕ, Varies Linearly

Furthermore, we can see that a radial line located a distance x from the fixed end will rotate through an angle ϕ(x). The angle, ϕ(x), is called the *angle of twist*, and will vary linearly with position x along the shaft as shown.

This is consistant with our observation the longitudinal lines remain straight after loading.



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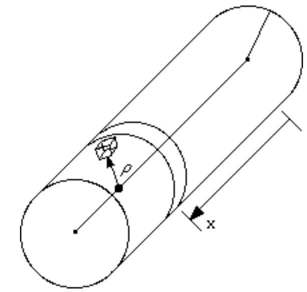
Focus on Small Element

To understand how this distortion strains the material, we will look at a small element a distance x from the fixed end and a distance ρ from the neutral axis.→



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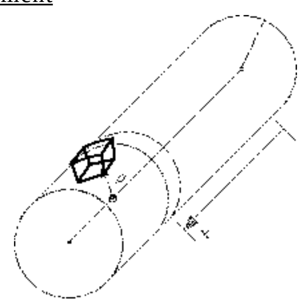

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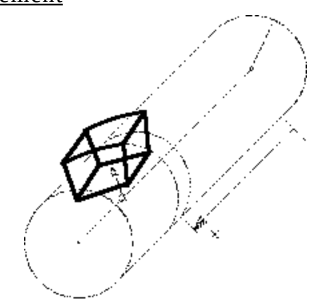

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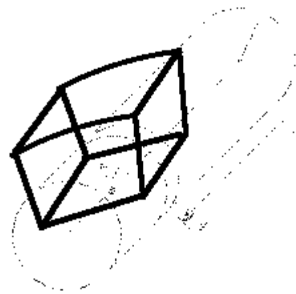

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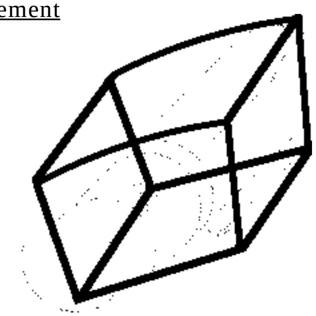

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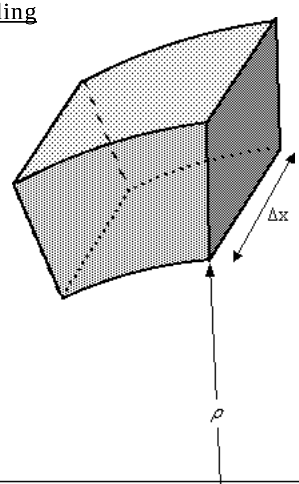
Apply Torsion Loading

How will this little element deform if we now apply a torsion load? →

(1) The front and rear surfaces will remain plane →, (2) the radial edges of the element will remain straight →, and (3) the longitudinal edges of the element will also remain straight due to the linear variation of the angle of twist, ϕ. →



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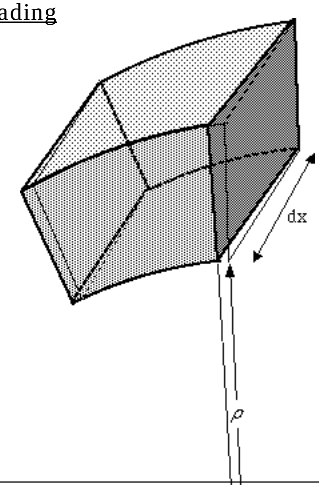

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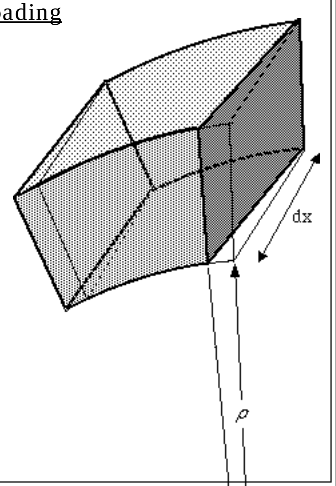

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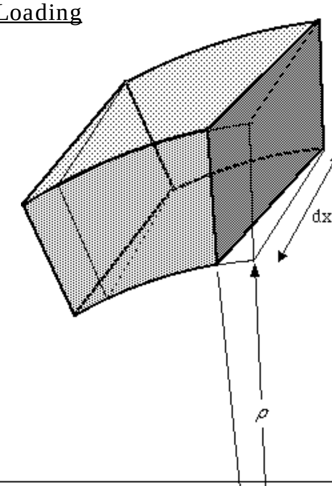

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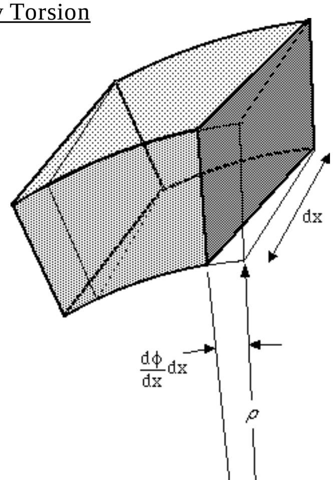
Strains Induced by Torsion

As a result of this deformation, both the front and back face of this element will undergo a rotation. The back face will rotate through the angle $\phi(x)$, and the front face will rotate through a larger angle,

$$\phi(x) + d\phi\ dx / dx$$

The difference in rotation between the two faces is then:

$$d\phi\ dx / dx$$



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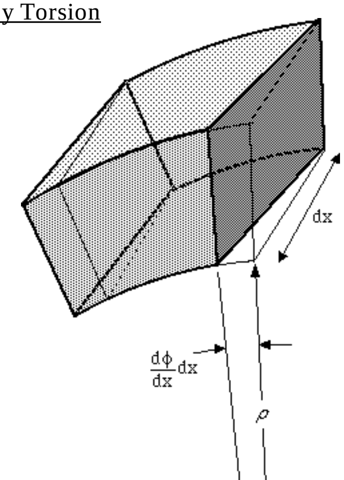
Strains Induced by Torsion

Looking at the element we can see that an angle α →, which was 90° before deformation, is now $\alpha' < 90°$ in the deformed element → . Furthermore, the difference between α and α' is by definition the *shear strain*, γ. →



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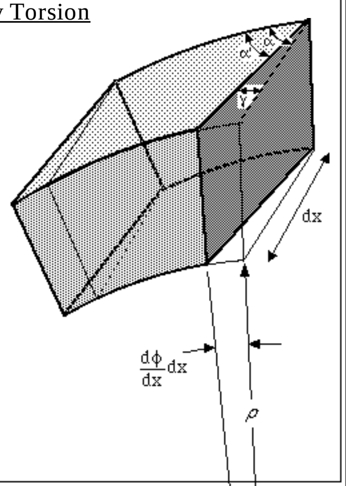
Strains Induced by Torsion

The shear strain, γ, can be related to the the length of the element, dx, and the difference in angle of rotation, $d\phi$, by looking at the length of the arc BC→. From the figure we have: →



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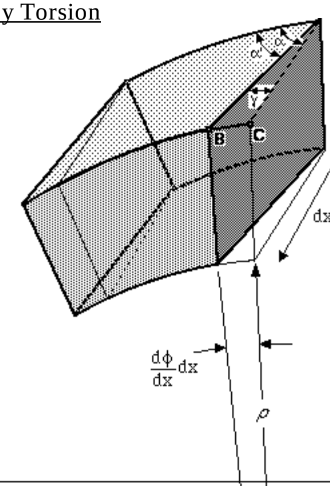
Strains Induced by Torsion

Solving for γ we get an expression for shear strain in terms of distance from the centroid of the section and angle of twist of the section.

$$\overline{BC} = \rho\, d\phi = dx\, \gamma$$

$$\gamma = \rho \frac{d\phi}{dx}$$



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Strains Induced by Torsion

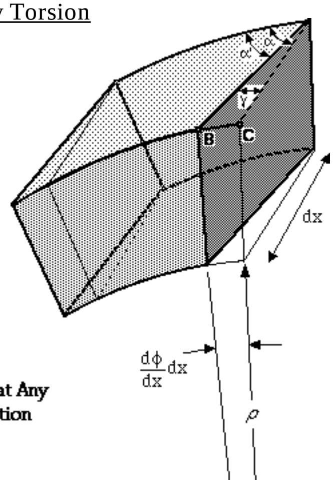
Noting that the angle of twist is constant at any section (all radial lines rotate the same amount) we now have an expression for shear strain which varies only with the distance from the centroid of the section, ρ.

In other words, the shear strain varies linearly along any radial line... →.

$$\overline{BC} = \rho\, d\phi = dx\, \gamma$$

$$\gamma = \rho \boxed{\frac{d\phi}{dx}}$$

Constant at Any Cross-Section



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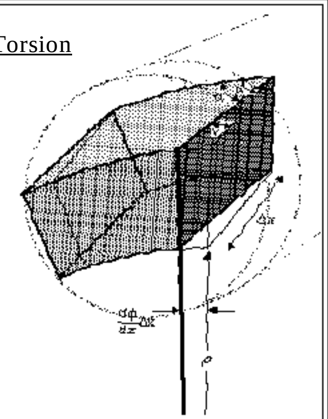

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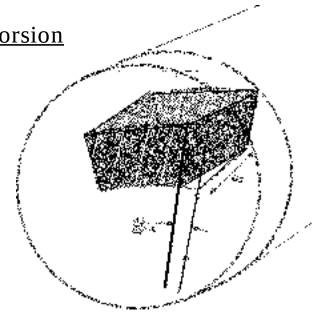

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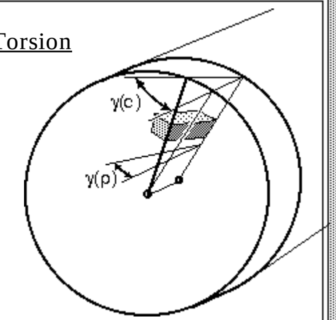

.... the shear strain varies from zero at the neutral axis →, to a maximum at the outside edge of the shaft →.

** note **

From now the symbol "c" refers to the distance from the centroid to the furthest point of the section.

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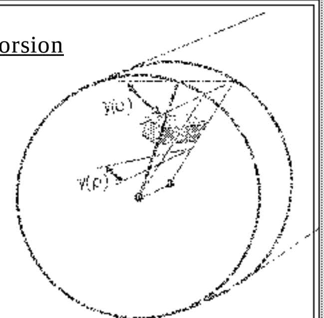
Strains Induced by Torsion



$$\gamma = \rho \frac{d\phi}{dx}$$

2) **Kinematics** (or geometry of deformations) for postulating displacements and developing strain-displacement relationships,

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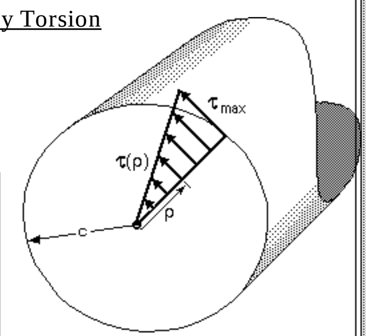
Stress Induced by Torsion

Hooke's Law:

$$\tau = G\gamma$$



We have just shown that a torque applied to a circular shaft will result in linear shear strains in the shaft varying from zero at the neutral axis to a maximum at the outer surface.

In order to relate these shear strains to shear stresses we need to recall the constitutive relationship between shear stess and shear strain; namely...

$$\tau = G\gamma$$

... where G, the shear modulus, is a

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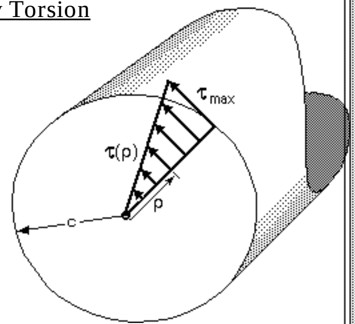
Stress Induced by Torsion

Hooke's Law :

$\tau = G\,\gamma$

Strain Varies Linearly

$\therefore$ Stress Varies Linearly



This relationship reveals to us that a linear variation in shear strain will lead to a corresponding linear variation in shear stress along any radial line of the cross section. The figure at the right shows a plot of the shear stress as a function of radius, ρ.

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Stress Induced by Torsion

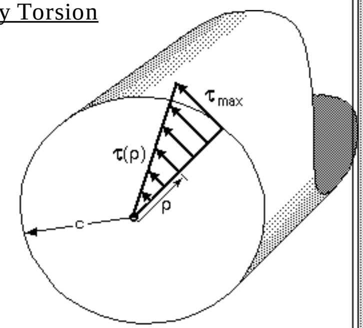
Hooke's Law :

$\tau = G\,\gamma$

Strain Varies Linearly

$\therefore$ Stress Varies Linearly

$$\tau = \frac{\rho}{c}\,\tau_{max}$$



As a result of the linear variation in shear stress we can express the shear stress at any distance ρ from the center of the shaft as shown above.

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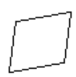
Longitudinal Stress



Recall from our discussion of stress at a point that a shear stress on any one face of an isolated element must, by reason of both force and moment equilibrium, also develop equal shear stresses on the three adjacent faces →. Consequently, if we isolate a vanishingly small element of material from the rest of the shaft → we should expect to see the following stresses → .

From this we can conclude that an internal torque, T, induces shear stresses in the plane of the cross section, and associated shear-stresses along each longitudinal plane →.

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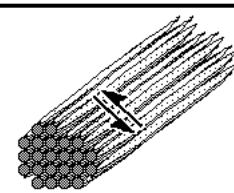
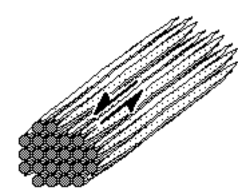
Longitudinal Stress

It is interesting to note that because of this axial shear stress, shafts made of wood tend to split along the axial plane when subjected to large torques.

This is because wood is an anisotropic material. Its shear strength parallel to its fibers (directed along the axis of the shaft) is much less than its shear strength perpendicular to the fibers.



Strong in Cross-Grain Shear



Weak in Longitudinal Shear

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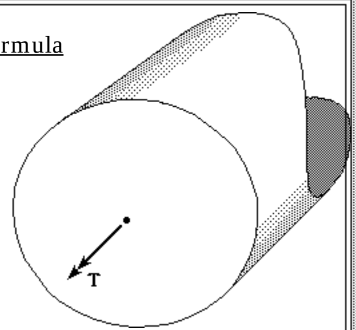
The Torsion Formula



Using the relationship we just derived for shear stress we will now require that the torque produced by the stress distribution over the entire cross section be equilvalent to the internal torque, T, at the section.

Specifically, each element of area dA →, located a distance ρ from the center →, is subjected to a differential force **dF** = τ dA → .

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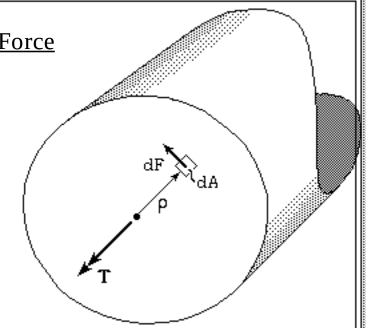
Differential Force

$$dF = \tau \cdot dA$$



The differential force can be calculated as the differential area times the shear stress acting on the area. If the differential area is small enough, it is alright to assume that the shear stress acting on the area is constant.

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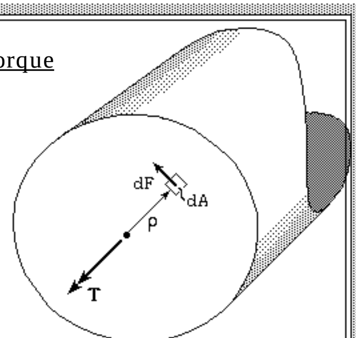
Differential Torque

$$dF = \tau \cdot dA$$

$$\text{Torque} = \text{Force} \cdot \text{Distance}$$



From somewhere in your studies you may recall that we calculate torque as force times distance.

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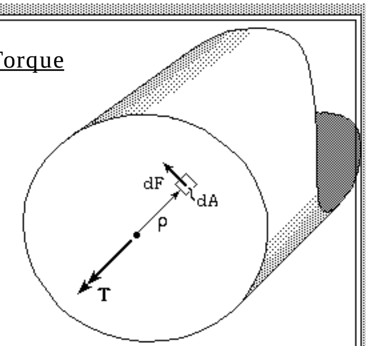
Differential Torque

$$dF = \tau \cdot dA$$

$$dT = \tau \, dA \cdot \rho$$



Therefore, to calculate the torque caused by the differential force about the centroid of the section we need only to multiply the force, τ dA, times it's distance from the centroid, ρ.

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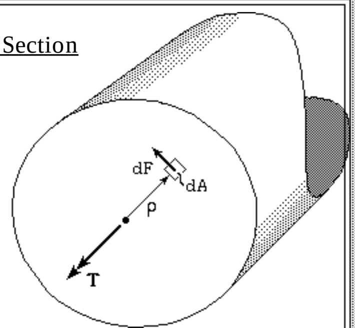
Total Torque on Section

$$dF = \tau \cdot dA$$
$$dT = \tau \, dA \cdot \rho$$
$$T = \int_A \rho \cdot \tau \, dA$$



To get the total torque caused by all the shear stresses in the cross-section, we integrate the expression for differential torque over the area of the cross-section.

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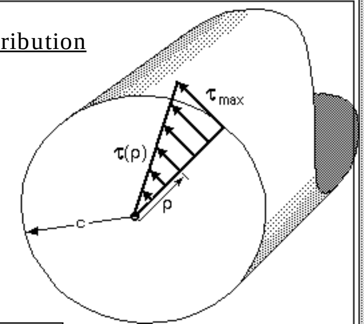
Recall Linear Stress Distribution

$$dF = \tau \cdot dA$$
$$dT = \tau \, dA \cdot \rho$$
$$T = \int_A \rho \cdot \tau \, dA$$
$$\tau = \frac{\rho}{c} \tau_{max}$$



Recall our expression for shear stress in terms of the maximum shear stress in the section.

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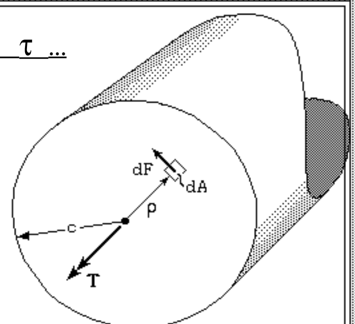
Sutstituting for τ ...

$$dF = \tau \cdot dA$$
$$dT = \tau \, dA \cdot \rho$$
$$T = \int_A \rho \cdot \tau \, dA$$
$$\tau = \frac{\rho}{c} \tau_{max}$$
$$T = \int_A \rho \cdot \frac{\rho}{c} \tau_{max} \, dA$$



We now substitute this expression for τ into the integral.

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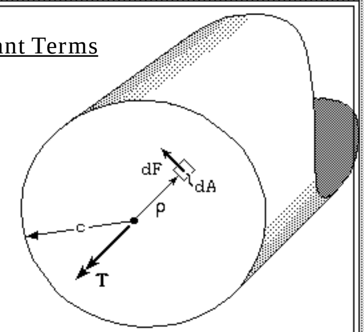
Bring out Constant Terms

$$dF = \tau \cdot dA$$
$$dT = \tau \, dA \cdot \rho$$
$$T = \int_A \rho \cdot \tau \, dA$$
$$T = \int_A \rho \cdot \frac{\rho}{c} \tau_{max} \, dA$$
$$T = \frac{\tau_{max}}{c} \int_A \rho^2 \, dA$$



Because the outer radius of the section and the maximum shear stress are constant for any section, we can bring them outisde of the area integral.

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Solve for Maximum Shear Stress

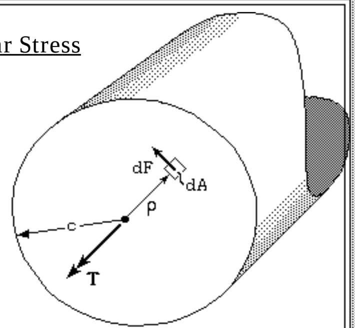
$$dF = \tau \cdot dA$$
$$dT = \tau\, dA \cdot \rho$$

$$T = \int_A \rho \cdot \tau\, dA$$

$$T = \int_A \rho \cdot \frac{\rho}{c}\, \tau_{max}\, dA$$

$$T = \frac{\tau_{max}}{c} \int_A \rho^2\, dA$$

$$\tau_{max} = \frac{T\,c}{\int_A \rho^2\, dA}$$



Solving for the τ_{max}, we have an expression for maximum shear stress in terms of internal torque and section geometry.

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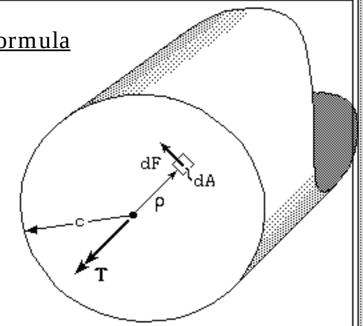
The Torsion Formula

$$\tau_{max} = \frac{T\,c}{\int_A \rho^2\, dA}$$



Using this formula, we can calculate the shear stress given the loading and the section geometry.

Do you recognize the integral in the denominator of the formula?

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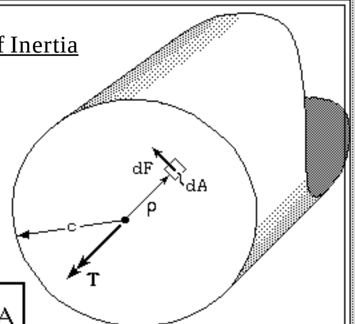
Polar Moment of Inertia

$$\tau_{max} = \frac{T\,c}{\int_A \rho^2\, dA}$$

$$\tau_{max} = \frac{T\,c}{J}$$

$$J = \int_A \rho^2\, dA$$



The second moment of area in a polar coordinate system is called the polar moment of inertia, J.

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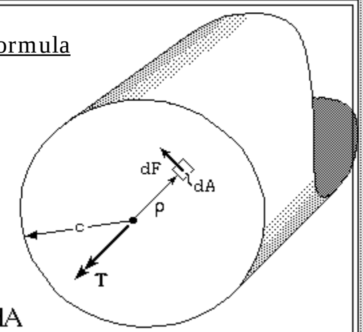
The Torsion Formula

$$\tau_{max} = \frac{T\,c}{\int_A \rho^2\, dA}$$

$$\tau_{max} = \frac{T\,c}{J}$$

$$J = \int_A \rho^2\, dA$$

$$\tau = \frac{T\,\rho}{J}$$



Using the fact that shear stress varies linearly with respect to radial distance, we can calculate the shear stress at any distance, ρ, from the centroid of the section as $T\rho / J$.

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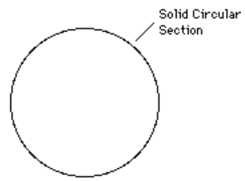
J for a Solid Circular Section



$$J = \int_A \rho^2 \, dA$$

Let's calculate the polar moment of inertia for a solid circular section.

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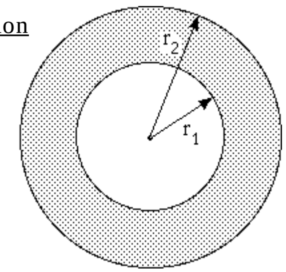
J for a Tubular Section

$$J = \int_A \rho^2 \, dA$$



Let us now consider a shaft whose cross section is tubular, with inner radius r_1 and outer radius r_2. As with the solid shaft, the *polar moment of inertia* is calculated as the second moment of area radially about the centroid of the section.

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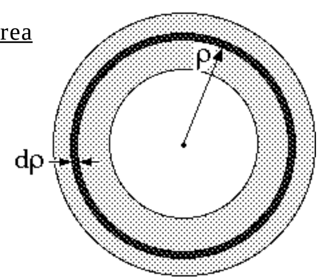
Differential Area

$$J = \int_A \rho^2 \, dA$$

$$J = \int_A \rho^2 \, 2\pi\rho \cdot d\rho$$



Recall that the differential area element we use in this integration is an annulus (i.e. a ring).

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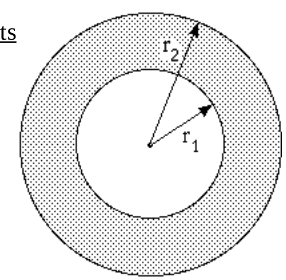
Integration Limits

$$J = \int_A \rho^2 \, dA$$

$$J = \int_A \rho^2 \, 2\pi\rho \cdot d\rho$$



The only difference between the tubular and the solid section are the integration limits.

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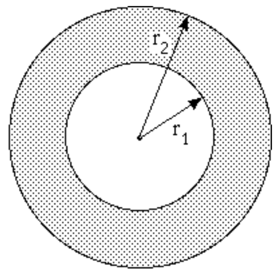
Integrating....



$$J = \int_A \rho^2 \, dA$$

$$J = \int_{r_1}^{r_2} \rho^2 \, 2\pi\rho \cdot d\rho$$

Performing the integration....

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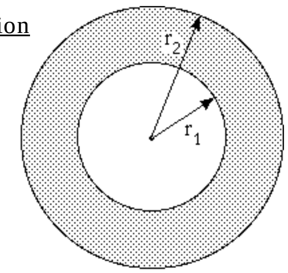
J for a Tubular Section



$$J = \int_A \rho^2 \, dA$$

$$J = \int_{r_1}^{r_2} \rho^2 \, 2\pi\rho \cdot d\rho$$

$$\boxed{J = \frac{\pi}{2} \left(r_2^4 - r_1^4 \right)}$$

Note that this expression is equal to J for the outer circle minus J for the inner circle.

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That's all For Now.

Soon to come....

DISPLACEMENTS !!

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