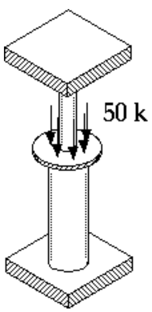
Statically Indeterminate Problems

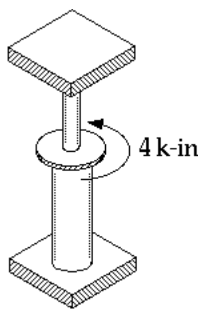
Axial

Torsion

Table of Contents

Calculate the Stresses

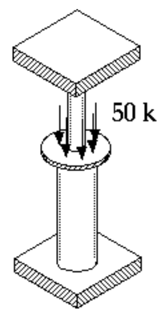


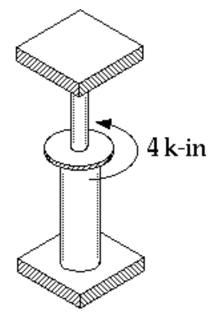


The same structure is loaded two different ways. On the left it is loaded by a 50 k axial force applied between the two elements. On the right it is loaded by a 4 k-in moment, again applied between the two members.

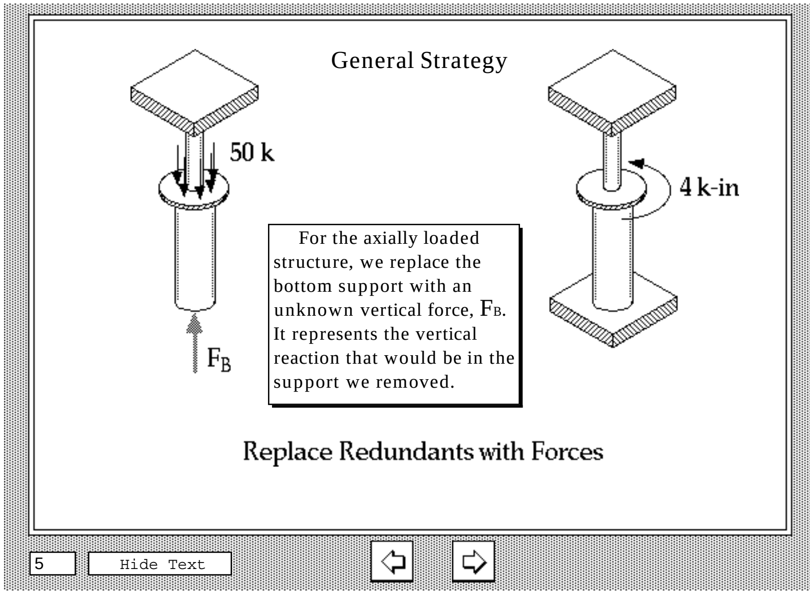
We want to calculate the stresses in the elements for both loading cases. Because the structure is statically indeterminate, we cannot solve for the stresses using equilibrium alone. How do we go about finding the stresses?

General Strategy





One strategy for solving indeterminate problems is to reduce the indeterminate structure to a determinate one by replacing supports with unknown


General Strategy
50 k
4 k-in
For the axially loaded structure, we replace the bottom support with an unknown vertical force, FB. It represents the vertical reaction that would be in the support we removed.
FB
Replace Redundants with Forces
5
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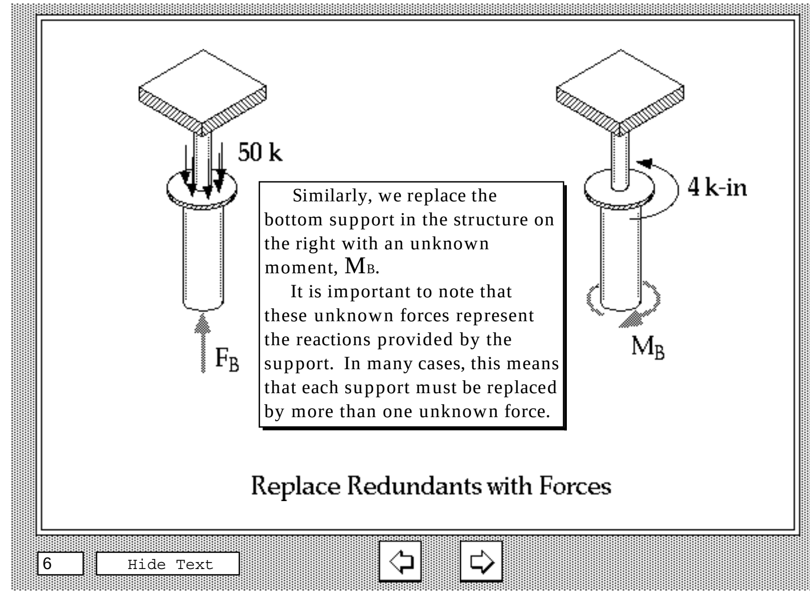
50 k
4 k-in
Similarly, we replace the bottom support in the structure on the right with an unknown moment, MB.
It is important to note that these unknown forces represent the reactions provided by the support. In many cases, this means that each support must be replaced by more than one unknown force.
FB
MB
Replace Redundants with Forces
6
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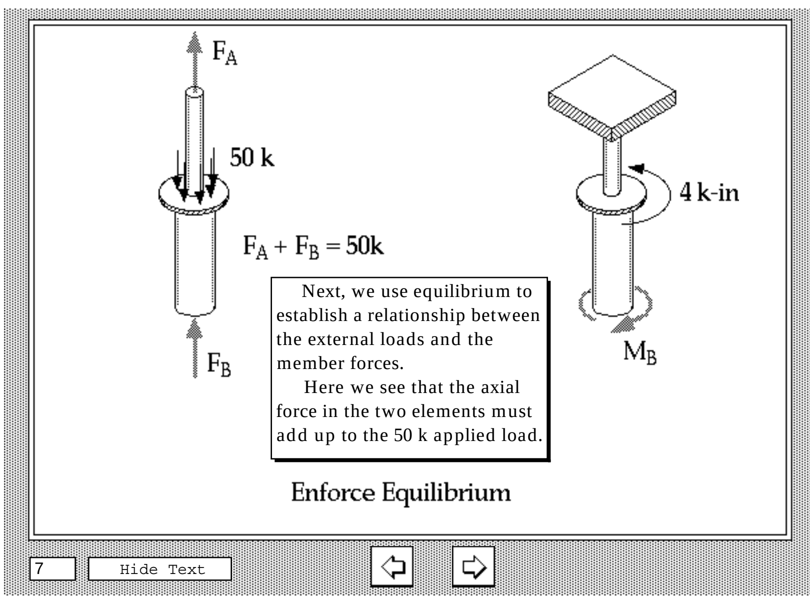
FA
50 k
4 k-in
FA + FB = 50k
Next, we use equilibrium to establish a relationship between the external loads and the member forces.
Here we see that the axial force in the two elements must add up to the 50 k applied load.
FB
MB
Enforce Equilibrium
7
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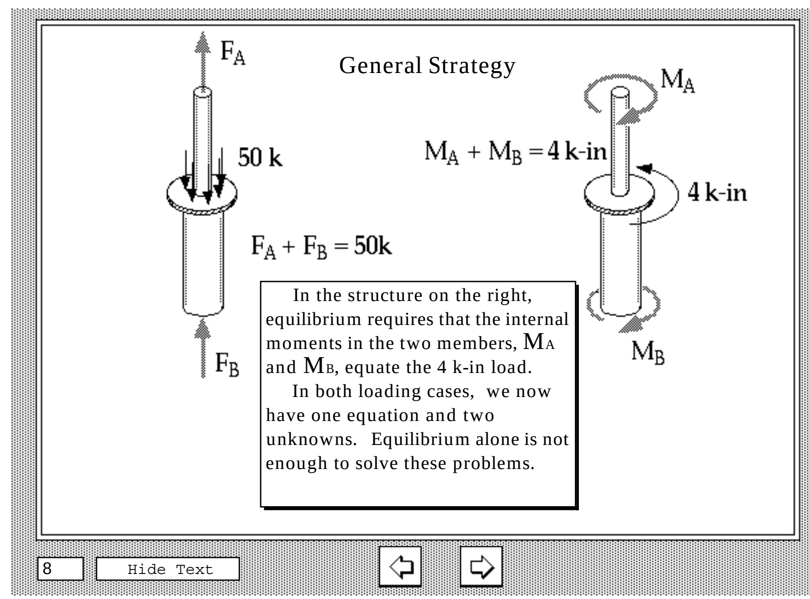
FA
General Strategy
MA
50 k
MA + MB = 4 k-in
4 k-in
FA + FB = 50k
In the structure on the right, equilibrium requires that the internal moments in the two members, MA and MB, equate the 4 k-in load.
In both loading cases, we now have one equation and two unknowns. Equilibrium alone is not enough to solve these problems.
FB
MB
8
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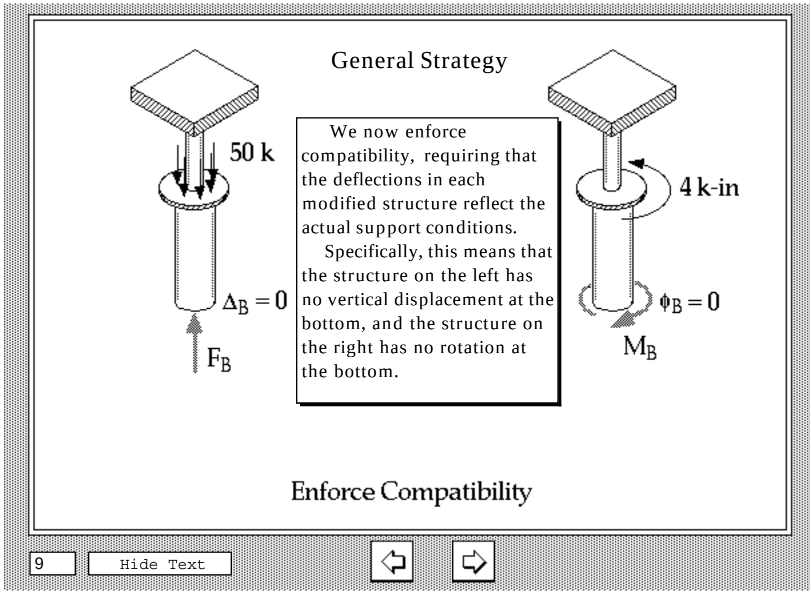
General Strategy
50 k
4 k-in
We now enforce compatibility, requiring that the deflections in each modified structure reflect the actual support conditions.
Specifically, this means that the structure on the left has no vertical displacement at the bottom, and the structure on the right has no rotation at the bottom.
ΔB = 0
ϕB = 0
FB
MB
Enforce Compatibility
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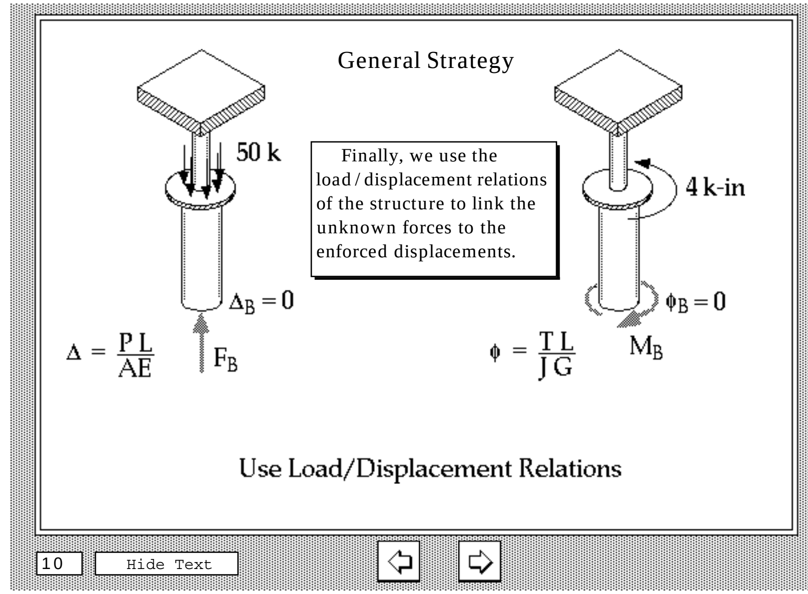
General Strategy
50 k
4 k-in
Finally, we use the load / displacement relations of the structure to link the unknown forces to the enforced displacements.
ΔB = 0
ϕB = 0
Δ = PL/AE
FB
ϕ = TL/JG
MB
Use Load/Displacement Relations
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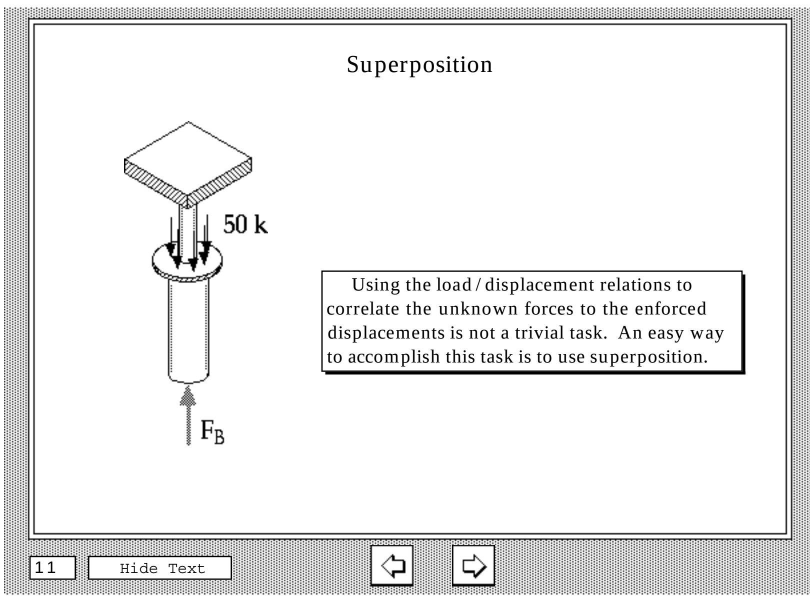
Superposition
50 k
Using the load / displacement relations to correlate the unknown forces to the enforced displacements is not a trivial task. An easy way to accomplish this task is to use superposition.
FB
11
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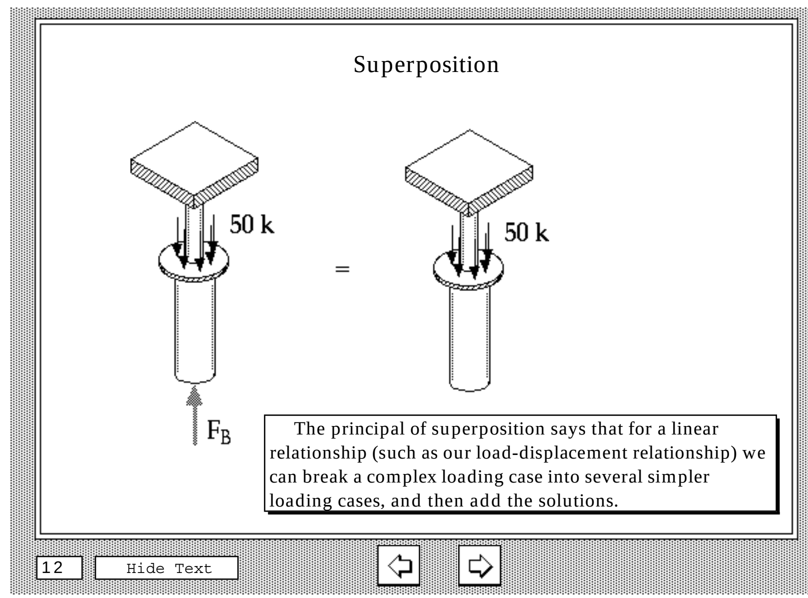
Superposition
50 k
=
50 k
FB
The principal of superposition says that for a linear relationship (such as our load-displacement relationship) we can break a complex loading case into several simpler loading cases, and then add the solutions.
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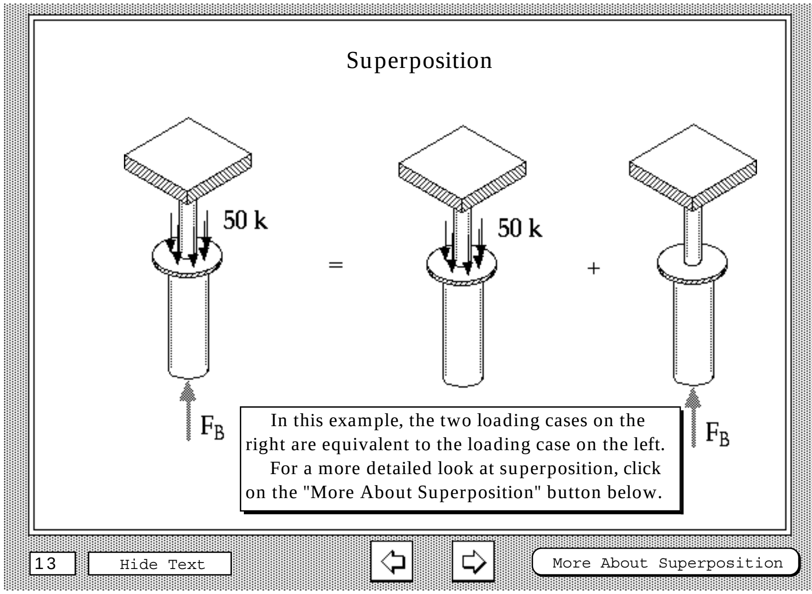
Superposition
50 k
=
50 k
+
F_B
F_B
In this example, the two loading cases on the right are equivalent to the loading case on the left.
For a more detailed look at superposition, click on the "More About Superposition" button below.
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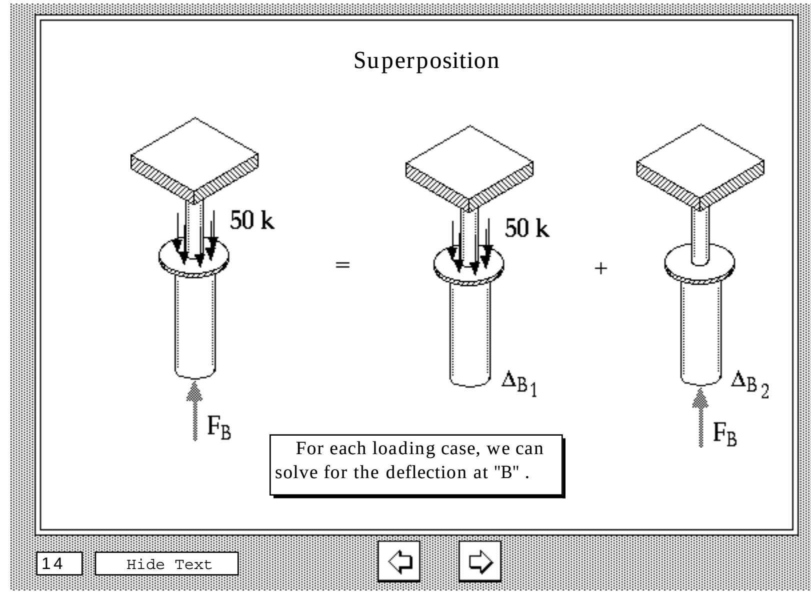
More About Superposition


Superposition
50 k
=
50 k
+
ΔB1
ΔB2
F_B
F_B
For each loading case, we can solve for the deflection at "B" .
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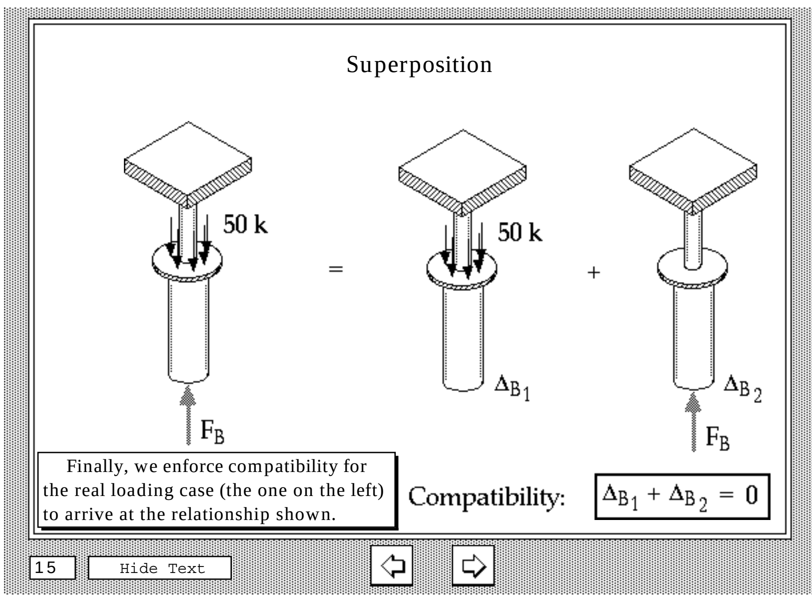
Superposition
50 k
=
50 k
+
ΔB1
ΔB2
F_B
F_B
Finally, we enforce compatibility for the real loading case (the one on the left) to arrive at the relationship shown.
Compatibility:
ΔB1 + ΔB2 = 0
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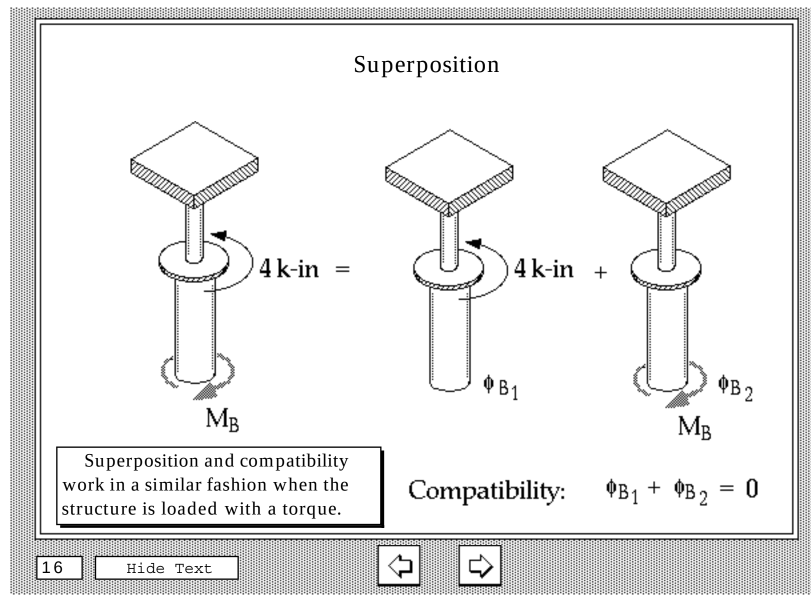
Superposition
4 k-in =
4 k-in +
M_B
φB1
φB2
M_B
Superposition and compatibility work in a similar fashion when the structure is loaded with a torque.
Compatibility:
φB1 + φB2 = 0
16
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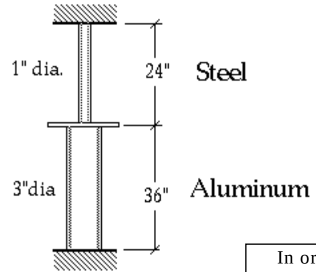
Details



In order to calculate the deflections necessary to solve this problem, we need to know (1) the geometery of the structure, (2) the geometery of cross-section, and (3) the material properties of the steel and aluminum from which the elements are made.

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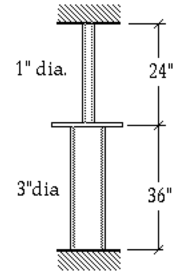
Details



Steel $\begin{cases} E_{St} = 30{,}000 \text{ ksi} \\ G_{St} = 11{,}000 \text{ ksi} \end{cases}$ $A_{St} = 0.785 \text{ in}^2$

Aluminum $\begin{cases} E_{Al} = 10{,}000 \text{ ksi} \\ G_{Al} = 3{,}900 \text{ ksi} \end{cases}$ $A_{Al} = 7.07 \text{ in}^2$

Here are the values for Young's Modulus, the shear modulus, and the cross-sectional area for each element.

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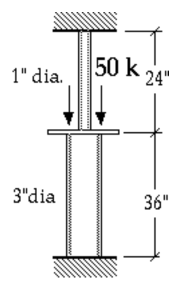
Solution: Axial Case



Let's begin with the case of axially loading.

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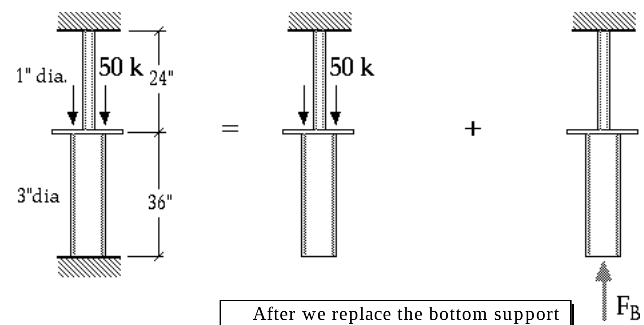
Solution: Axial Case



After we replace the bottom support with an unknown force, F_B, the principle of superposition allows us to split up the loading into two separate cases.

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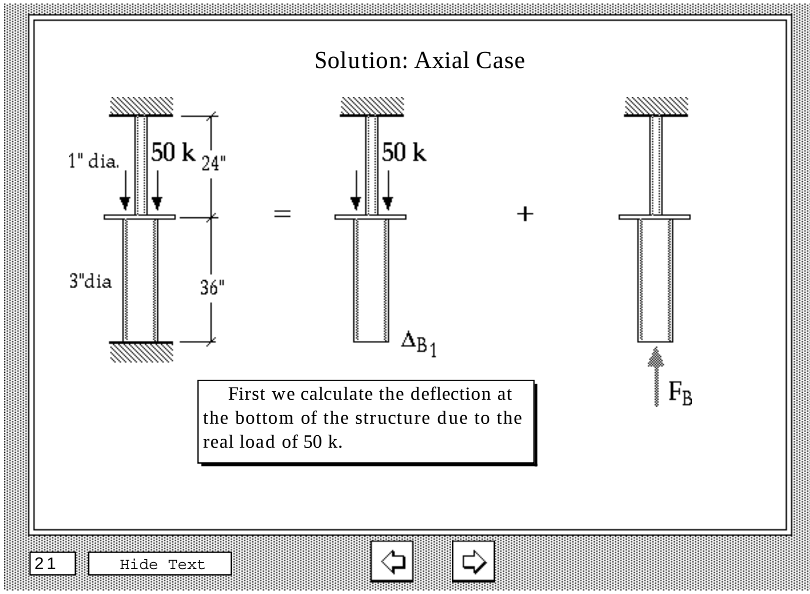
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
Δ_{B_1}
+
F_B
First we calculate the deflection at the bottom of the structure due to the real load of 50 k.
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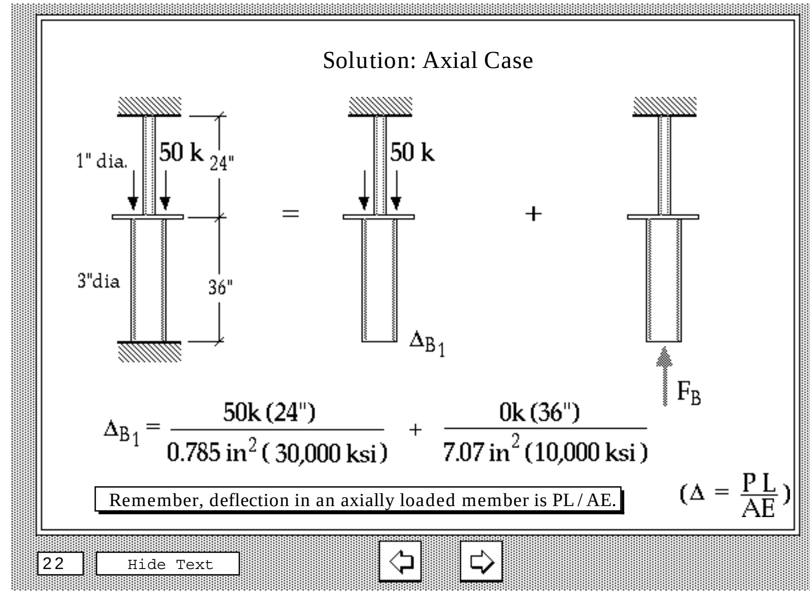
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
Δ_{B_1}
+
F_B
$\Delta_{B_1} = \frac{50k\,(24")}{0.785\ in^2\,(30{,}000\ ksi)} + \frac{0k\,(36")}{7.07\ in^2\,(10{,}000\ ksi)}$
Remember, deflection in an axially loaded member is PL / AE.
$(\Delta = \frac{PL}{AE})$
22
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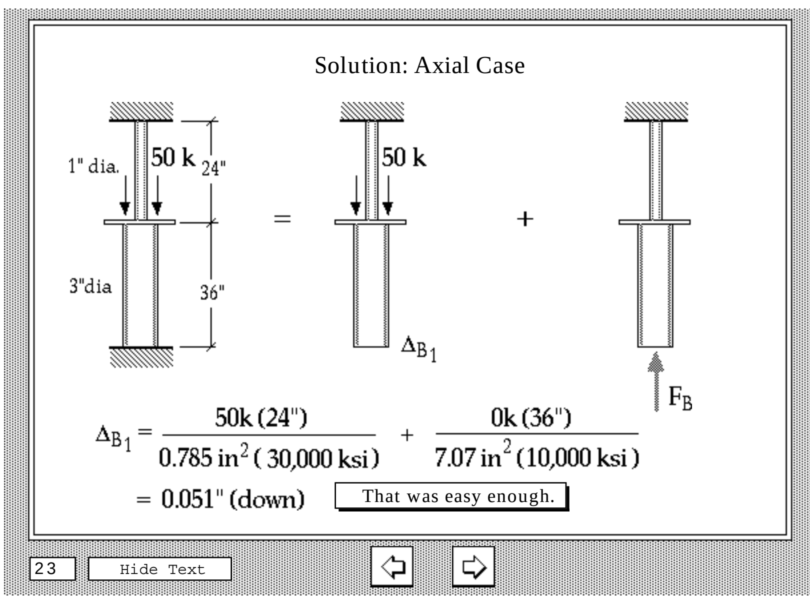
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
Δ_{B_1}
+
F_B
$\Delta_{B_1} = \frac{50k\,(24")}{0.785\ in^2\,(30{,}000\ ksi)} + \frac{0k\,(36")}{7.07\ in^2\,(10{,}000\ ksi)}$
$= 0.051"$ (down)
That was easy enough.
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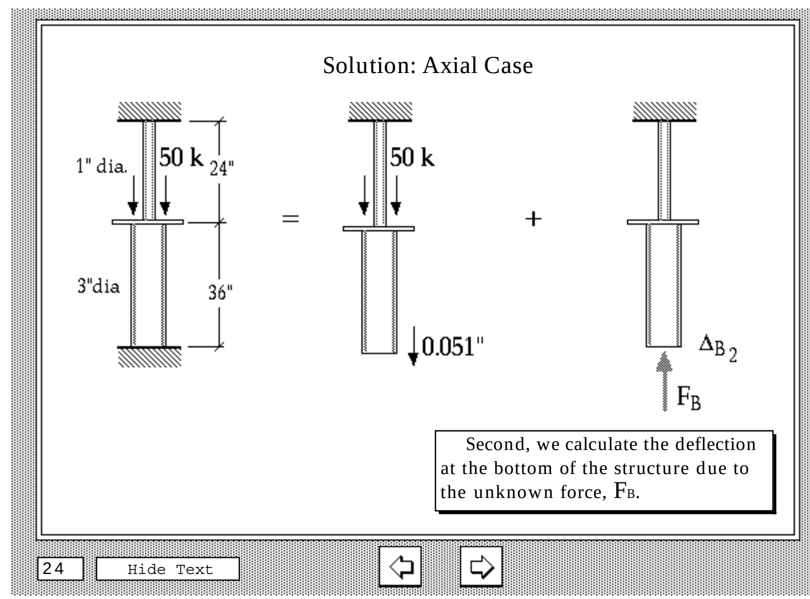
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
0.051"
+
Δ_{B_2}
F_B
Second, we calculate the deflection at the bottom of the structure due to the unknown force, F_B.
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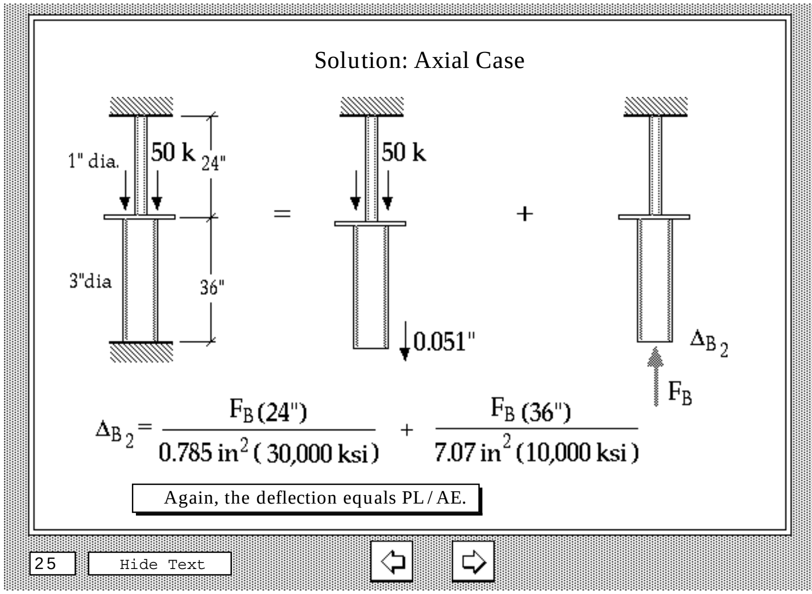
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
+
0.051"
ΔB 2
FB
ΔB 2 = FB (24") / 0.785 in2 (30,000 ksi) + FB (36") / 7.07 in2 (10,000 ksi)
Again, the deflection equals PL / AE.
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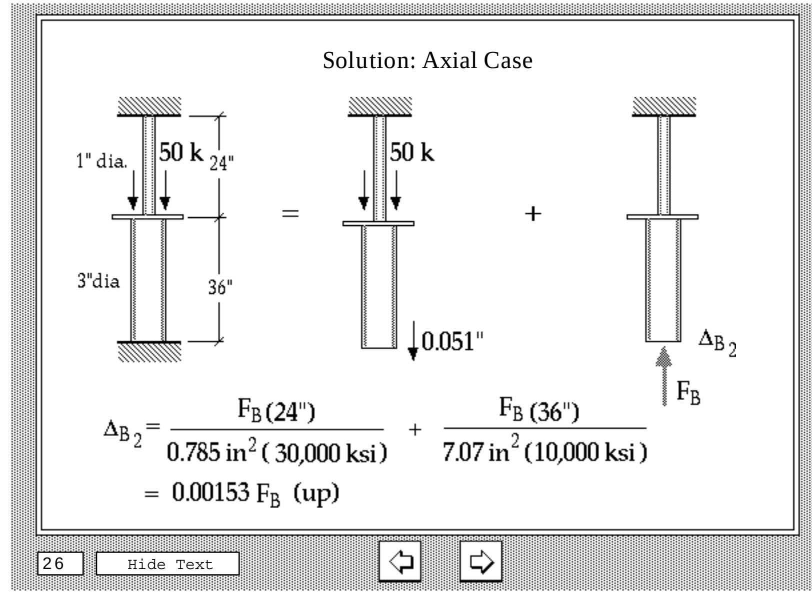
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
+
0.051"
ΔB 2
FB
ΔB 2 = FB (24") / 0.785 in2 (30,000 ksi) + FB (36") / 7.07 in2 (10,000 ksi)
= 0.00153 FB (up)
26
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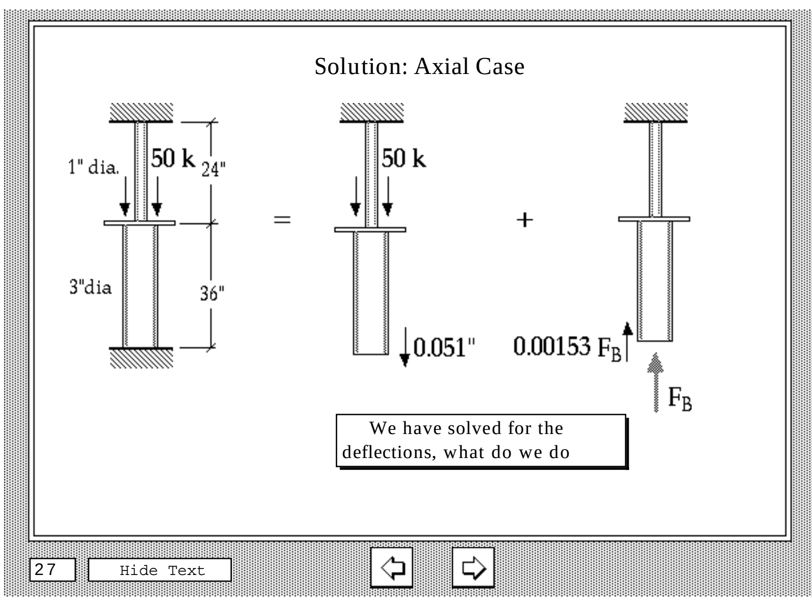
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
+
0.051"
0.00153 FB
FB
We have solved for the deflections, what do we do
27
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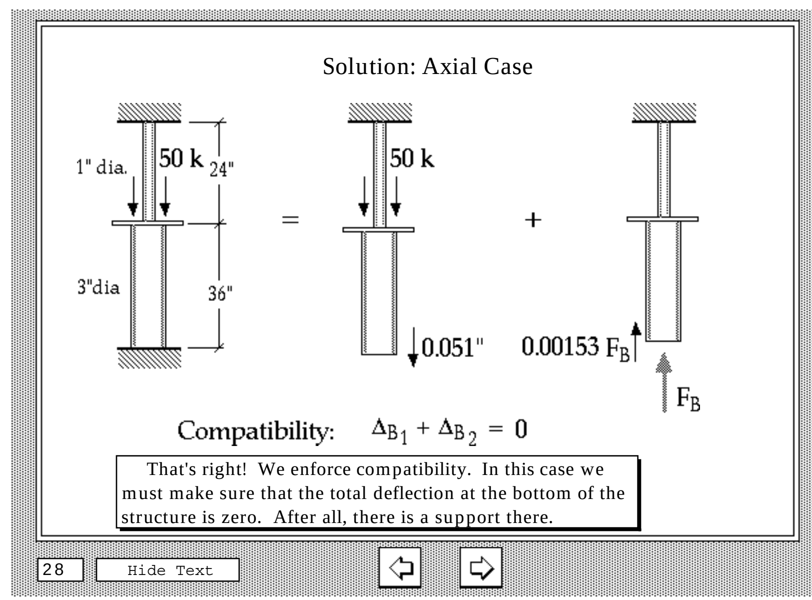
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
+
0.051"
0.00153 FB
FB
Compatibility: ΔB1 + ΔB2 = 0
That's right! We enforce compatibility. In this case we must make sure that the total deflection at the bottom of the structure is zero. After all, there is a support there.
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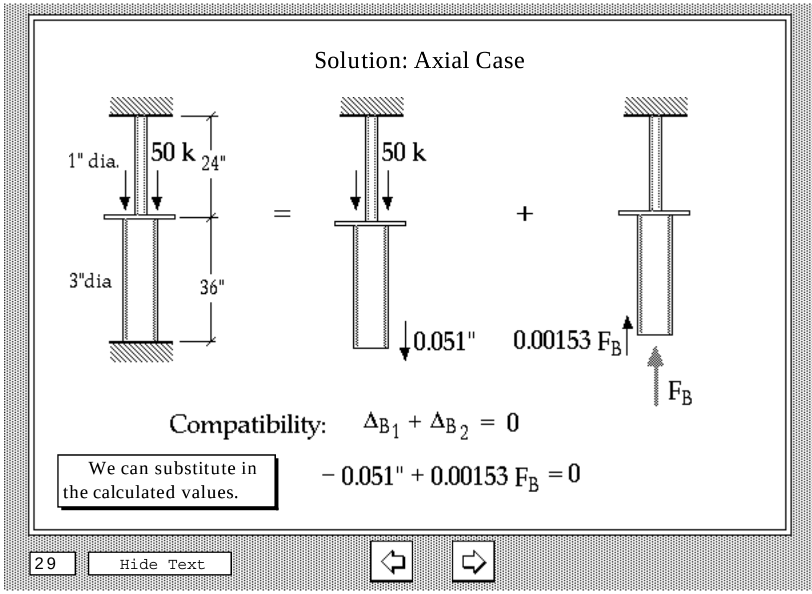
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
0.051"
+
0.00153 F_B
F_B
Compatibility: ΔB1 + ΔB2 = 0
We can substitute in the calculated values.
− 0.051" + 0.00153 F_B = 0
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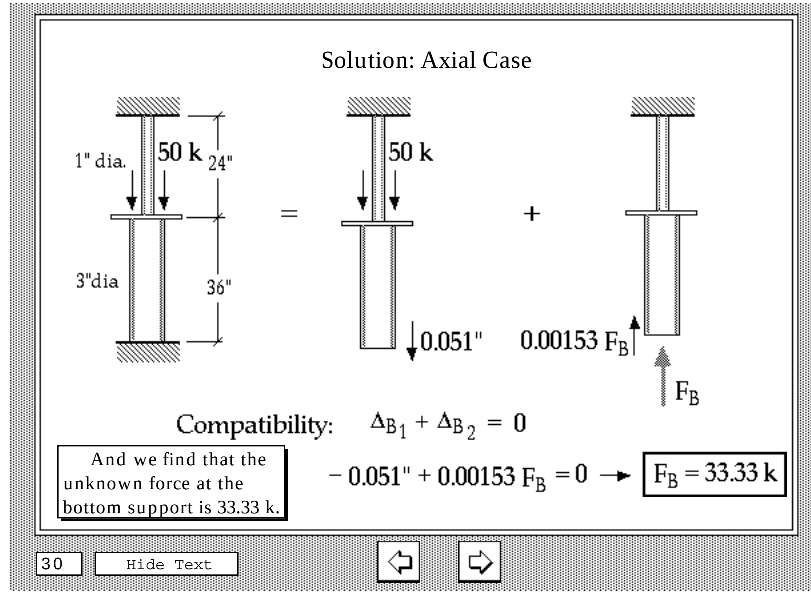
Solution: Axial Case
1" dia.
50 k
24"
3"dia
36"
=
50 k
0.051"
+
0.00153 F_B
F_B
Compatibility: ΔB1 + ΔB2 = 0
And we find that the unknown force at the bottom support is 33.33 k.
− 0.051" + 0.00153 F_B = 0
F_B = 33.33 k
30
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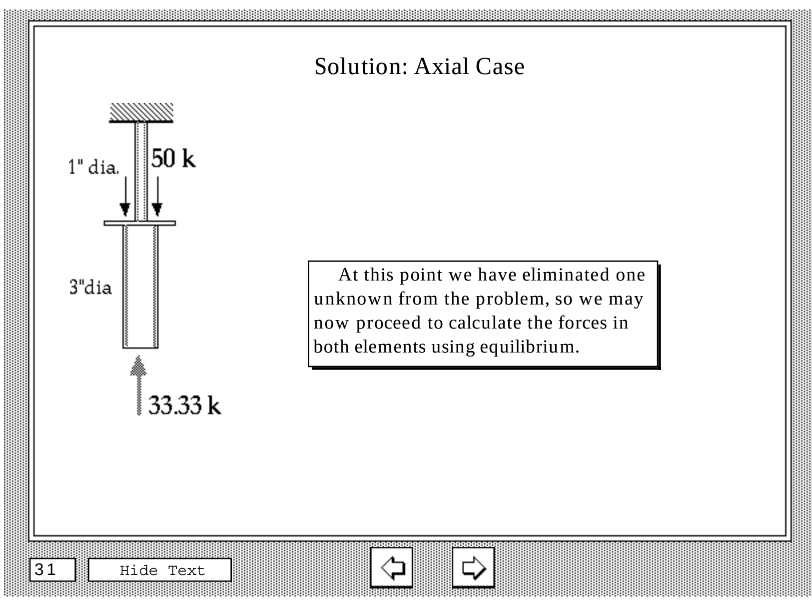
Solution: Axial Case
1" dia.
50 k
3"dia
33.33 k
At this point we have eliminated one unknown from the problem, so we may now proceed to calculate the forces in both elements using equilibrium.
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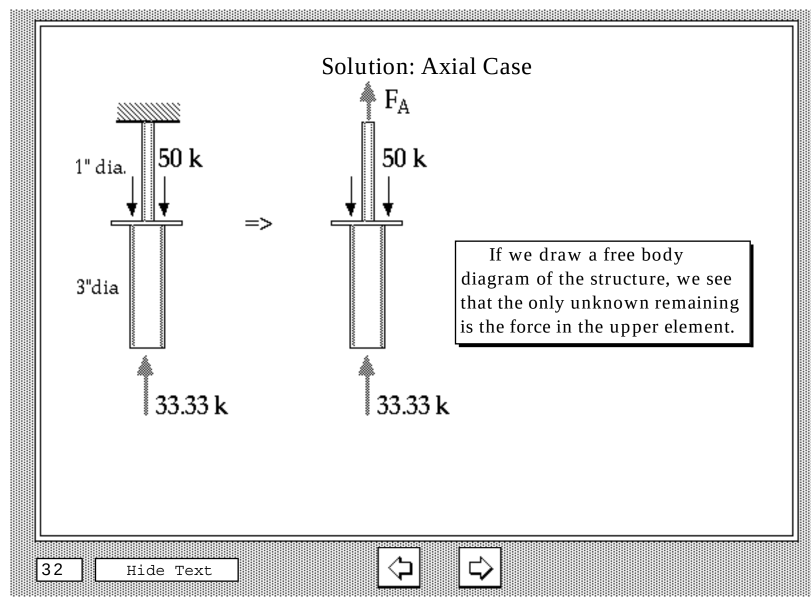
Solution: Axial Case
F_A
1" dia.
50 k
3"dia
33.33 k
=>
50 k
33.33 k
If we draw a free body diagram of the structure, we see that the only unknown remaining is the force in the upper element.
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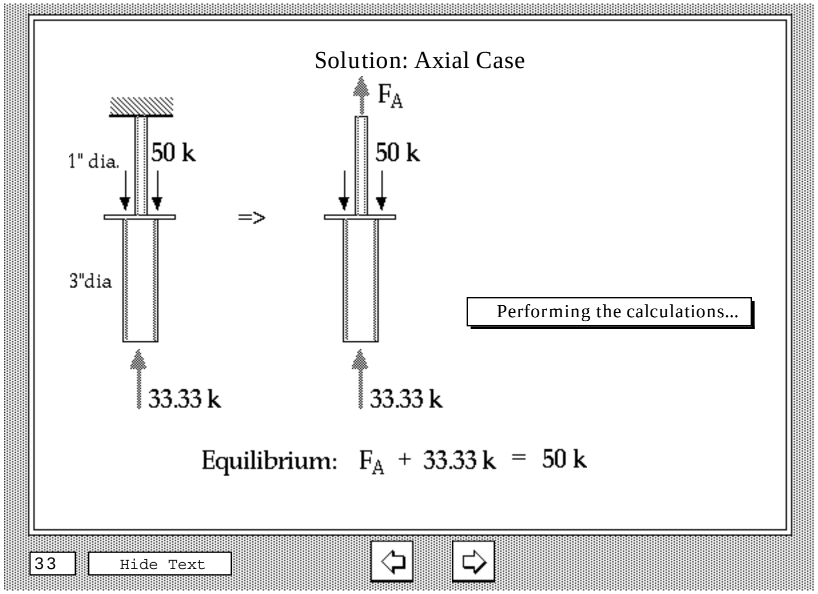
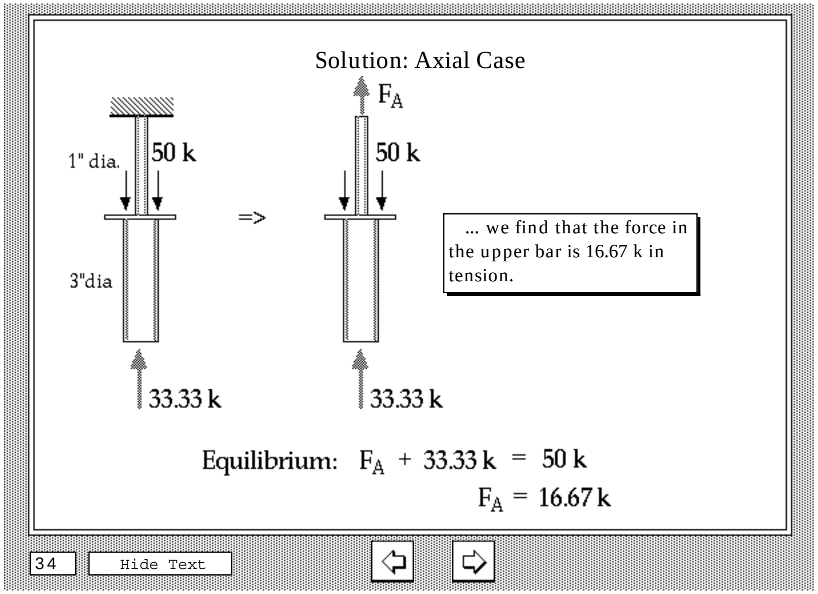

Solution: Axial Case
F_A
1" dia.
50 k
50 k
=>
3"dia
Performing the calculations...
33.33 k
33.33 k
Equilibrium: F_A + 33.33 k = 50 k
33
Hide Text


Solution: Axial Case
F_A
1" dia.
50 k
50 k
=>
... we find that the force in the upper bar is 16.67 k in tension.
3"dia
33.33 k
33.33 k
Equilibrium: F_A + 33.33 k = 50 k
F_A = 16.67 k
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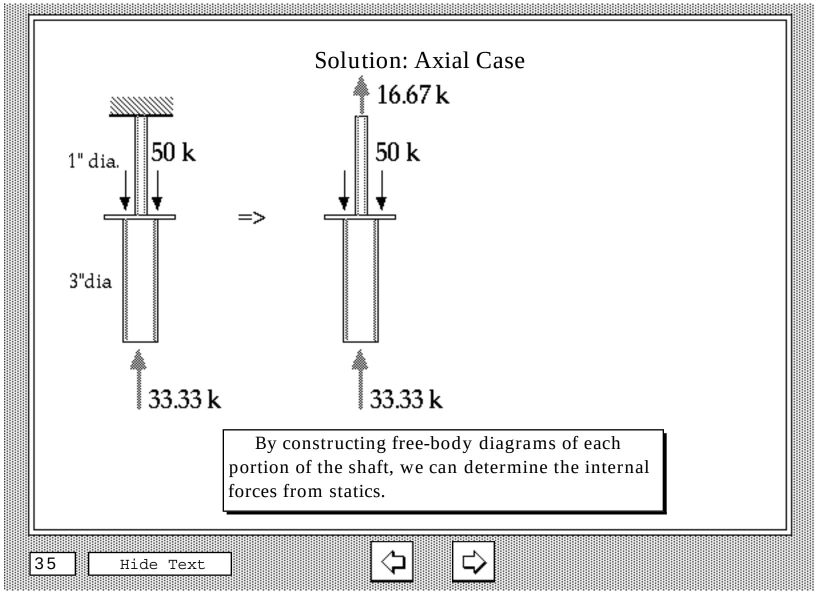
Solution: Axial Case
16.67 k
1" dia.
50 k
50 k
=>
3"dia
33.33 k
33.33 k
By constructing free-body diagrams of each portion of the shaft, we can determine the internal forces from statics.
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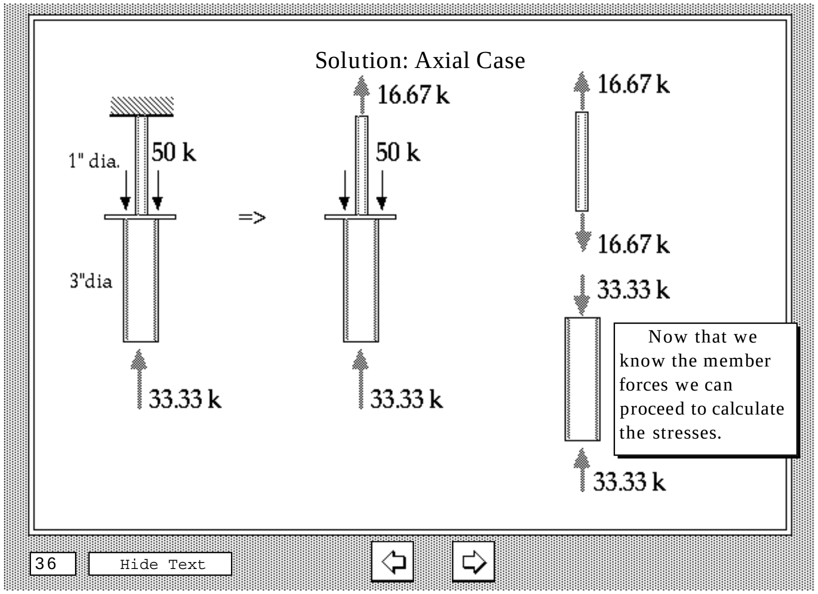
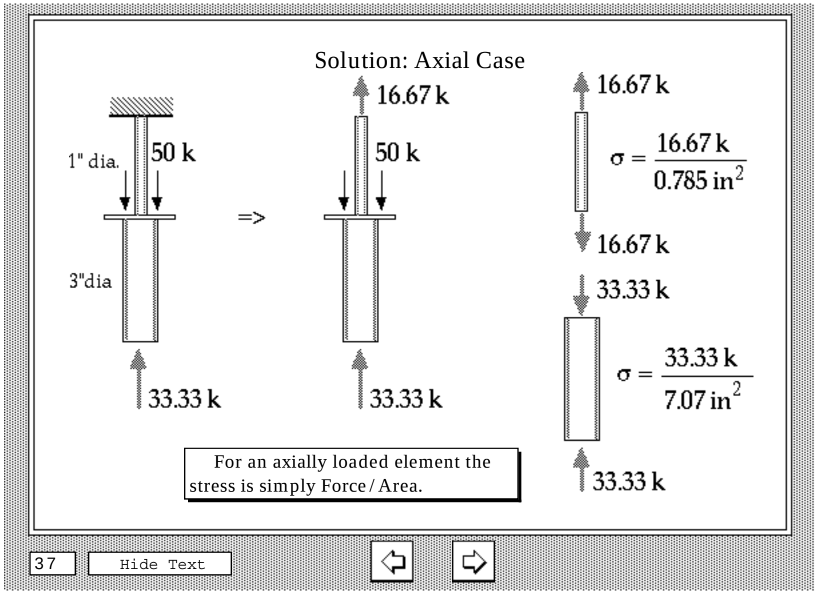

Solution: Axial Case
16.67 k
16.67 k
1" dia.
50 k
50 k
=>
16.67 k
33.33 k
3"dia
Now that we know the member forces we can proceed to calculate the stresses.
33.33 k
33.33 k
33.33 k
36
Hide Text


Solution: Axial Case
16.67 k
50 k
1" dia.
=>
3"dia
33.33 k
σ = 16.67 k / 0.785 in²
33.33 k
σ = 33.33 k / 7.07 in²
For an axially loaded element the stress is simply Force / Area.
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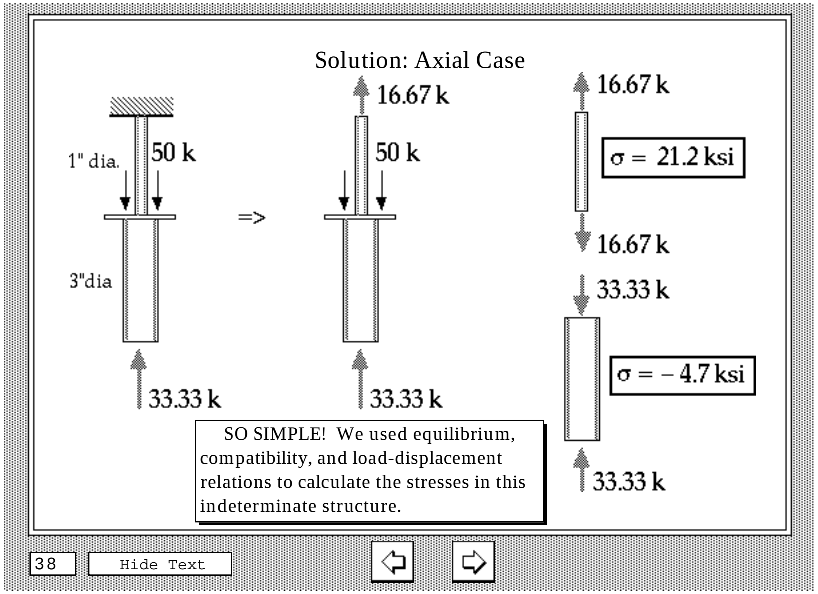
Solution: Axial Case
16.67 k
50 k
1" dia.
=>
3"dia
33.33 k
σ = 21.2 ksi
σ = – 4.7 ksi
SO SIMPLE! We used equilibrium, compatibility, and load-displacement relations to calculate the stresses in this indeterminate structure.
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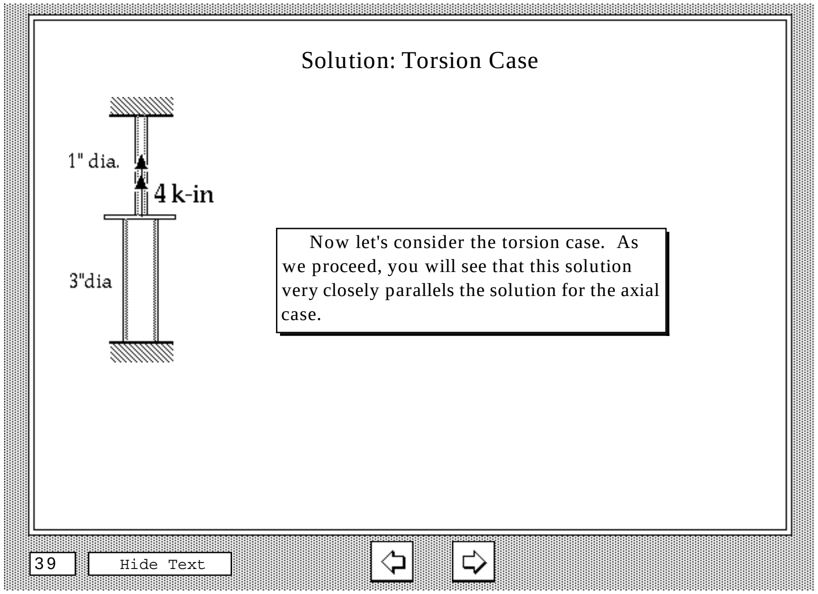
Solution: Torsion Case
1" dia.
4 k-in
3"dia
Now let's consider the torsion case. As we proceed, you will see that this solution very closely parallels the solution for the axial case.
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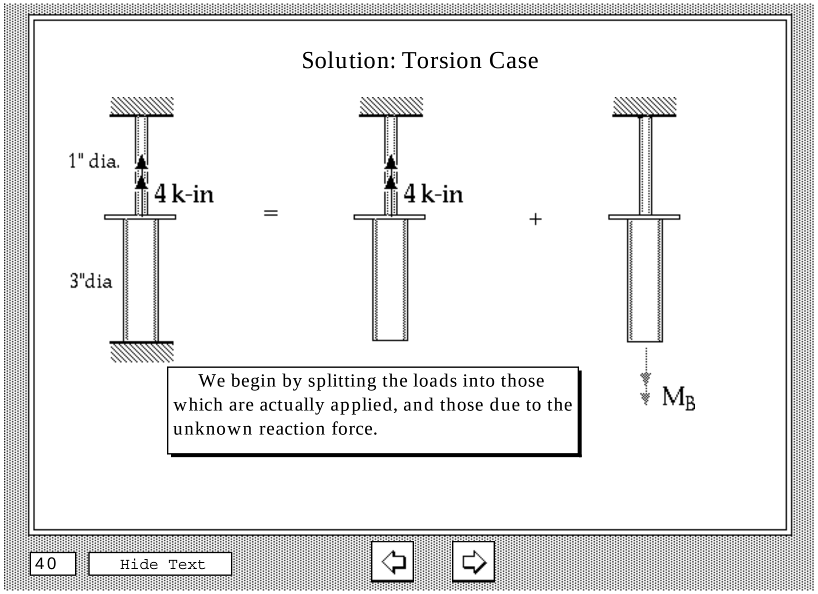
Solution: Torsion Case
1" dia.
4 k-in
=
+
3"dia
M_B
We begin by splitting the loads into those which are actually applied, and those due to the unknown reaction force.
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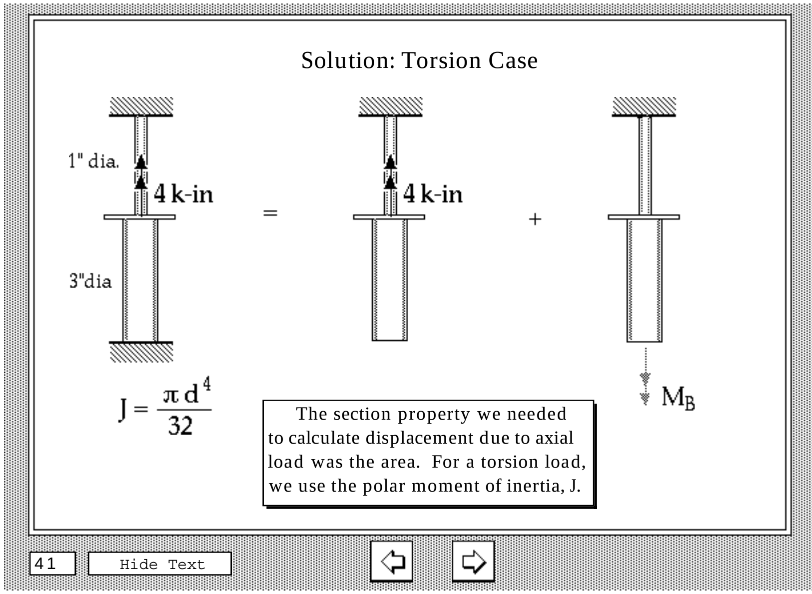
Solution: Torsion Case
1" dia.
4 k-in
=
4 k-in
+
3"dia
J = π d^4 / 32
M_B
The section property we needed to calculate displacement due to axial load was the area. For a torsion load, we use the polar moment of inertia, J.
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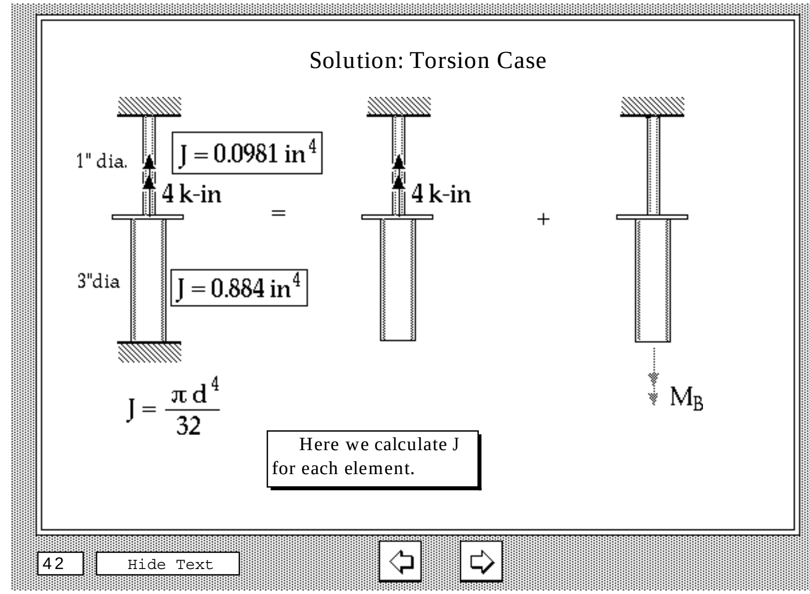
Solution: Torsion Case
1" dia.
J = 0.0981 in^4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in^4
J = π d^4 / 32
M_B
Here we calculate J for each element.
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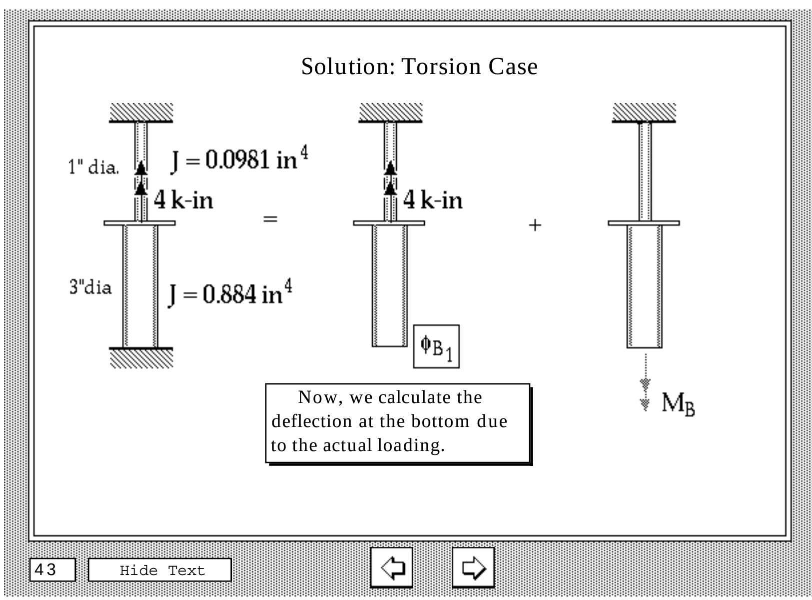
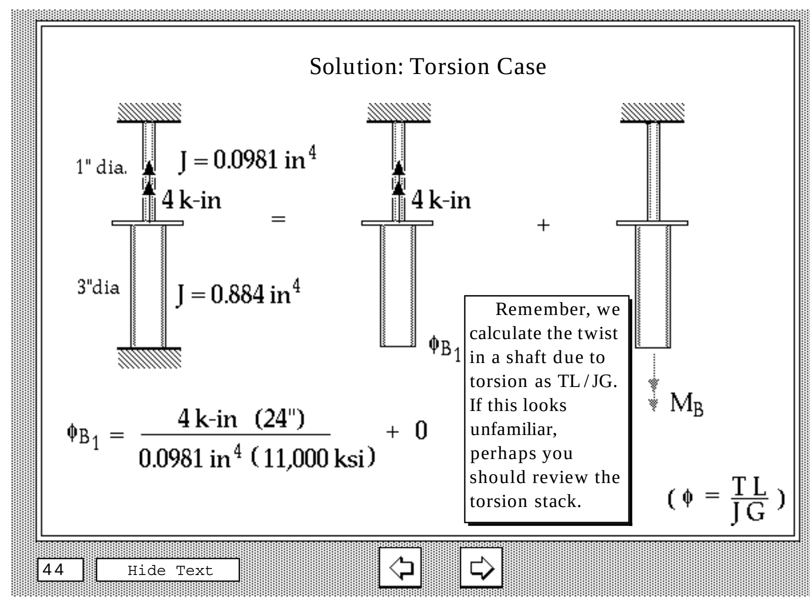
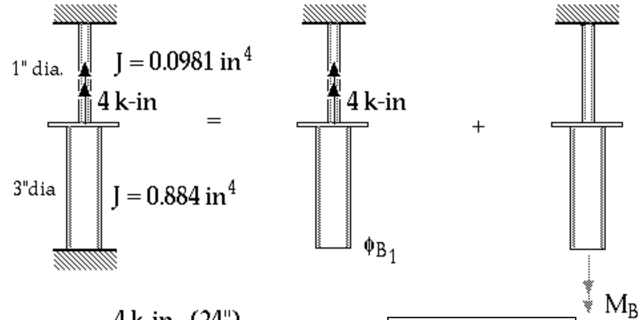

Solution: Torsion Case
1" dia.
J = 0.0981 in^4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in^4
φB1
M_B
Now, we calculate the deflection at the bottom due to the actual loading.
43
Hide Text


Solution: Torsion Case
1" dia.
J = 0.0981 in^4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in^4
φB1
M_B
φB1 = 4 k-in (24") / 0.0981 in^4 (11,000 ksi) + 0
Remember, we calculate the twist in a shaft due to torsion as TL/JG. If this looks unfamiliar, perhaps you should review the torsion stack.
(φ = TL / JG)
44
Hide Text

Solution: Torsion Case



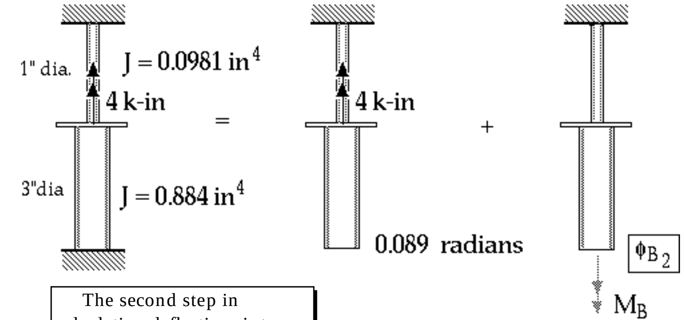
$$\phi_{B_1} = \frac{4\text{ k-in } (24")}{0.0981\text{ in}^4\,(11{,}000\text{ ksi})} + 0 = 0.089\text{ radians}$$

As is usually the case in this course, the math is really simple.

$$\left(\phi = \frac{T\,L}{J\,G}\right)$$

45 Hide Text

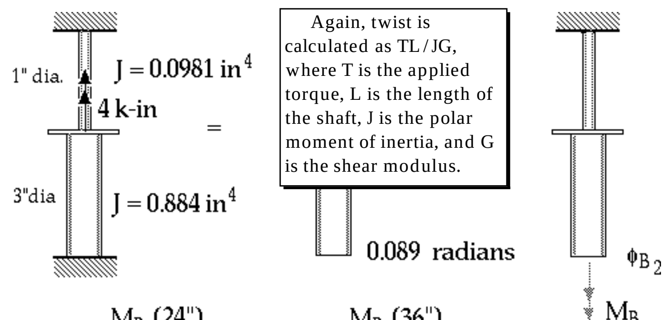
Solution: Torsion Case



The second step in calculating deflections is to find the twist at the bottom of the structure due to the unknown moment, M_B.

46 Hide Text

Solution: Torsion Case

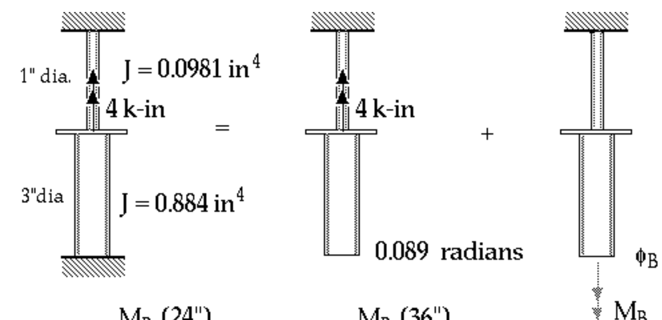


Again, twist is calculated as TL/JG, where T is the applied torque, L is the length of the shaft, J is the polar moment of inertia, and G is the shear modulus.

$$\phi_{B_2} = \frac{M_B\,(24")}{0.0981\text{ in}^4\,(11{,}000\text{ ksi})} + \frac{M_B\,(36")}{0.884\text{ in}^4\,(3{,}900\text{ ksi})}$$

47 Hide Text

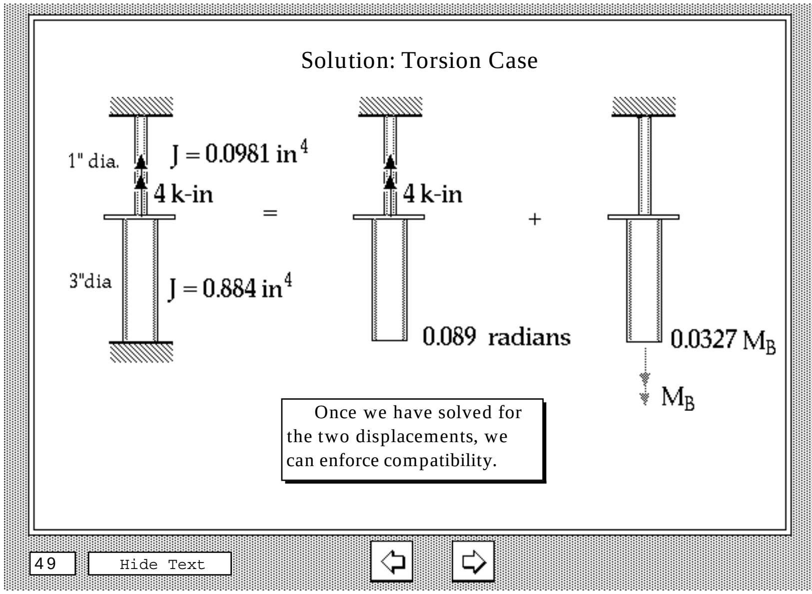
Solution: Torsion Case

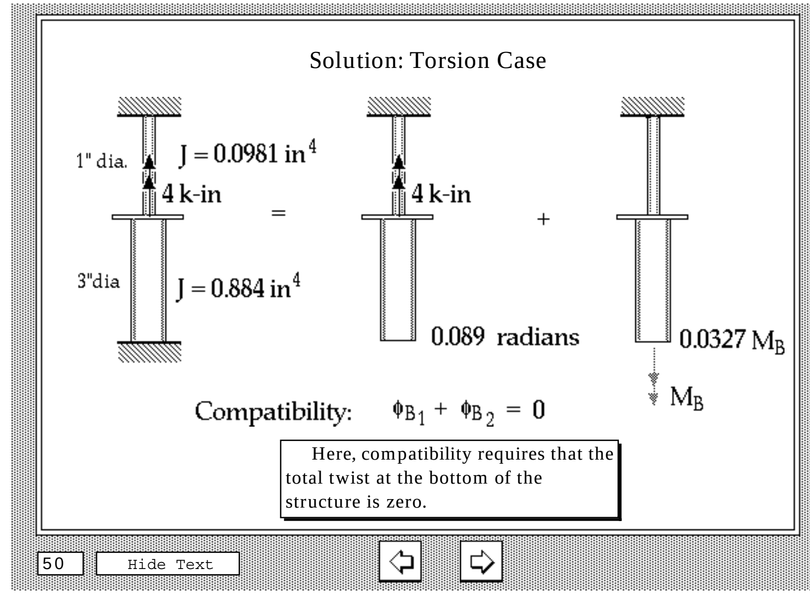


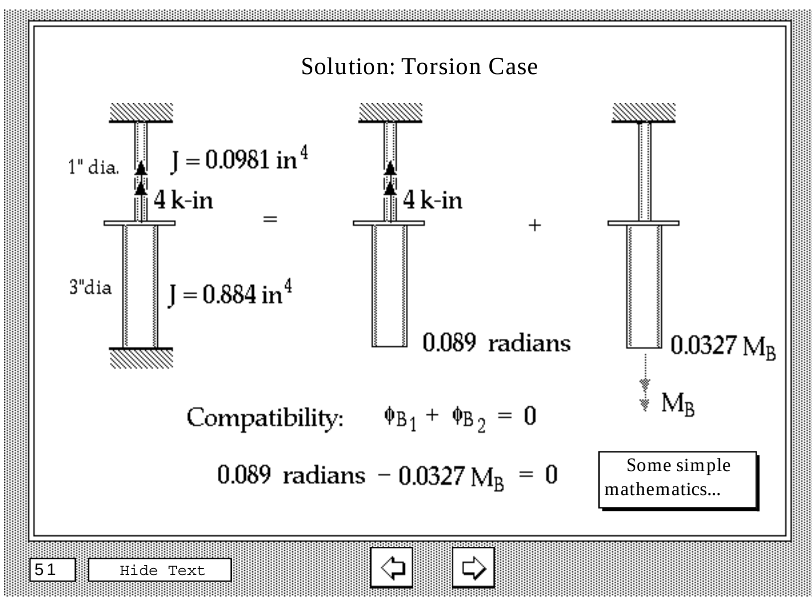
$$\phi_{B_2} = \frac{M_B\,(24")}{0.0981\text{ in}^4\,(11{,}000\text{ ksi})} + \frac{M_B\,(36")}{0.884\text{ in}^4\,(3{,}900\text{ ksi})} = 0.0327\,M_B$$

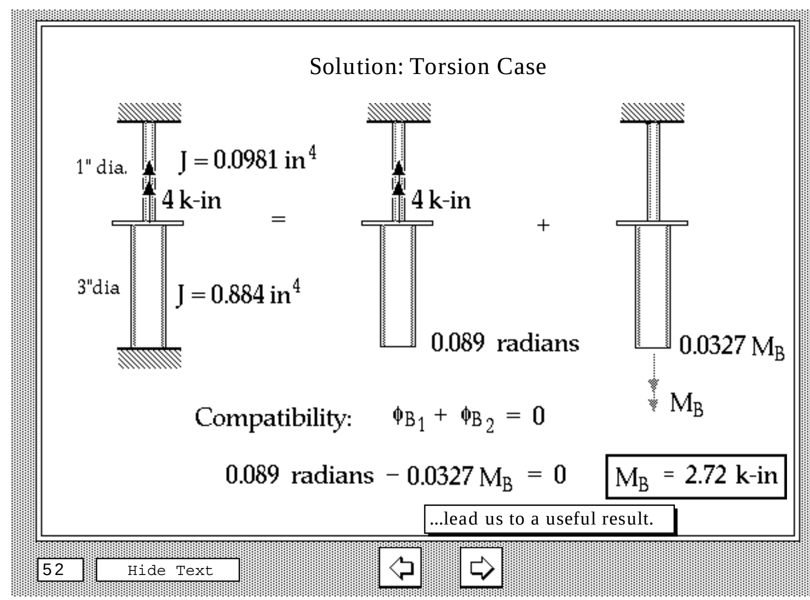
The result of this calculation is the twist at 'B' in terms of the unknown reaction, M_B.

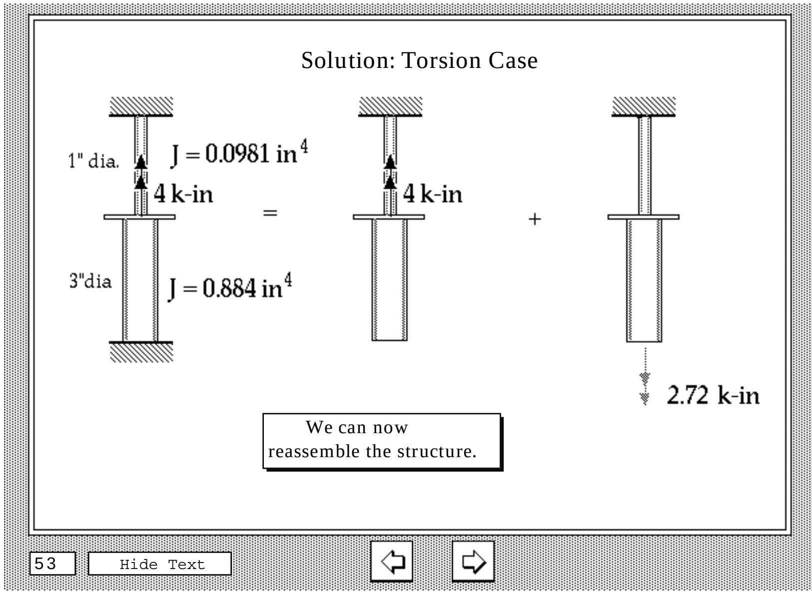
48 Hide Text

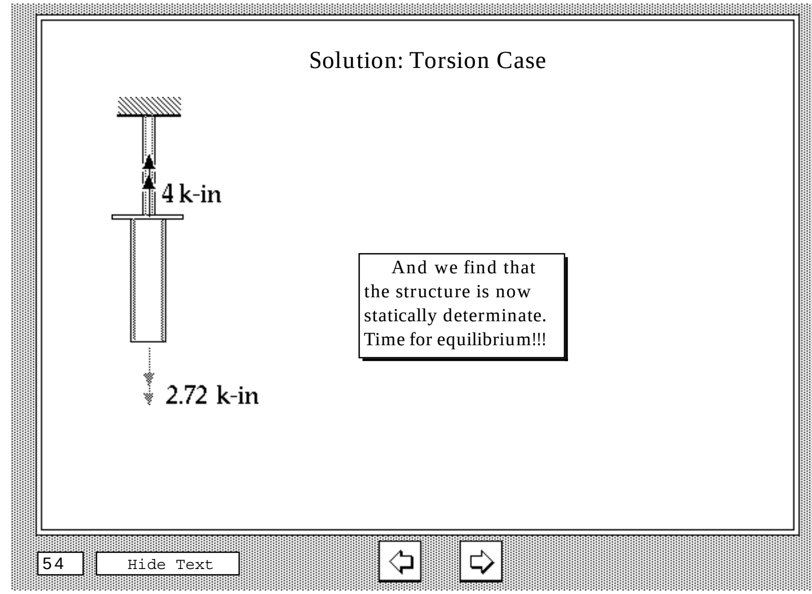

Solution: Torsion Case
1" dia.
J = 0.0981 in4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in4
0.089 radians
0.0327 MB
MB
Once we have solved for the two displacements, we can enforce compatibility.
49
Hide Text

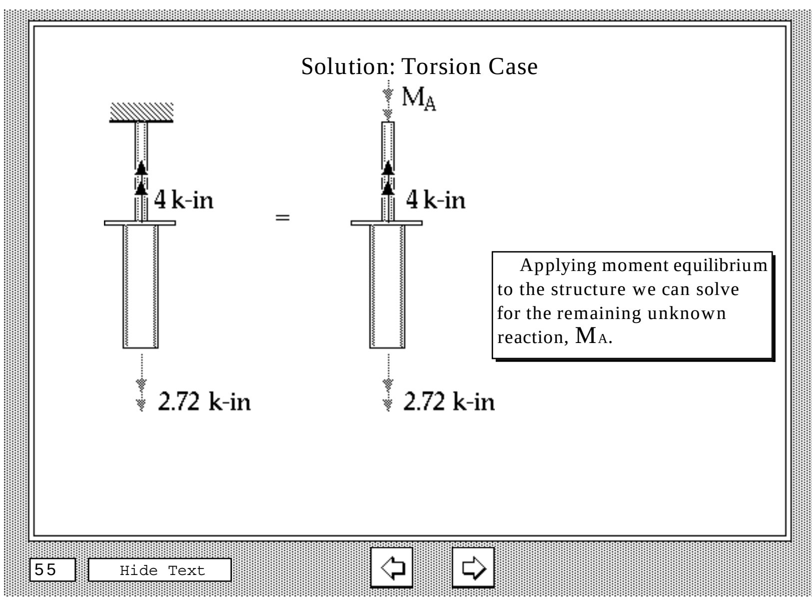

Solution: Torsion Case
1" dia.
J = 0.0981 in4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in4
0.089 radians
0.0327 MB
MB
Compatibility: φB1 + φB2 = 0
Here, compatibility requires that the total twist at the bottom of the structure is zero.
50
Hide Text

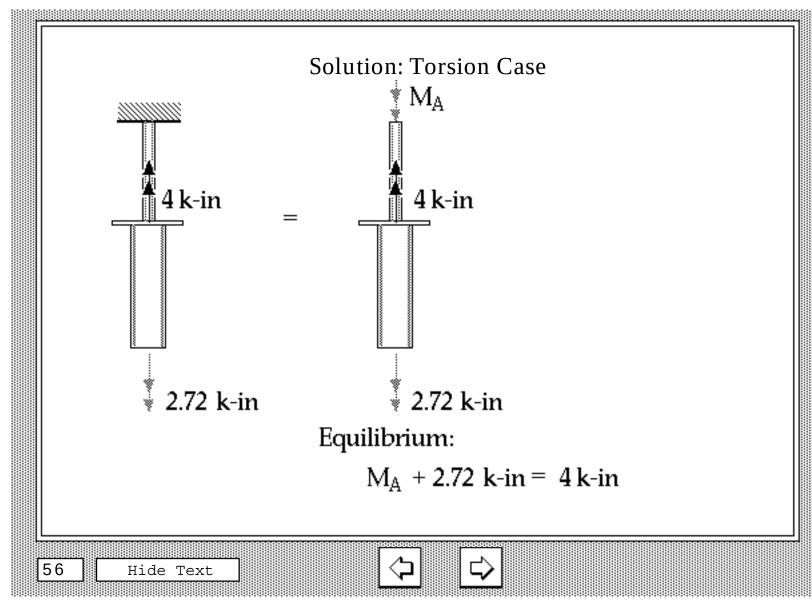

Solution: Torsion Case
1" dia.
J = 0.0981 in4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in4
0.089 radians
0.0327 MB
MB
Compatibility: φB1 + φB2 = 0
0.089 radians − 0.0327 MB = 0
Some simple mathematics...
51
Hide Text

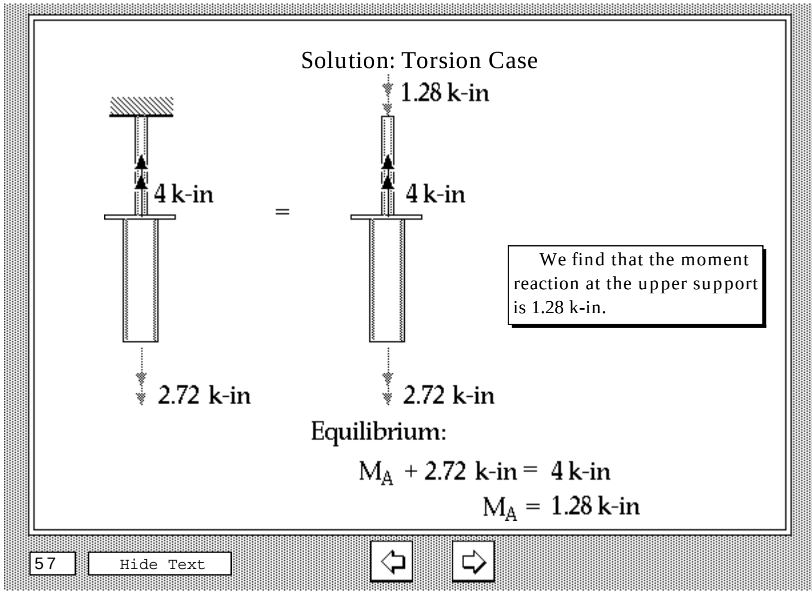

Solution: Torsion Case
1" dia.
J = 0.0981 in4
4 k-in
=
4 k-in
+
3"dia
J = 0.884 in4
0.089 radians
0.0327 MB
MB
Compatibility: φB1 + φB2 = 0
0.089 radians − 0.0327 MB = 0
MB = 2.72 k-in
...lead us to a useful result.
52
Hide Text

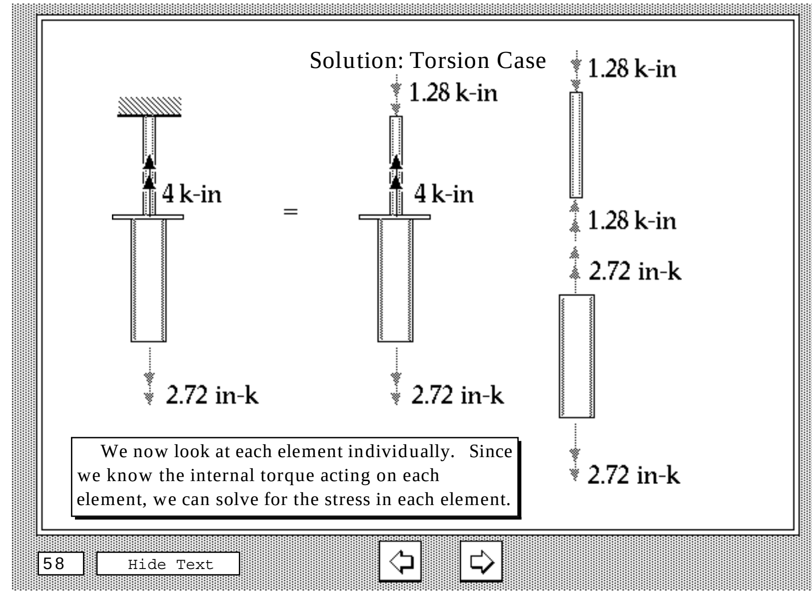

Solution: Torsion Case
1" dia.
J = 0.0981 in4
4 k-in
3"dia
J = 0.884 in4
=
4 k-in
+
2.72 k-in
We can now reassemble the structure.
53
Hide Text

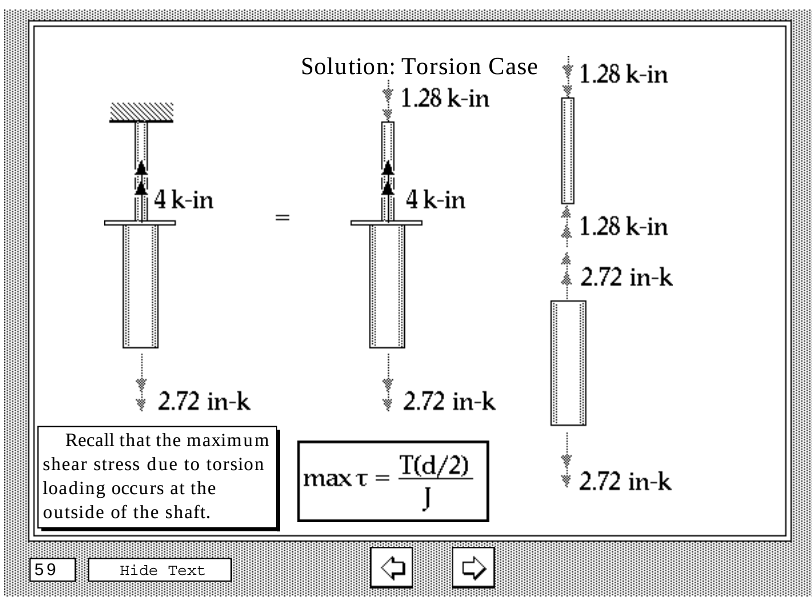

Solution: Torsion Case
4 k-in
And we find that the structure is now statically determinate. Time for equilibrium!!!
2.72 k-in
54
Hide Text

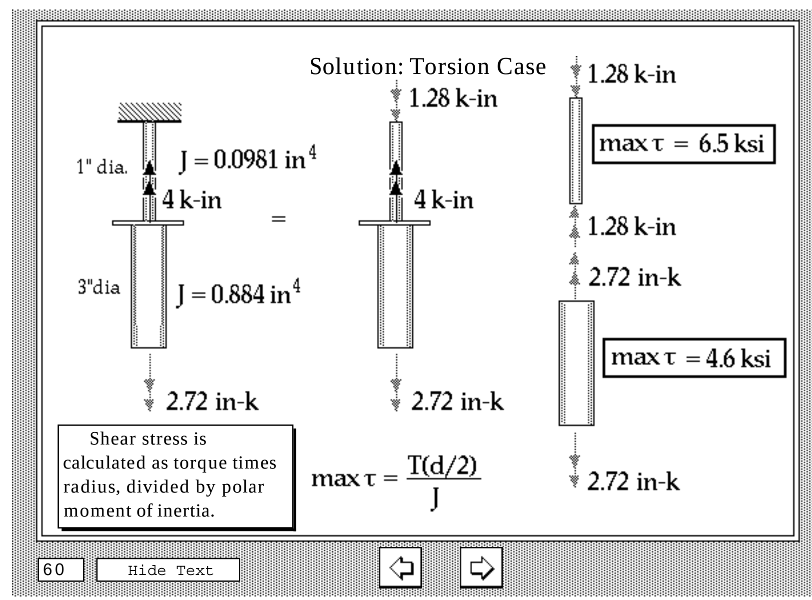

Solution: Torsion Case
MA
4 k-in
=
4 k-in
Applying moment equilibrium to the structure we can solve for the remaining unknown reaction, MA.
2.72 k-in
2.72 k-in
55
Hide Text


Solution: Torsion Case
MA
4 k-in
=
4 k-in
2.72 k-in
2.72 k-in
Equilibrium:
MA + 2.72 k-in = 4 k-in
56
Hide Text

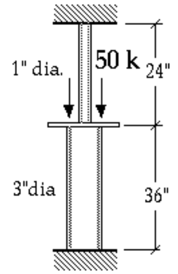

Solution: Torsion Case
1.28 k-in
4 k-in
=
4 k-in
2.72 k-in
2.72 k-in
We find that the moment reaction at the upper support is 1.28 k-in.
Equilibrium:
M_A + 2.72 k-in = 4 k-in
M_A = 1.28 k-in
57
Hide Text


Solution: Torsion Case
1.28 k-in
1.28 k-in
4 k-in
=
4 k-in
1.28 k-in
2.72 in-k
2.72 in-k
2.72 in-k
2.72 in-k
We now look at each element individually. Since we know the internal torque acting on each element, we can solve for the stress in each element.
58
Hide Text


Solution: Torsion Case
1.28 k-in
1.28 k-in
4 k-in
=
4 k-in
1.28 k-in
2.72 in-k
2.72 in-k
2.72 in-k
Recall that the maximum shear stress due to torsion loading occurs at the outside of the shaft.
max τ = T(d/2) / J
2.72 in-k
59
Hide Text


Solution: Torsion Case
1.28 k-in
1.28 k-in
max τ = 6.5 ksi
1" dia.
J = 0.0981 in^4
4 k-in
=
4 k-in
1.28 k-in
2.72 in-k
3"dia
J = 0.884 in^4
max τ = 4.6 ksi
2.72 in-k
2.72 in-k
Shear stress is calculated as torque times radius, divided by polar moment of inertia.
max τ = T(d/2) / J
2.72 in-k
60
Hide Text

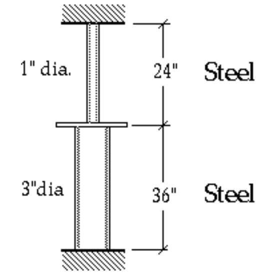
Another Look at the Axial Case



Let's now solve the axial problem a second time, but change the materials used to make the structure and observe the effect.

61 Hide Text

Both Sections made of Steel !

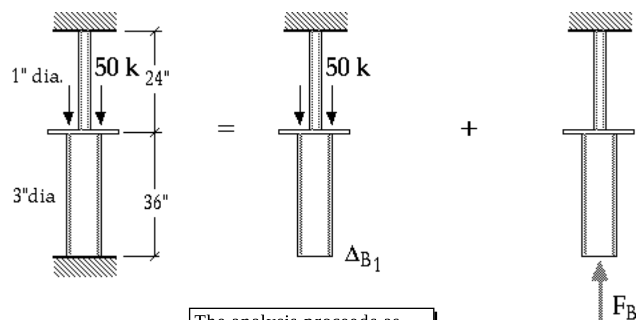


$E_{St} = 30{,}000$ ksi
$G_{St} = 11{,}000$ ksi

This time, both sections are made of steel.

62 Hide Text

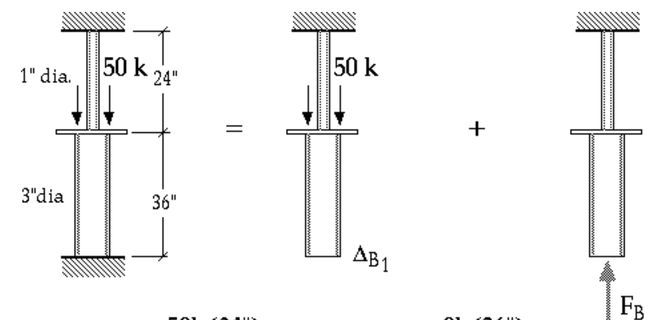
Superposition



The analysis proceeds as before.

63 Hide Text

Deflection for First Loading Case



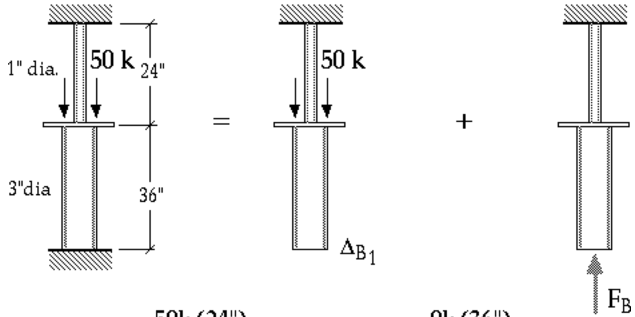
$$\Delta_{B_1} = \frac{50k\,(24'')}{0.785\ in^2\,(30{,}000\ ksi)} + \frac{0k\,(36'')}{7.07\ in^2\,(30{,}000\ ksi)}$$

These equations are as before, except now E is 30,000 throughout.

$$\left(\Delta = \frac{P\,L}{AE}\right)$$

64 Hide Text

Deflection for First Loading Case

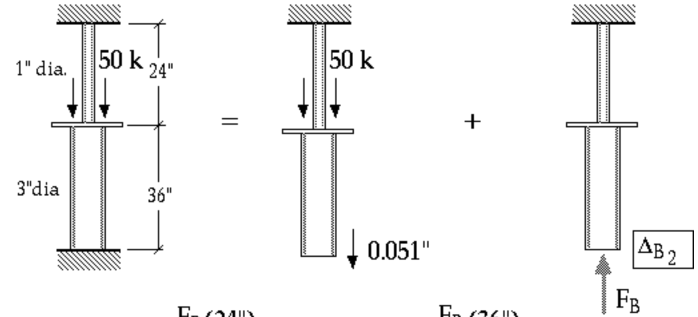


$$\Delta_{B_1} = \frac{50k\,(24")}{0.785\ in^2\,(\,30{,}000\ ksi)} + \frac{0k\,(36")}{7.07\ in^2\,(\,30{,}000\ ksi)}$$

$= 0.051"$ (down)

This deflection must be countered by F_B.

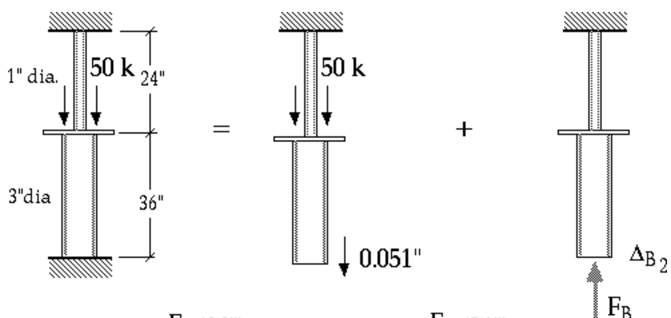
Deflection for Second Loading Case



$$\Delta_{B_2} = \frac{F_B\,(24")}{0.785\ in^2\,(\,30{,}000\ ksi)} + \frac{F_B\,(36")}{7.07\ in^2\,(\,30{,}000\ ksi)}$$

The displacement due to F_B can be calculated as shown.

Deflection for Second Loading Case

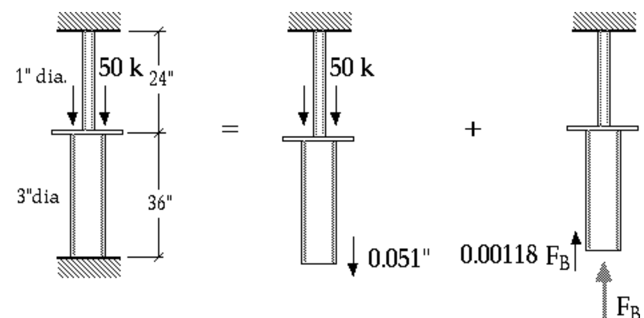


$$\Delta_{B_2} = \frac{F_B\,(24")}{0.785\ in^2\,(\,30{,}000\ ksi)} + \frac{F_B\,(36")}{7.07\ in^2\,(\,30{,}000\ ksi)}$$

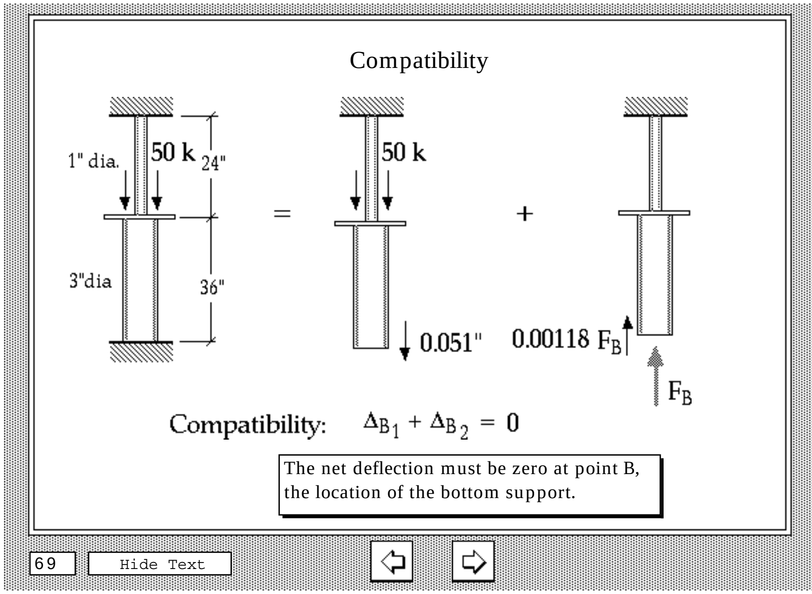
$= 0.00118\ F_B$ (up)

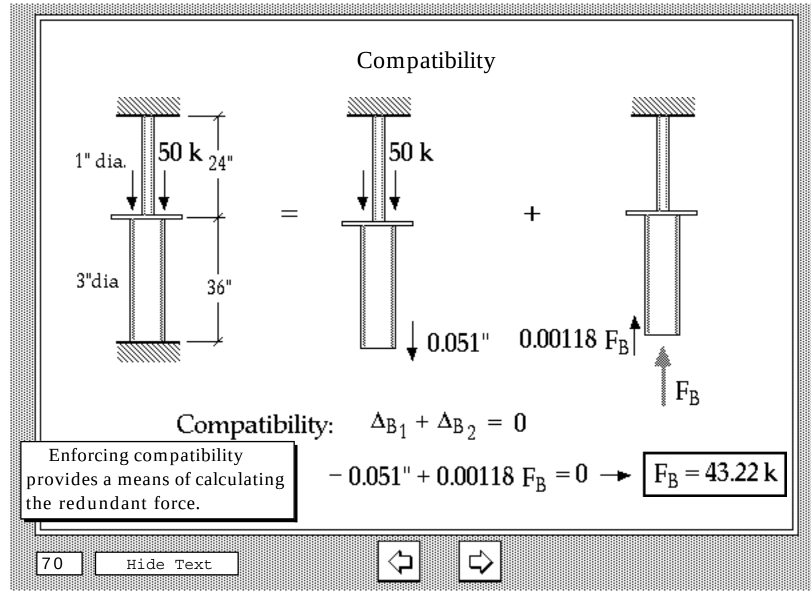
After crunching the numbers.

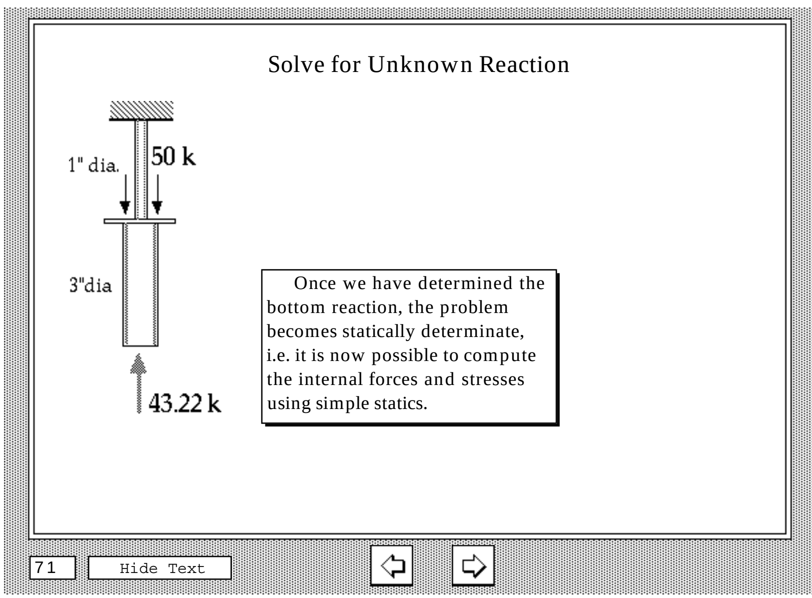
Deflection for Second Loading Case

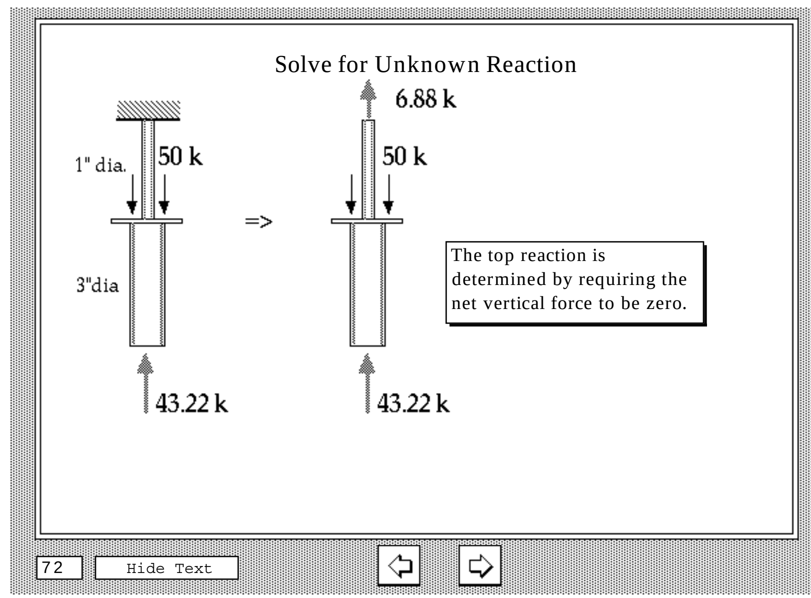


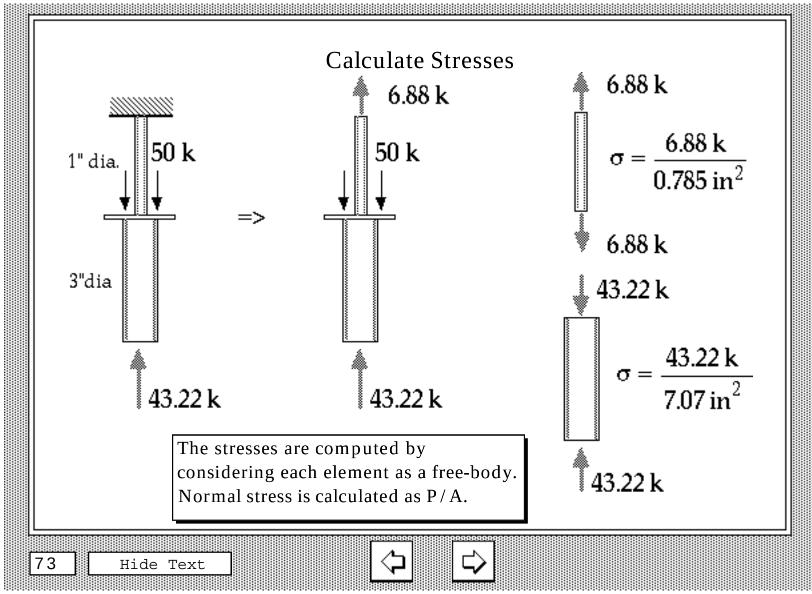
Now we enforce compatibility !!

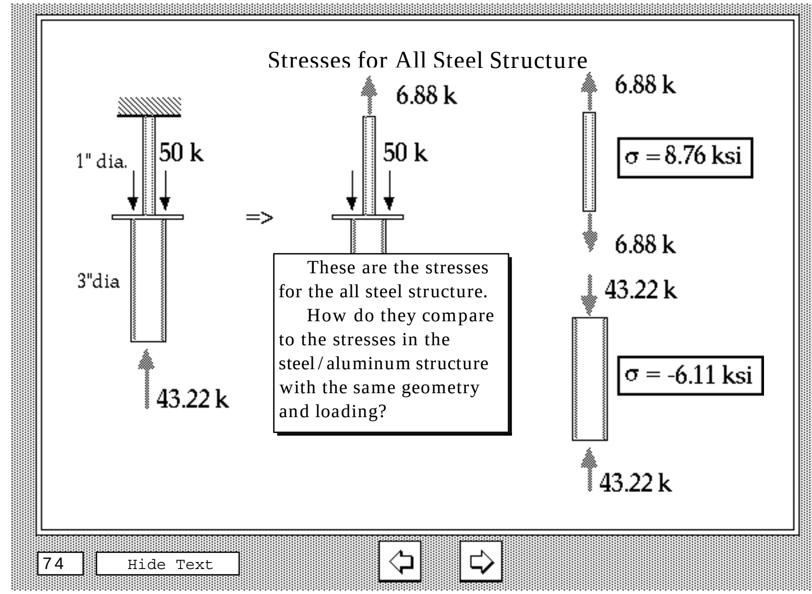

Compatibility
1" dia.
50 k
24"
3"dia
36"
=
50 k
0.051"
+
0.00118 F_B
F_B
Compatibility: Δ_B1 + Δ_B2 = 0
The net deflection must be zero at point B, the location of the bottom support.
69
Hide Text

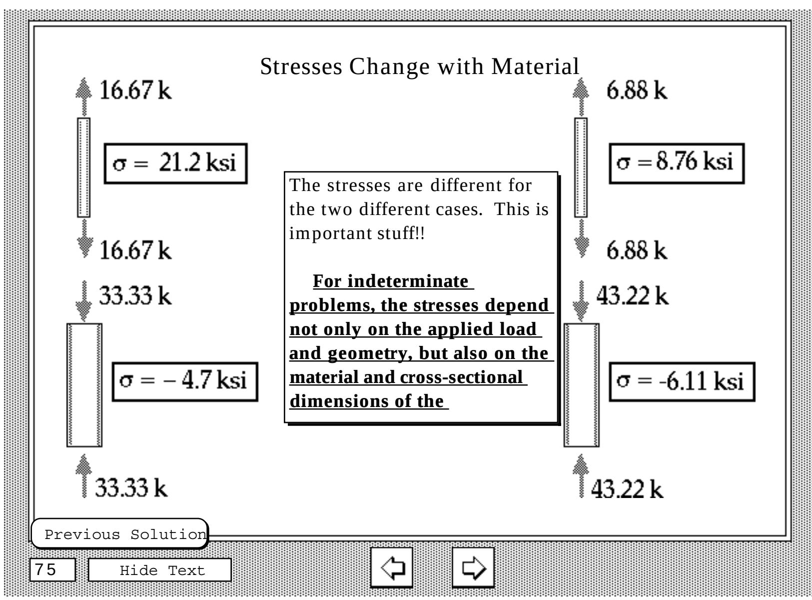

Compatibility
1" dia.
50 k
24"
3"dia
36"
=
50 k
0.051"
+
0.00118 F_B
F_B
Compatibility: Δ_B1 + Δ_B2 = 0
Enforcing compatibility provides a means of calculating the redundant force.
− 0.051" + 0.00118 F_B = 0
F_B = 43.22 k
70
Hide Text

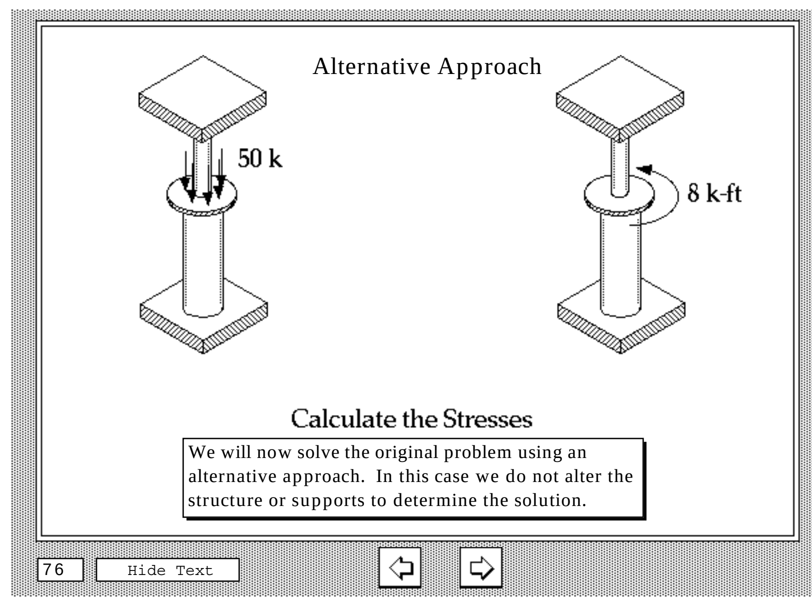

Solve for Unknown Reaction
1" dia.
50 k
3"dia
43.22 k
Once we have determined the bottom reaction, the problem becomes statically determinate, i.e. it is now possible to compute the internal forces and stresses using simple statics.
71
Hide Text


Solve for Unknown Reaction
1" dia.
50 k
3"dia
43.22 k
=>
6.88 k
50 k
43.22 k
The top reaction is determined by requiring the net vertical force to be zero.
72
Hide Text

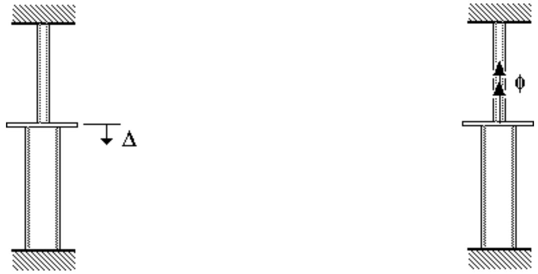

Calculate Stresses
6.88 k
1" dia.
50 k
=>
50 k
3"dia
43.22 k
43.22 k
6.88 k
σ = 6.88 k / 0.785 in²
6.88 k
43.22 k
σ = 43.22 k / 7.07 in²
43.22 k
The stresses are computed by considering each element as a free-body. Normal stress is calculated as P / A.
73
Hide Text


Stresses for All Steel Structure
6.88 k
1" dia.
50 k
=>
50 k
3"dia
43.22 k
These are the stresses for the all steel structure. How do they compare to the stresses in the steel / aluminum structure with the same geometry and loading?
6.88 k
σ = 8.76 ksi
6.88 k
43.22 k
σ = -6.11 ksi
43.22 k
74
Hide Text


Stresses Change with Material
16.67 k
σ = 21.2 ksi
16.67 k
33.33 k
σ = – 4.7 ksi
33.33 k
The stresses are different for the two different cases. This is important stuff!!
For indeterminate problems, the stresses depend not only on the applied load and geometry, but also on the material and cross-sectional dimensions of the
6.88 k
σ = 8.76 ksi
6.88 k
43.22 k
σ = -6.11 ksi
43.22 k
Previous Solution
75
Hide Text


Alternative Approach
50 k
8 k-ft
Calculate the Stresses
We will now solve the original problem using an alternative approach. In this case we do not alter the structure or supports to determine the solution.
76
Hide Text

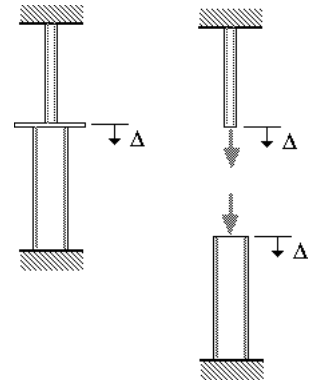
Consider Displacements



The previous method of solution is called the **force method**, because our unknown was the force or moment at a redundant support. This time we will use a displacement as our unknown, and so this approach is called a **displacement method**.

For brevity, we will focus on the axial problem only; the torsion solution would proceed in an entirely similar fashion.

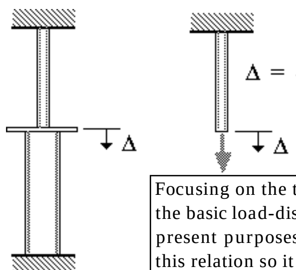
Calculate Internal Forces



Our basic strategy is to assume a displacement of Δ as shown, and then calculate the resulting force in each piece or component of the structure.

We implicitly assume that both the top and bottom pieces are subjected to the same Δ, so we are using **compatibility** right away rather than saving it for the end.

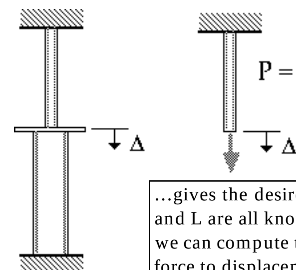
Calculate Internal Forces



$$\Delta = \frac{P\,L}{AE}$$

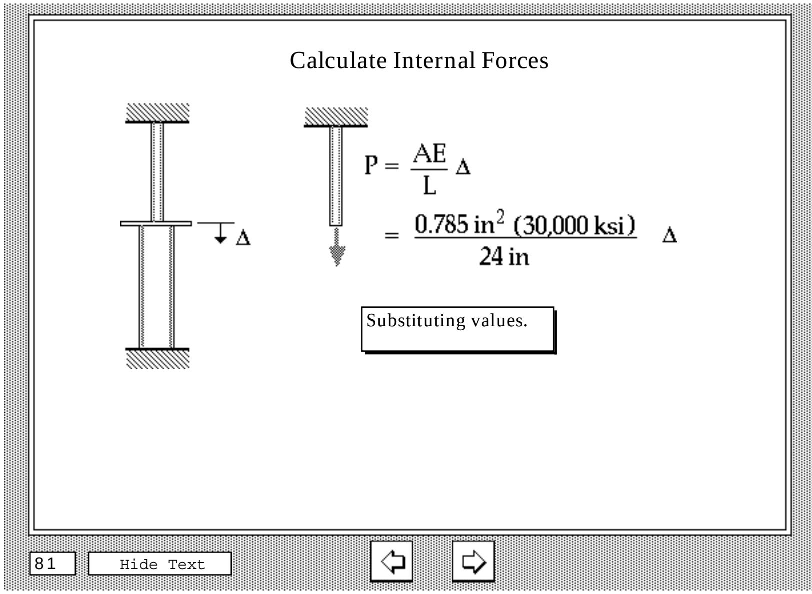
Focusing on the top piece first, we recall the basic load-displacement relation. For present purposes, we need to rearrange this relation so it gives force in terms of displacement. Solving for P…

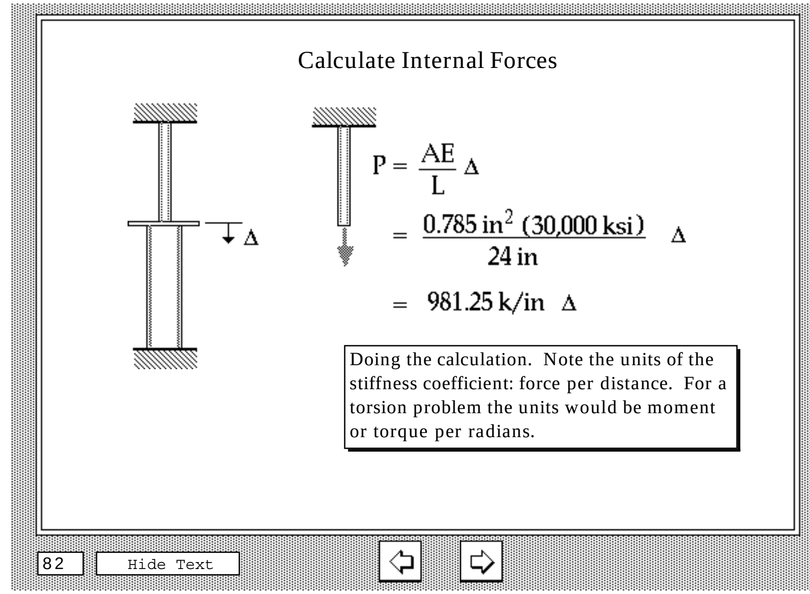
Calculate Internal Forces

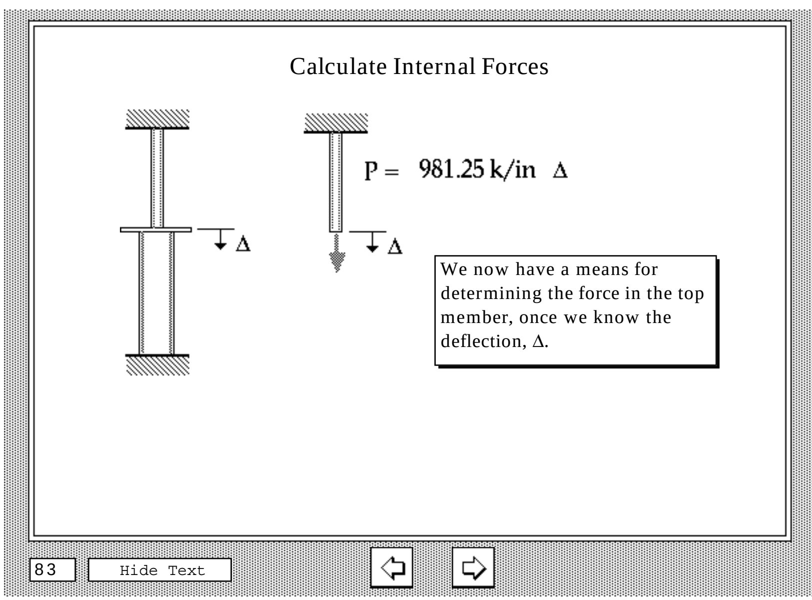


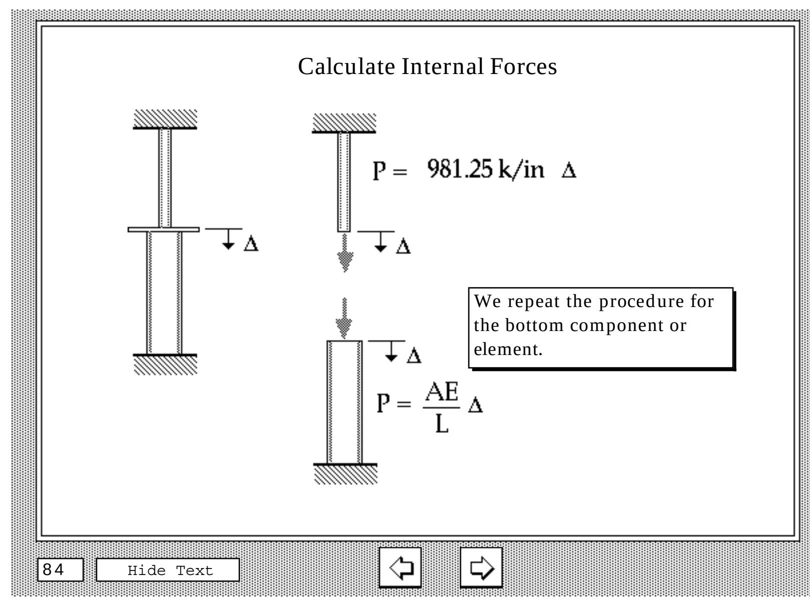
$$P = \frac{AE}{L}\,\Delta$$

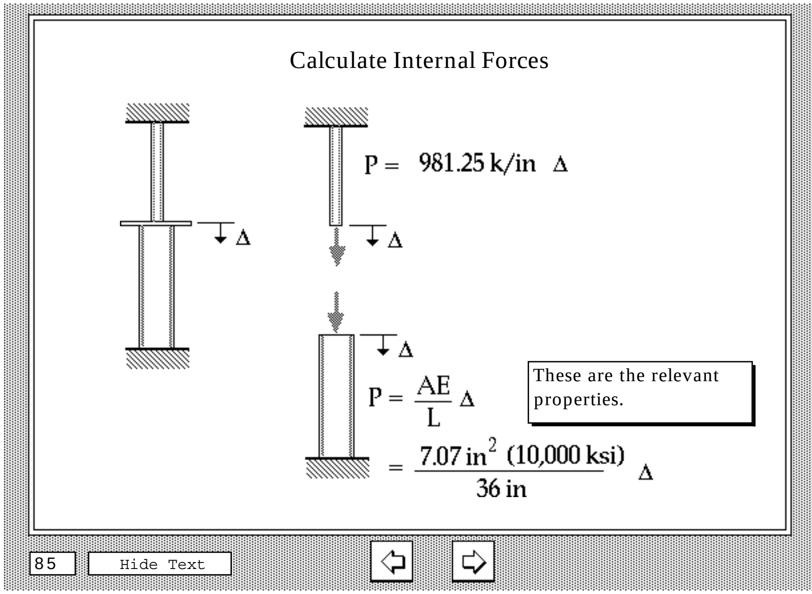
…gives the desired result. Since A, E and L are all known element properties, we can compute the coefficient relating force to displacement. This coefficient is called the **stiffness** of the element.

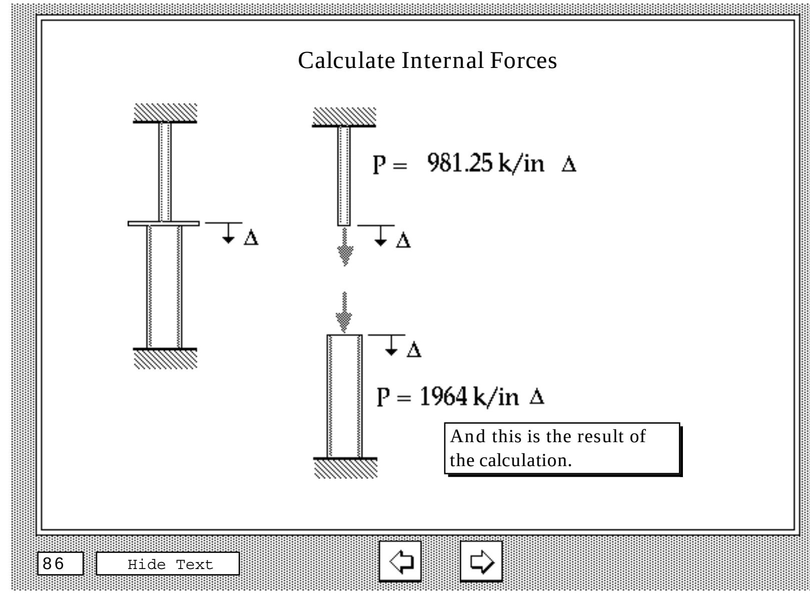

Calculate Internal Forces
Δ
P = AE/L Δ
= 0.785 in² (30,000 ksi) / 24 in Δ
Substituting values.
81
Hide Text


Calculate Internal Forces
Δ
P = AE/L Δ
= 0.785 in² (30,000 ksi) / 24 in Δ
= 981.25 k/in Δ
Doing the calculation. Note the units of the stiffness coefficient: force per distance. For a torsion problem the units would be moment or torque per radians.
82
Hide Text


Calculate Internal Forces
Δ
P = 981.25 k/in Δ
Δ
We now have a means for determining the force in the top member, once we know the deflection, Δ.
83
Hide Text


Calculate Internal Forces
Δ
P = 981.25 k/in Δ
Δ
Δ
P = AE/L Δ
We repeat the procedure for the bottom component or element.
84
Hide Text


Calculate Internal Forces
P = 981.25 k/in Δ
Δ
Δ
Δ
P = AE/L Δ
= 7.07 in^2 (10,000 ksi) / 36 in Δ
These are the relevant properties.
85
Hide Text


Calculate Internal Forces
P = 981.25 k/in Δ
Δ
Δ
Δ
P = 1964 k/in Δ
And this is the result of the calculation.
86
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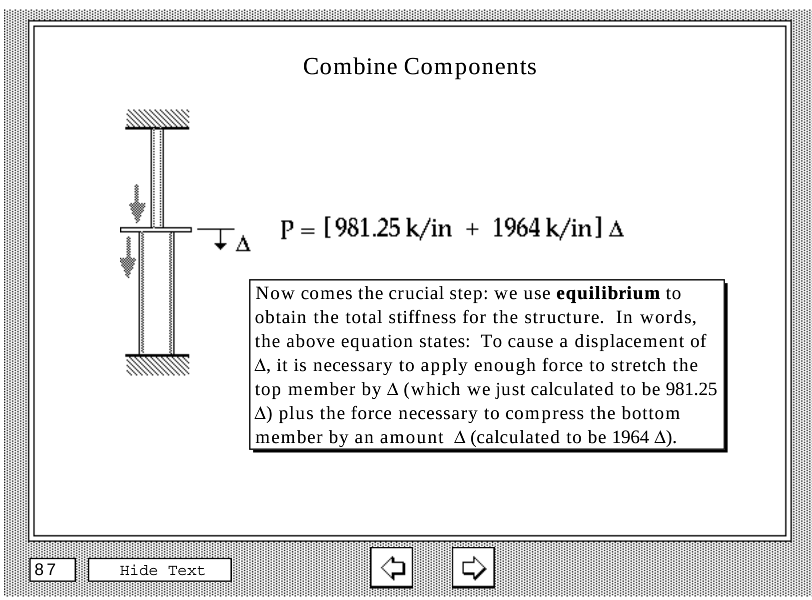
Combine Components
Δ
P = [981.25 k/in + 1964 k/in] Δ
Now comes the crucial step: we use **equilibrium** to obtain the total stiffness for the structure. In words, the above equation states: To cause a displacement of Δ, it is necessary to apply enough force to stretch the top member by Δ (which we just calculated to be 981.25 Δ) plus the force necessary to compress the bottom member by an amount Δ (calculated to be 1964 Δ).
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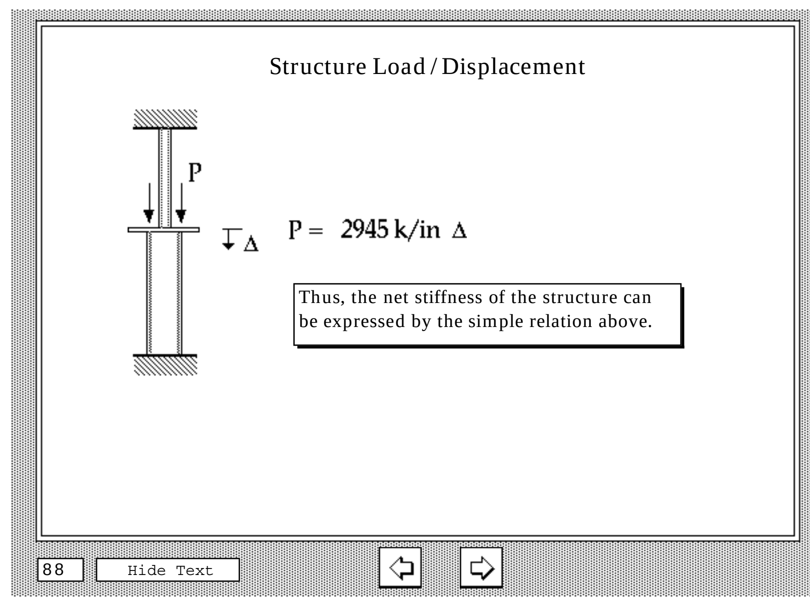
Structure Load / Displacement
P
Δ
P = 2945 k/in Δ
Thus, the net stiffness of the structure can be expressed by the simple relation above.
88
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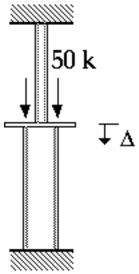
Given P



$$P = 2945 \text{ k/in } \Delta$$

Of course, we do not know Δ, we know P. You can see that this is no problem, however, since our relation can be solved easily for Δ in terms of

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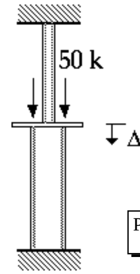
Solve for Δ



$$P = 2945 \text{ k/in } \Delta$$

$$50 \text{ k} = 2945 \text{ k/in } \Delta$$

Putting in the given load, P = 50k, ...

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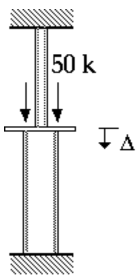
Solve for Δ



$$P = 2945 \text{ k/in } \Delta$$

$$50 \text{ k} = 2945 \text{ k/in } \Delta$$

$$\Delta = \frac{50 \text{ k}}{2945 \text{ k/in}}$$

and solving for Δ ...

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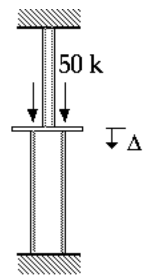
Solve for Δ



$$P = 2945 \text{ k/in } \Delta$$

$$50 \text{ k} = 2945 \text{ k/in } \Delta$$

$$\Delta = \frac{50 \text{ k}}{2945 \text{ k/in}} = \boxed{0.017 \text{ in}}$$

...gives this result. The hard part of the solution is now over. Since we already expressed the member forces in terms of Δ, we need only do a few simple substitutions to calculate the stresses.

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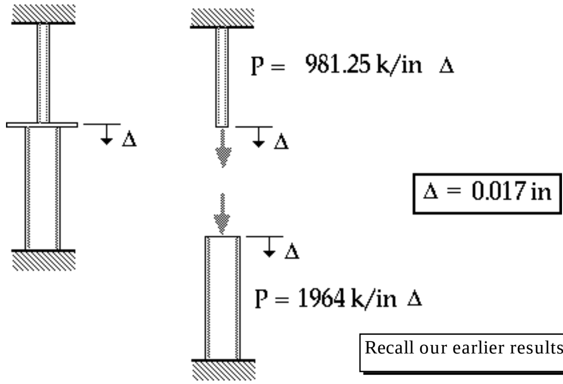
Calculate Internal Forces



Recall our earlier results.

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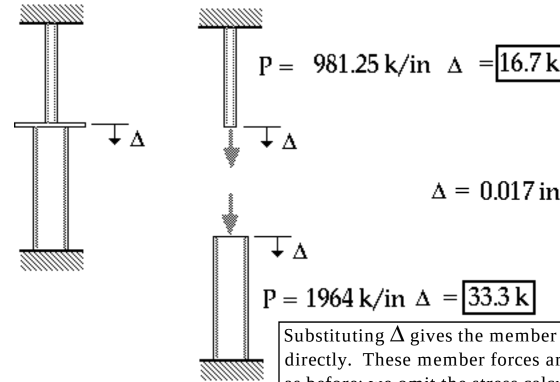
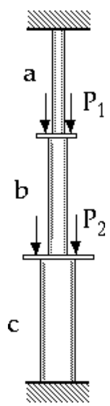
Calculate Internal Forces



Substituting Δ gives the member forces directly. These member forces are the same as before; we omit the stress calculation since it is no different from the previous solution.

94 Hide Text Previous Solution

More General Case



The displacement method is a general and powerful means for solving all kinds of stress analysis problems. The basic process of breaking a structure into pieces, calculating the force contribution of each piece due to a set of specified displacements, and then assembling these various force contributions can be computerized quite elegantly. In a computerized context, the approach is called the **Finite Element Method**, since a given body is analyzed in terms of a bunch of finite-sized pieces or elements.

To give a flavor of how this approach generalizes, consider the structure shown to the left. To solve this problem using the original force method, the procedure is identical to the previous case. To use the displacement method, however, there are a few additional nuances.

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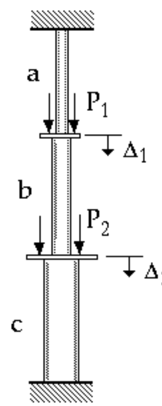
Assume Displacements



Instead of identifying a single unknown displacement, it is necessary to work with two unknown displacements. The more complex the structure, the more unknown displacements need to be included.

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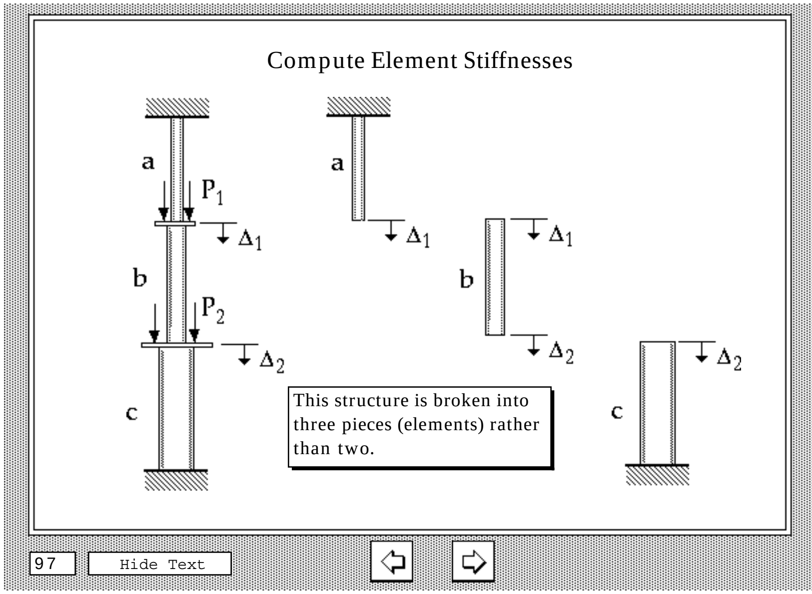
Compute Element Stiffnesses
a
P1
Δ1
b
P2
Δ2
c
This structure is broken into three pieces (elements) rather than two.
97
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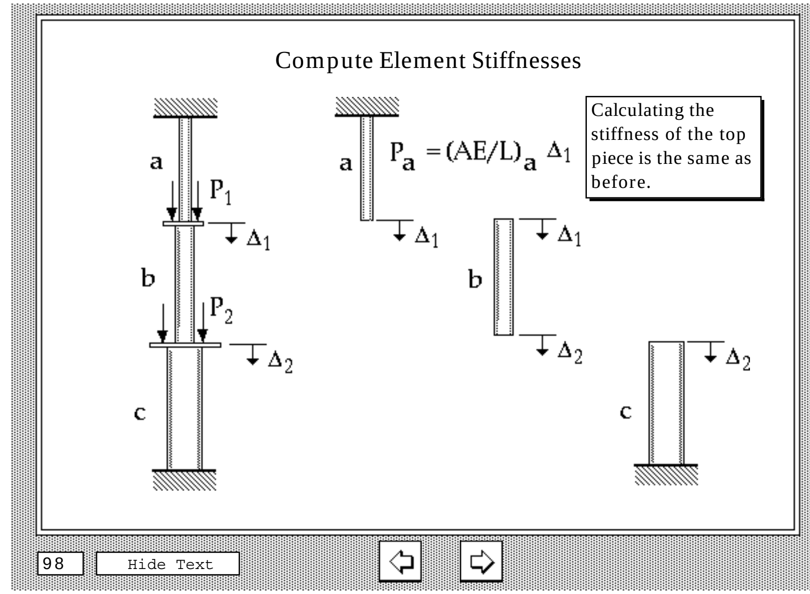
Compute Element Stiffnesses
Calculating the stiffness of the top piece is the same as before.
$P_a = (AE/L)_a \Delta_1$
a
P1
Δ1
b
P2
Δ2
c
98
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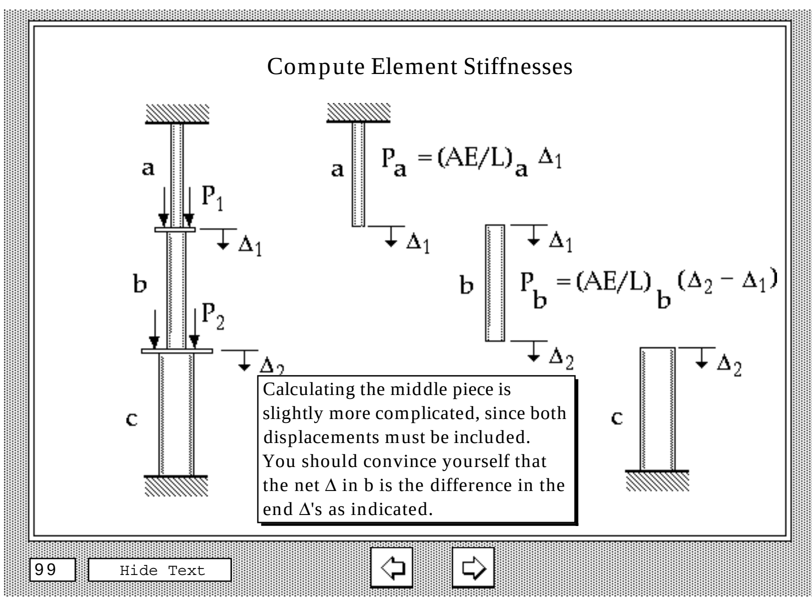
Compute Element Stiffnesses
$P_a = (AE/L)_a \Delta_1$
$P_b = (AE/L)_b (\Delta_2 - \Delta_1)$
Calculating the middle piece is slightly more complicated, since both displacements must be included. You should convince yourself that the net Δ in b is the difference in the end Δ's as indicated.
99
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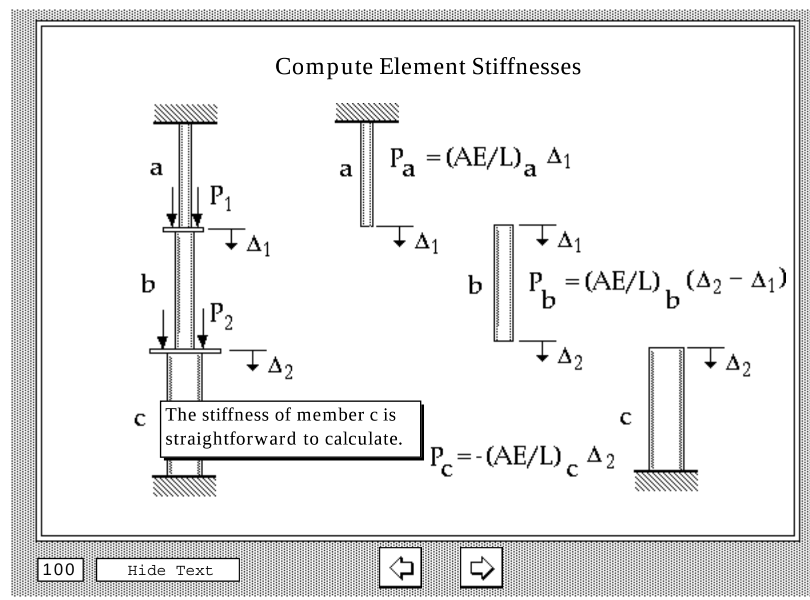
Compute Element Stiffnesses
$P_a = (AE/L)_a \Delta_1$
$P_b = (AE/L)_b (\Delta_2 - \Delta_1)$
The stiffness of member c is straightforward to calculate.
$P_c = -(AE/L)_c \Delta_2$
100
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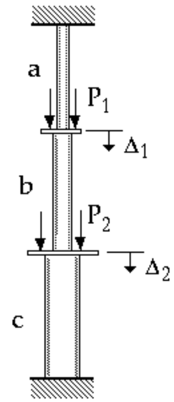
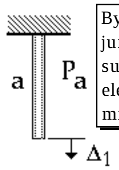
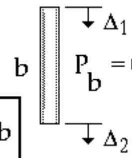
Equilibrium





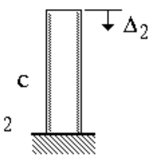
By equilibrium, the net force at each junction of the structure must be the sum of the forces in each connecting element. Can you see why P_b has a minus sign in the P_1 equati

Why?



$$P_b = (AE/L)_b\,(\Delta_2 - \Delta_1)$$

$$P_1 = P_a - P_b$$
$$P_2 = P_c + P_b$$



$$P_c = -(AE/L)_c\,\Delta_2$$

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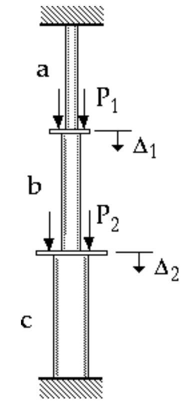
Combine Results



$$P_a = (AE/L)_a\,\Delta_1$$

These various results need to be assembled.

$$P_b = (AE/L)_b\,(\Delta_2 - \Delta_1)$$

$$P_1 = P_a - P_b$$
$$P_2 = P_c + P_b$$

$$P_c = -(AE/L)_c\,\Delta_2$$

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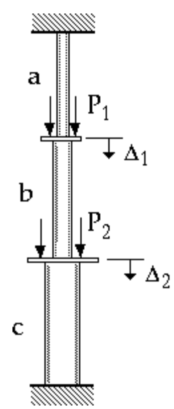
Combine Results



$$P_1 = (AE/L)_a\,\Delta_1 - (AE/L)_b\,(\Delta_2 - \Delta_1)$$
$$P_2 = (AE/L)_c\,\Delta_2 + (AE/L)_b\,(\Delta_2 - \Delta_1)$$

Direct substitution leads to these equations, which can be rearranged…

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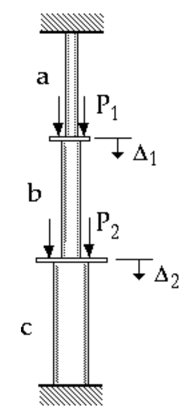
Two Equations: Two Unknowns



$$P_1 = [(AE/L)_a + (AE/L)_b]\,\Delta_1 - (AE/L)_b\,\Delta_2$$
$$P_2 = -(AE/L)_b\,\Delta_1 + [(AE/L)_c + (AE/L)_b]\,\Delta_2$$

…to this form, which can be recognized as a simple system of two equations with two unknowns (remember that we assume we know the loads and the member properties, but the displacements are unknown).

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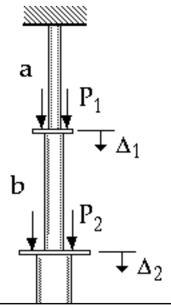
Expressed in Matrix Format



$$P_1 = [(AE/L)_a + (AE/L)_b]\,\Delta_1 - (AE/L)_b\,\Delta_2$$

$$P_2 = -(AE/L)_b\,\Delta_1 + [(AE/L)_c + (AE/L)_b]\,\Delta_2$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

These equations can be written in matrix form as shown. The matrix with the stiffness coefficients, kij, is called a stiffness matrix. This matrix characterizes the elastic behavior of the structure.

To solve a particular problem, it is necessary to invert the stiffness matrix. This can be accomplished easily by a computer, even for very large matrices.

Summary

Do not worry if you do not feel that you have thoroughly mastered all the concepts and methods covered in this stack. In subsequent courses you will revisit these topics, and have a chance to explore them much more fully. You will also learn systematic methods for handling very complex systems. Nevertheless, the two basic notions of adjusting redundant forces to satisfy compatibility and adjusting displacements to satisfy equilibrium will be at the heart of these more advanced treatments.

For the simple problems presented here, note the inherent similarity between the torsion and axial problems, and the nearly identical solution process used to generate internal stresses.

The End