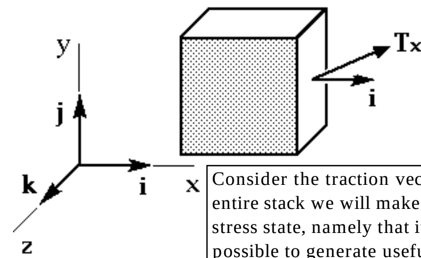
STRESS IV

- Principal Stresses
- Maximum Shear Stress
- Mohr's Circle

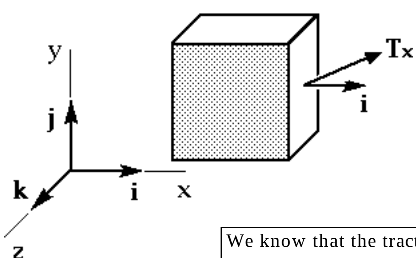
In this stack we will investigate additional features of stress. To go straight to any of the topics listed above, click on the corresponding text. Choose "Stack Contents" from the "Navigation" menu to return to this page at any time.

• Principal Stresses



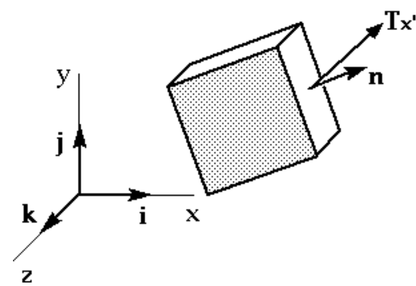
Consider the traction vector on the x-face as shown. For this entire stack we will make an important limitation on our stress state, namely that it is **2-Dimensional**. (This makes it possible to generate useful results without relying on results from linear algebra, which not all students have taken.) The traction vector shown lies in the x-y plane, and we will change the orientation of the block by rotating about the z-axis only.

• Principal Stresses

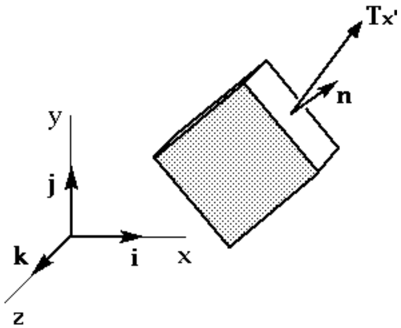


We know that the traction vector on a given surface depends on the orientation of the surface. We might ask if we can find an orientation such that the traction vector is parallel to the normal.

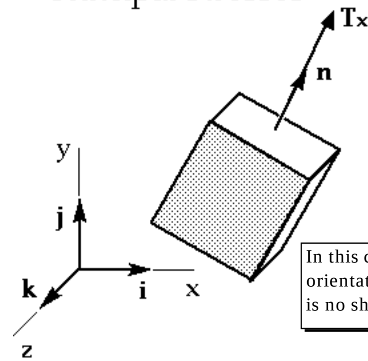
• Principal Stresses



• Principal Stresses

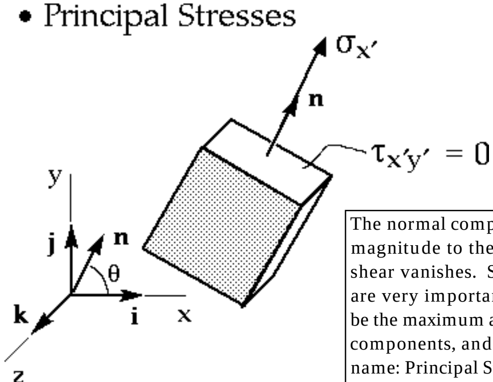


• Principal Stresses



In this case it appears we have found such an orientation. Note that for this orientation there is no shear component.

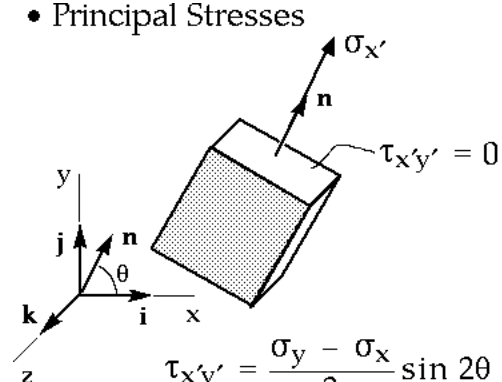
• Principal Stresses



The normal component, $\sigma_{x'}$, is equal in magnitude to the traction vector and the shear vanishes. Such stress components are very important since they turn out to be the maximum and minimum normal components, and so we give them a special name: Principal Stresses.

To calculate such stress components it is necessary to determine the proper block orientation. For this we will need the stress transformation equation for shear.

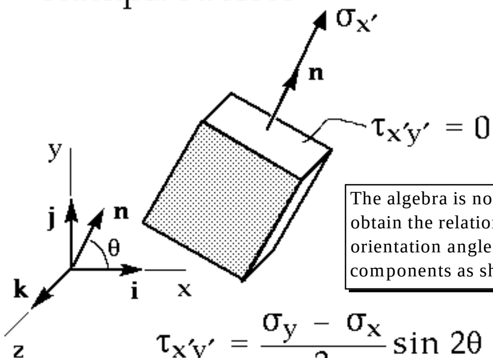
• Principal Stresses



$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

By definition, the principal stresses occur on planes for which the shear stress vanishes. Therefore, we can use this equation to solve for the θ for which $\tau_{x'y'} = 0$.

- Principal Stresses

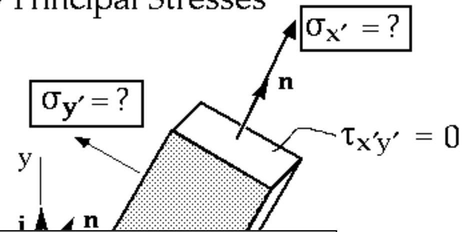


The algebra is not difficult, and we can obtain the relation between the principal orientation angle and the basic stress components as shown.

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = 0 \longrightarrow \boxed{\tan 2\theta_p = -\frac{2\tau_{xy}}{\sigma_y - \sigma_x}}$$

- Principal Stresses



We now can calculate the orientation of the principal plane, but we still need to determine the corresponding stresses. For this purpose we will use the remaining stress transformation relations.

$$\tan 2\theta_p = -\frac{2\tau_{xy}}{\sigma_y - \sigma_x}$$

- Principal Stresses

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\,\tau_{xy}\sin\theta\cos\theta$$

These are the stress transformation equations from before. We now need to do a little algebra...

- Principal Stresses

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \cancel{2\,\tau_{xy}\sin\theta\cos\theta}$$

$$+\ \sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \cancel{2\,\tau_{xy}\sin\theta\cos\theta}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \sigma_x \sin^2\theta + \sigma_y \cos^2\theta$$

First we add these two relations.

- Principal Stresses

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \cancel{\sin\theta\cos\theta}$$
$$+\ \sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\,\tau_{xy} \cancel{\sin\theta\cos\theta}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \sigma_x \sin^2\theta + \sigma_y \cos^2\theta$$

$$= (\sigma_x + \sigma_y)(\cos^2\theta + \sin^2\theta)$$

We can simplify the right hand side by factoring.

- Principal Stresses

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \cancel{\sin\theta\cos\theta}$$
$$+\ \sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\,\tau_{xy} \cancel{\sin\theta\cos\theta}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \sigma_x \sin^2\theta + \sigma_y \cos^2\theta$$

$$= (\sigma_x + \sigma_y)(\cos^2\theta + \sin^2\theta)$$

$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

This can be simplified further to this result. Let's clean up a little and see what this means.

- Principal Stresses

$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y} \quad \text{1st Invariant}$$

This result states that regardless of the orientation of the stress block, the sum of the normal components does not change. Since this quantity does not vary with our point of view, we call it an **invariant**. Such quantities are very important, since they provide a means of capturing the important features of a physical quantity without reference to any particular coordinate system. For **vectors**, the important invariant is the length. For stress **tensors** there are actually three invariants, but in two dimensions we only need to use two of them. The first invariant is what we just calculated; the second we will state without proof...

- Principal Stresses

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \quad \text{1st Invariant}$$

$$\boxed{\sigma_{x'}\sigma_{y'} - \tau^2_{x'y'} = \sigma_x\sigma_y - \tau^2_{xy}} \quad \text{2nd Invariant}$$

...but those of you with a background in linear algebra might recognize this as the determinant of the 2x2 matrix of the stress components. The relation above could be verified by direct substitution into the transformation equations, but the algebra would get rather involved.

We can use these two invariants to find the principal stresses. In particular we substitute the principal stress expressions on the left hand side.

$$\text{2-D Stress Tensor} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

$$\sigma_1 \sigma_2 - \tau_{12}^2 = \sigma_x \sigma_y - \tau_{xy}^2$$

By definition, we know that the shear stress is zero in the principal directions.

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

$$\sigma_1 \sigma_2 - \cancel{\tau_{12}^2}^{\,0} = \sigma_x \sigma_y - \tau_{xy}^2$$

We can simplify the above into a simple system of two equations and two unknowns.

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \text{(i)}$$

$$\sigma_1 \sigma_2 = \sigma_x \sigma_y - \tau_{xy}^2 \qquad \text{(ii)}$$

These equations can be solved for the unknowns, σ_1 and σ_2, in terms of the known quantities σ_x, σ_y, and τ_{xy}.

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \text{(i)}$$

$$\sigma_1 \sigma_2 = \sigma_x \sigma_y - \tau_{xy}^2 \qquad \text{(ii)}$$

(i) $\longrightarrow$ (ii)

We substitute equation (i) into equaiton (ii).

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \text{(i)}$$

$$\sigma_1 \, \sigma_2 = \sigma_x \sigma_y - \tau_{xy}^2 \qquad \text{(ii)}$$

(i) ⟶ (ii)

$$\sigma_1 \, (\sigma_x + \sigma_y - \sigma_1) = \sigma_x \sigma_y - \tau_{xy}^2$$

We now have a single equation for σ_1. We can expand this equation to put it into a recognizable form.

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \text{(i)}$$

$$\sigma_1 \, \sigma_2 = \sigma_x \sigma_y - \tau_{xy}^2 \qquad \text{(ii)}$$

(i) ⟶ (ii)

$$\sigma_1 \, (\sigma_x + \sigma_y - \sigma_1) = \sigma_x \sigma_y - \tau_{xy}^2$$

$$\sigma_1^2 - (\sigma_x + \sigma_y)\,\sigma_1 + (\sigma_x \sigma_y - \tau_{xy}^2) = 0$$

This is a standard quadratic equation.

• Principal Stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \text{(i)}$$

$$\sigma_1 \, \sigma_2 = \sigma_x \sigma_y - \tau_{xy}^2 \qquad \text{(ii)}$$

(i) ⟶ (ii)

$$\sigma_1 \, (\sigma_x + \dots$$

The solution can be obtained from the quadratic formula. In fact, it turns out we get both σ_1 and σ_2 from this quadratic equation.

Let's clean up a little and look at our result.

$$\sigma_1^2 - (\sigma_x + \sigma_y)\,\sigma_1 + (\sigma_x \sigma_y - \tau_{xy}^2) = 0$$

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - \left(\sigma_x \sigma_y - \tau_{xy}^2\right)}}$$

• Principal Stresses

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

After reducing the term under the square root, we arrive at what we have been persuing: an equation for calculating principal stresses in terms of the basic stress components.

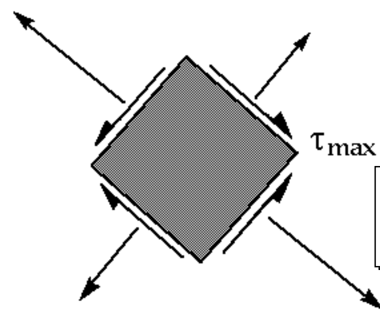
- Principal Stresses

$$\tan 2\theta_p = -\frac{2\tau_{xy}}{\sigma_y - \sigma_x}$$ Direction

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$ Magnitudes

Together with our previous equation for the principal direction, this provides a general means for calculating principal stress information. It is not difficult to show that the term in the square root is always positive, so it turns out that for any stress state, we can find principal stresses and directions; i.e. directions for which the shear stress vanishes.

- Maximum In-plane Shear Stress



σ_x = ✓

τ_{yx} = ✓

σ_y = ✓

τ_{max} = ?

We have seen how to calculate the principal normal stresses, but what about maximum / minimum shear stress?

- Maximum In-plane Shear Stress

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

To determine a way of calculating the maximum shear stress in terms of a given set of basic components, σ_x, σ_y, and τ_{xy} , we begin with the stress transformation equation for shear.

- Maximum In-plane Shear Stress

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2} \sin 2\hat{\theta} \qquad (@\ \hat{\theta} = \theta - \theta_p)$$

Since this transformation equation can be used with respect to any original coordinate system, we choose the principal system as our reference system without loss of generality. This makes our calculations easy; we just need to remember that any angle measurements are made with respect to the principal axes rather than the original x-y system. Thus, we introduce theta hat to account for the necessary offset.

• Maximum In-plane Shear Stress

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2}\sin 2\hat{\theta} \qquad (@\ \hat{\theta} = \theta - \theta_p)$$

$$\max\ \tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2} \qquad (@\ \hat{\theta} = \pm 45°)$$

We already know from earlier work how to find the maximum of something times the sine of 2 times an angle. This leads us to the expression for the maximum shear above. We still need to express this result in terms of σ_x, σ_y, and τ_{xy}, however.

• Maximum In-plane Shear Stress

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2}\sin 2\hat{\theta} \qquad (@\ \hat{\theta} = \theta - \theta_p)$$

$$\max\ \tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2} \qquad (@\ \hat{\theta} = \pm 45°)$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

This can be done by means of our earlier relation between the principal stresses and σ_x, σ_y, and τ_{xy}. Substituting this into the maximum shear equation above gives the desired result.

• Maximum In-plane Shear Stress

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2}\sin 2\hat{\theta} \qquad (@\ \hat{\theta} = \theta - \theta_p)$$

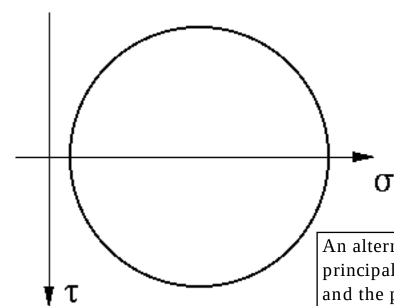
$$\max\ \tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2} \qquad (@\ \hat{\theta} = \pm 45°)$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\boxed{\max\ \tau_{x'y'} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Voilá!

• Mohr's Circle



$\sigma_x = \checkmark$

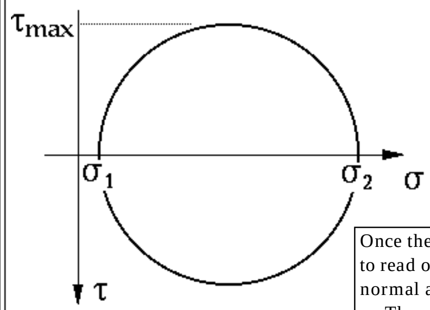
$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$

An alternative method for determining principal stresses, the maximum shear stress and the principal directions is to construct a plot of all the stress component combinations for a given set of σ_x, σ_y, and τ_{xy}. The resulting plot is a circle as shown, called Mohr's circle after Otto von Mohr, who first published these ideas.

• Mohr's Circle

$\sigma_x = \checkmark$
$\tau_{yx} = \checkmark$
$\sigma_y = \checkmark$

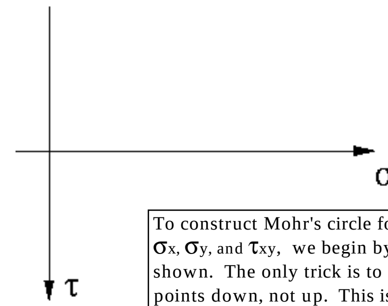


Once the circle is constructed, it is simple to read off the maximum and minimum normal and shear stresses directly.

The main advantages of Mohr's circle are its pictorial nature and the fact that you do not need to remember a bunch of equations to construct it.

• Mohr's Circle

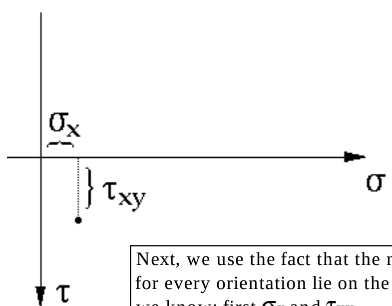
$\sigma_x = \checkmark$
$\tau_{yx} = \checkmark$
$\sigma_y = \checkmark$



To construct Mohr's circle for a given set of stresses, σ_x, σ_y, and τ_{xy}, we begin by laying out a set of axes as shown. The only trick is to note that the shear axis points down, not up. This is an arbitrary convention that makes some later results come out a little easier.

• Mohr's Circle

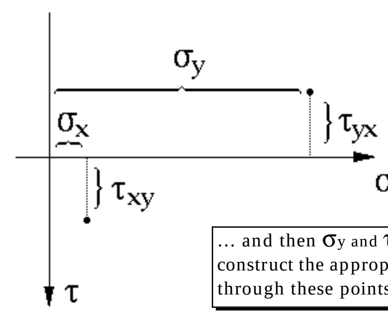
$\sigma_x = \checkmark$
$\tau_{yx} = \checkmark$
$\sigma_y = \checkmark$



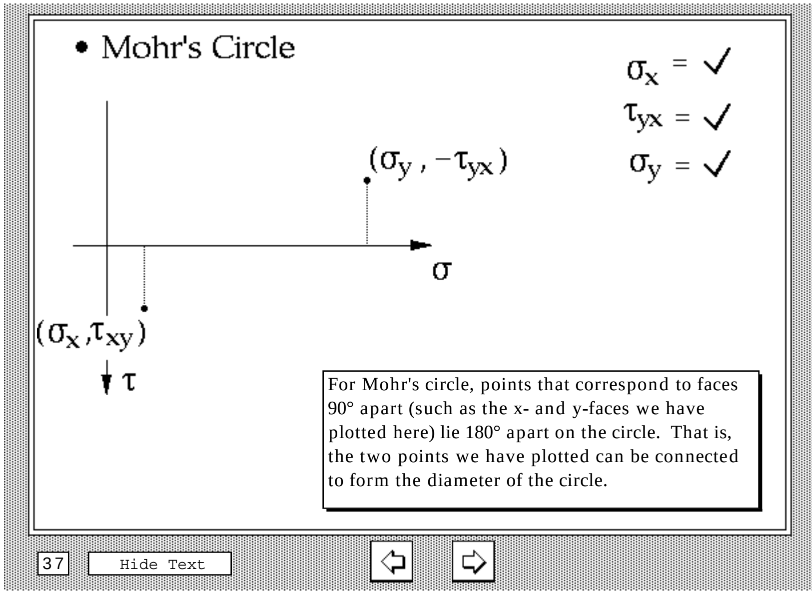
Next, we use the fact that the normal and shear components for every orientation lie on the circle, and so we plot the points we know: first σ_x and τ_{xy}...

• Mohr's Circle

$\sigma_x = \checkmark$
$\tau_{yx} = \checkmark$
$\sigma_y = \checkmark$



... and then σ_y and τ_{yx}. We now need to construct the appropriate circle passing through these points.


• Mohr's Circle
σ_x = ✓
τ_{yx} = ✓
σ_y = ✓
$(\sigma_y , -\tau_{yx})$
σ
(σ_x ,τ_{xy})
τ
For Mohr's circle, points that correspond to faces 90° apart (such as the x- and y-faces we have plotted here) lie 180° apart on the circle. That is, the two points we have plotted can be connected to form the diameter of the circle.
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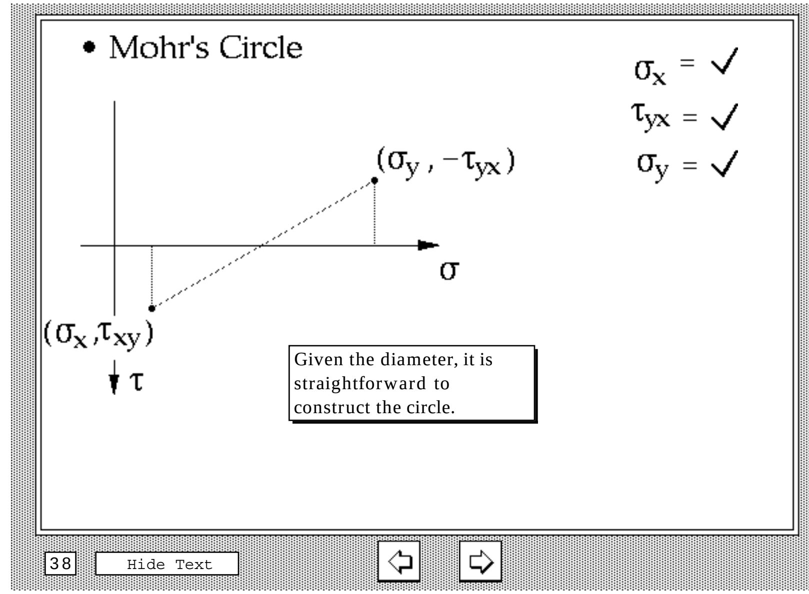
• Mohr's Circle
σ_x = ✓
τ_{yx} = ✓
σ_y = ✓
$(\sigma_y , -\tau_{yx})$
σ
(σ_x ,τ_{xy})
τ
Given the diameter, it is straightforward to construct the circle.
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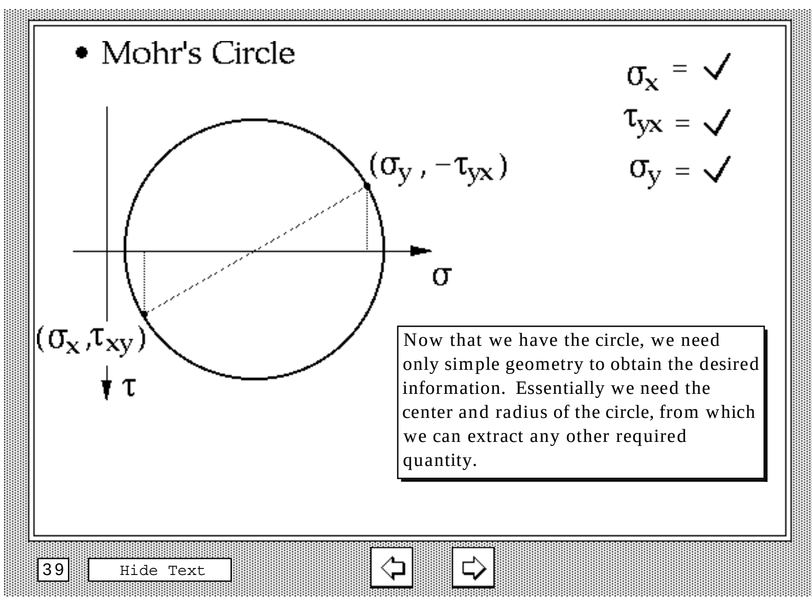
• Mohr's Circle
σ_x = ✓
τ_{yx} = ✓
σ_y = ✓
$(\sigma_y , -\tau_{yx})$
σ
(σ_x ,τ_{xy})
τ
Now that we have the circle, we need only simple geometry to obtain the desired information. Essentially we need the center and radius of the circle, from which we can extract any other required quantity.
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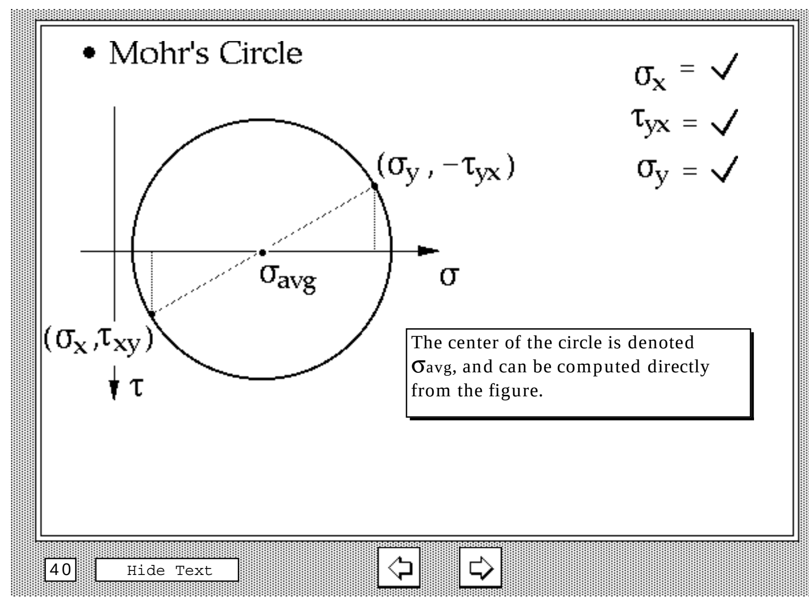
• Mohr's Circle
σ_x = ✓
τ_{yx} = ✓
σ_y = ✓
$(\sigma_y , -\tau_{yx})$
σ_{avg}
σ
(σ_x ,τ_{xy})
τ
The center of the circle is denoted σ_{avg}, and can be computed directly from the figure.
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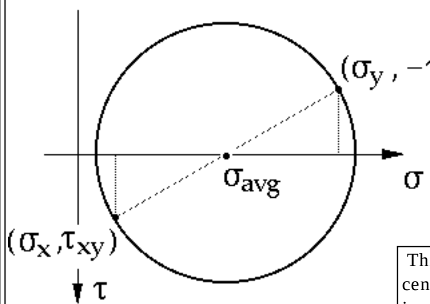
• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\boxed{\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}}$$

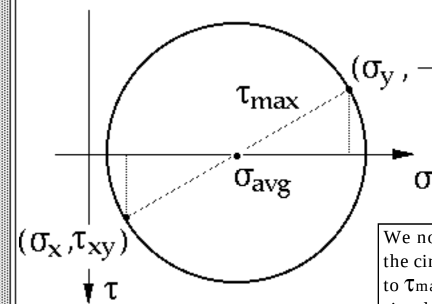
This gives the location of the circle's center (note the relation to the first invariant we discussed earlier).

• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

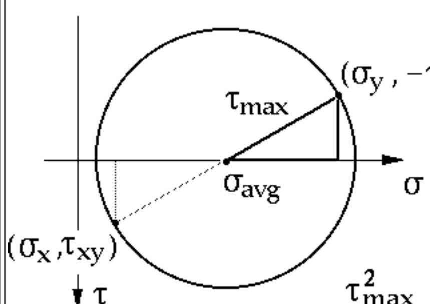
We now need to compute the radius of the circle, which as we saw earlier is equal to τ_{max}. We can calculate this using simple right triangle relations.

• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max}^2 = (\sigma_y - \sigma_{avg})^2 + \tau_{xy}^2$$

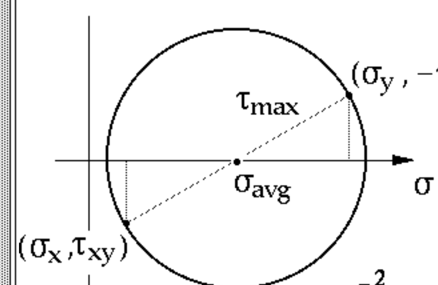
Using the Pythagorean theorem for the triangle shown gives the relation above.

• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max}^2 = (\sigma_y - \sigma_{avg})^2 + \tau_{xy}^2$$

Substituting for σ_{avg} and taking the square root of both sides leads to this result. This is the same result we arrived at earlier for maximum shear stress.

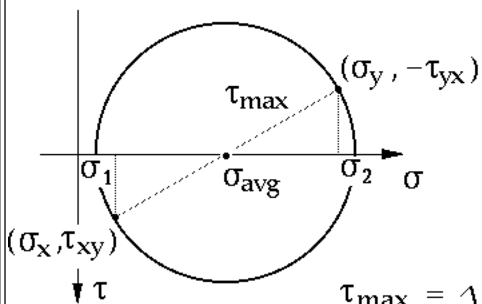
$$\boxed{\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}}$$

• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

We now need to calculate σ_1 and σ_2 as shown. Given the center and radius of the circle, this is quite simple to do.

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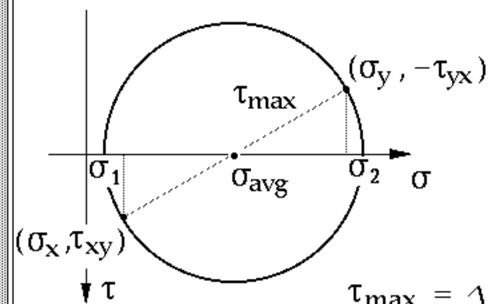
• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

We have this simple result.

$$\boxed{\sigma_{1,2} = \sigma_{avg} \pm \tau_{max}}$$

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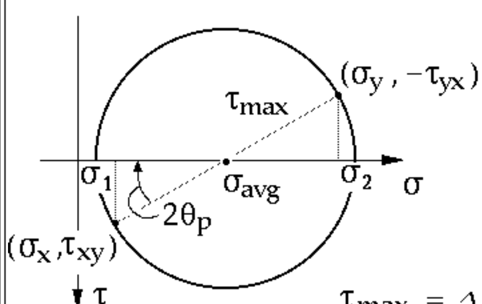
• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

To determine the orientation angle corresponding to the principal stresses, we use the relation shown on the figure. Note the factor of 2.

$$\sigma_{1,2} = \sigma_{avg} \pm \tau_{max}$$

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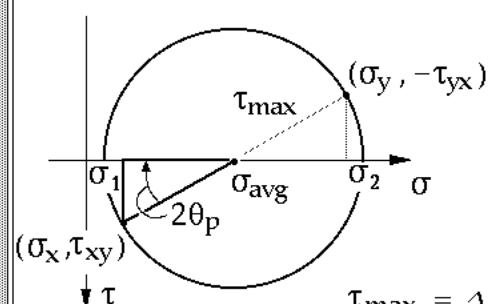
• Mohr's Circle

$\sigma_x = \checkmark$

$\tau_{yx} = \checkmark$

$\sigma_y = \checkmark$



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$\boxed{\tan(2|\theta_p|) = \frac{2\tau_{xy}}{|\sigma_y - \sigma_x|}}$$

This equation can be determined from the triangle shown. The direction of the angle is as shown on the figure.

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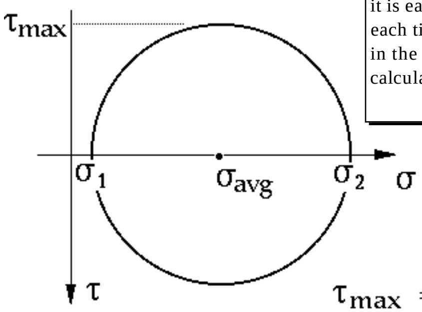
• Mohr's Circle

Although the equations below were determined using Mohr's circle, in practice it is easier to simply start from scratch each time, and to use the numerical values in the problem to construct the circle and calculate the geometric properties.



$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan(2|\theta_p|) = \frac{2\tau_{xy}}{|\sigma_y - \sigma_x|} \qquad \sigma_{1,2} = \sigma_{avg} \pm \tau_{max}$$

• Problem Solving Tips

Stress Transformation Examples

Problem solving techniques are best illustrated using an example problem, as presented in a the stack "Stress Transformation Examples". The important ideas from this stack are the basic notions of principal stresses, maximum shear stresses, and Mohr's circle being a plot of displaying the complete set of stress component/orientation combinations.