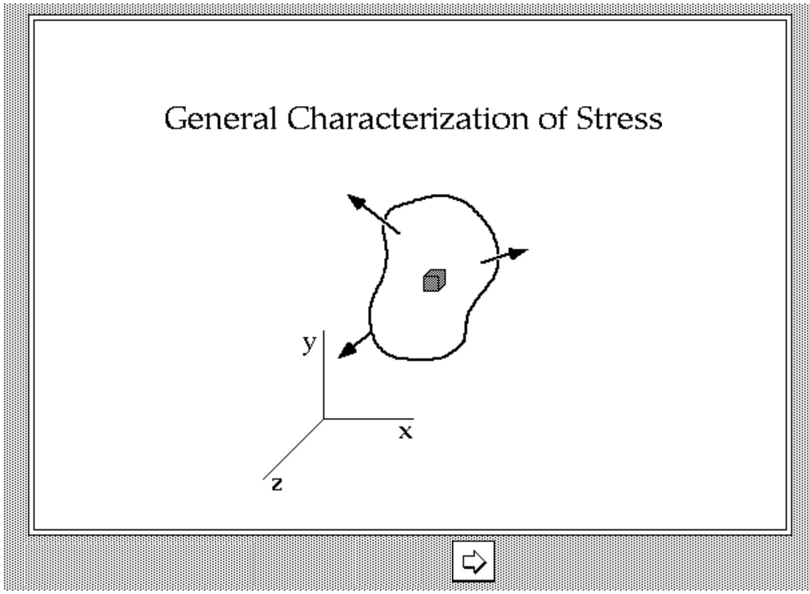
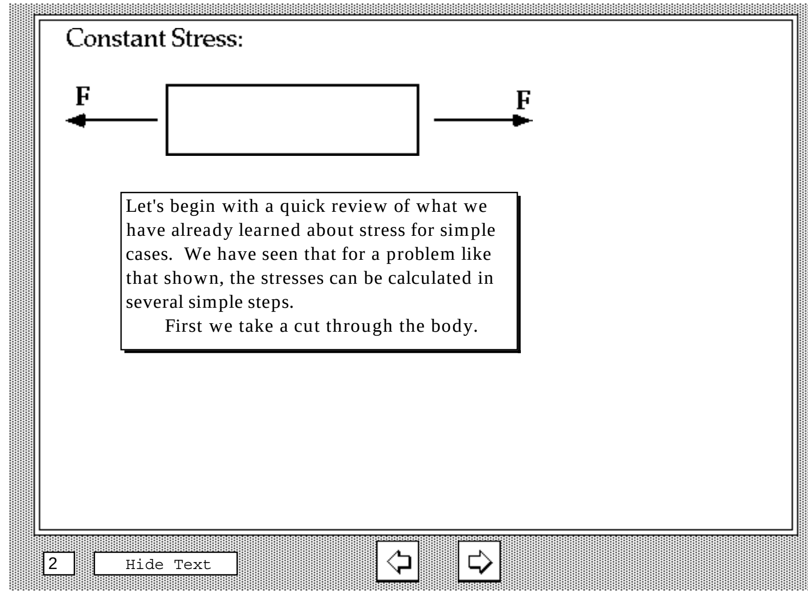
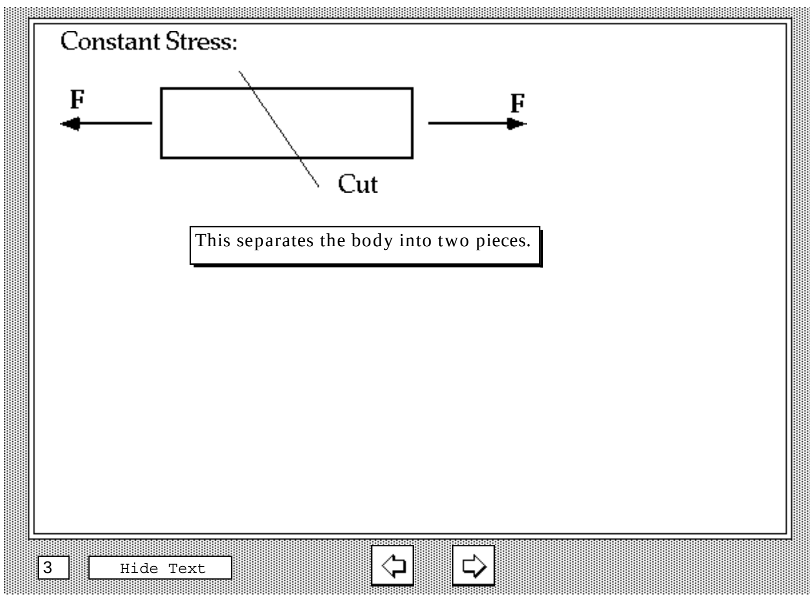

General Characterization of Stress
y
x
z


Constant Stress:
F
F
Let's begin with a quick review of what we have already learned about stress for simple cases. We have seen that for a problem like that shown, the stresses can be calculated in several simple steps.
First we take a cut through the body.
2
Hide Text


Constant Stress:
F
F
Cut
This separates the body into two pieces.
3
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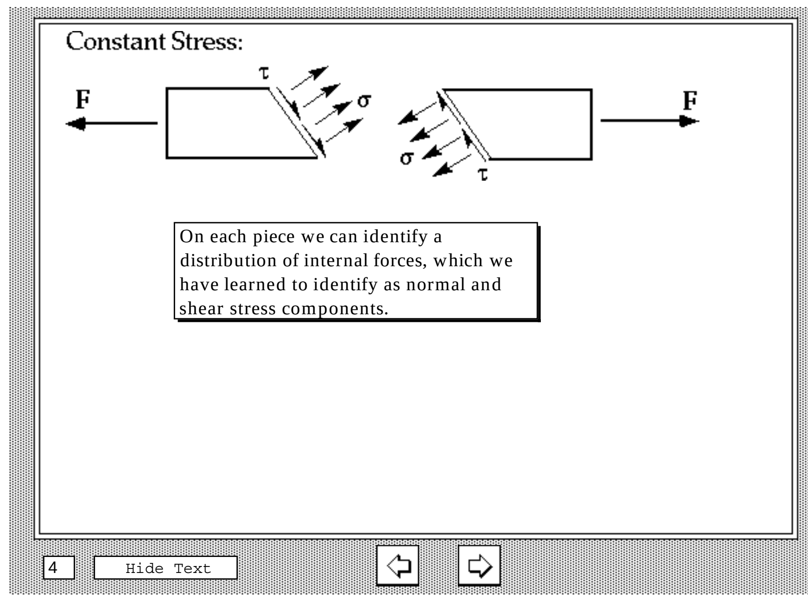
Constant Stress:
F
τ
σ
σ
τ
F
On each piece we can identify a distribution of internal forces, which we have learned to identify as normal and shear stress components.
4
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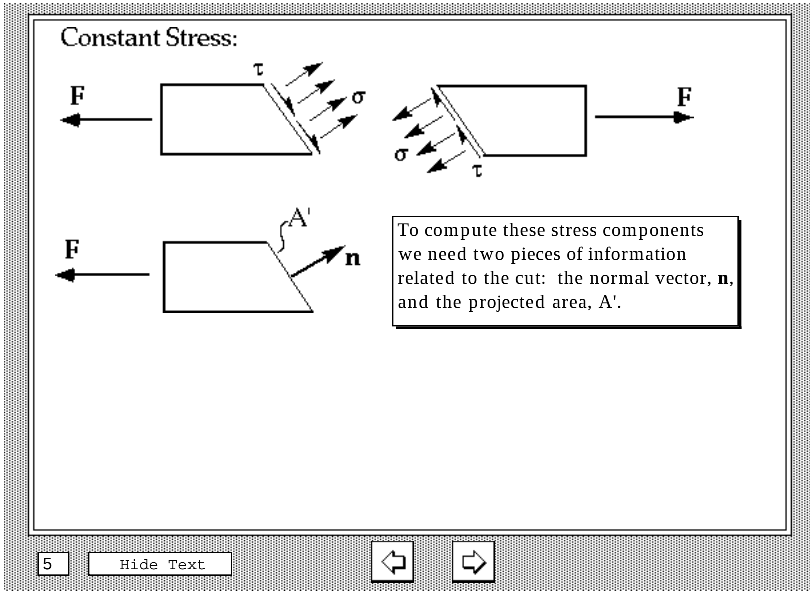
Constant Stress:
F
τ
σ
F
σ
τ
A'
F
n
To compute these stress components we need two pieces of information related to the cut: the normal vector, **n**, and the projected area, A'.
5
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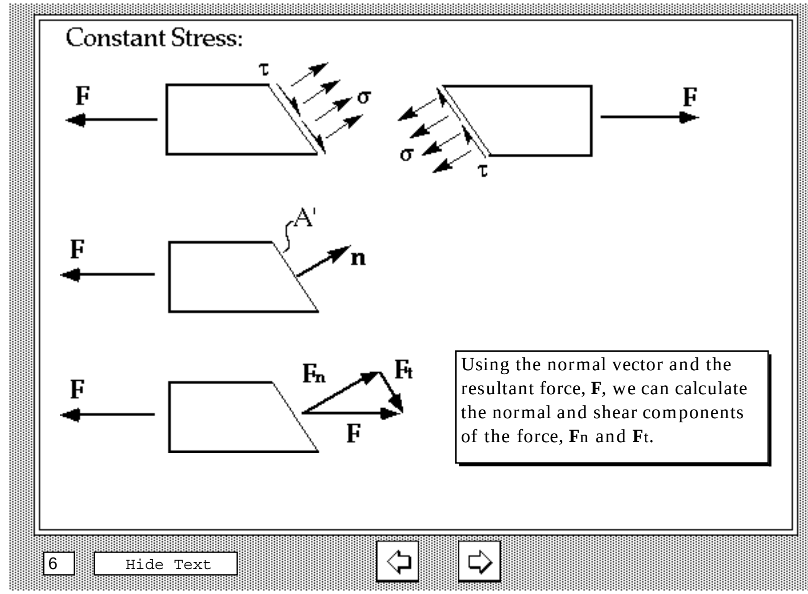
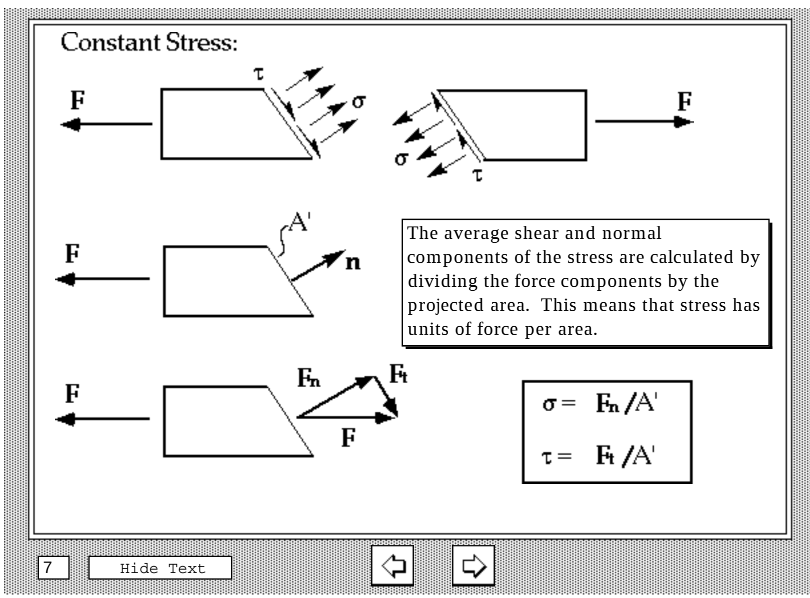
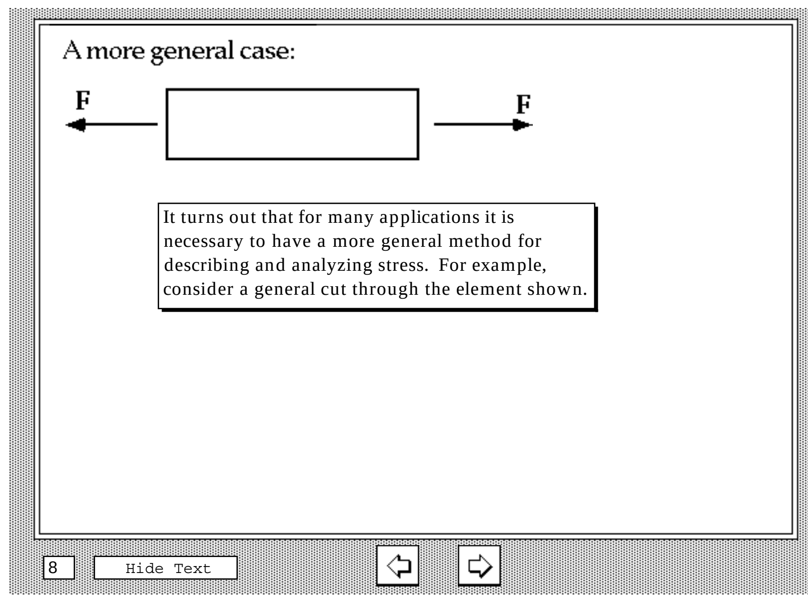
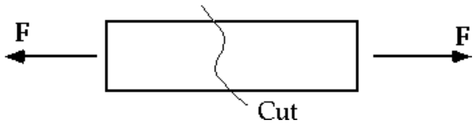

Constant Stress:
F
τ
σ
F
σ
τ
A'
F
n
F
Fn
Ft
F
Using the normal vector and the resultant force, **F**, we can calculate the normal and shear components of the force, **F**n and **F**t.
6
Hide Text


Constant Stress:
F
τ
σ
F
σ
τ
A'
F
n
The average shear and normal components of the stress are calculated by dividing the force components by the projected area. This means that stress has units of force per area.
F
Fn
Ft
F
σ = Fn /A'
τ = Ft /A'
7
Hide Text


A more general case:
F
F
It turns out that for many applications it is necessary to have a more general method for describing and analyzing stress. For example, consider a general cut through the element shown.
8
Hide Text

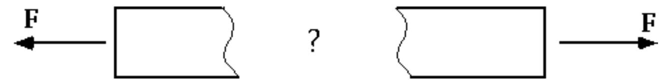
A more general case:



As before, we consider the cut pieces as free bodies.

9 Hide Text

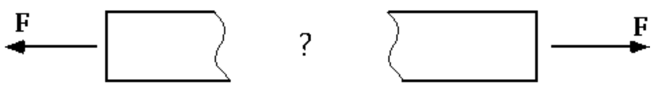
A more general case:

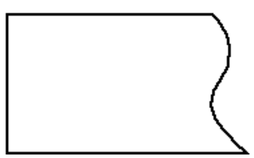


Now we must examine what happens on this general cut. In the Real World material will automatically experience stresses on cuts in every possible direction. If the stresses exceed the material strength on one of these cuts it will fail. As engineers we need to be able to investigate this possibility.

10 Hide Text

A more general case:

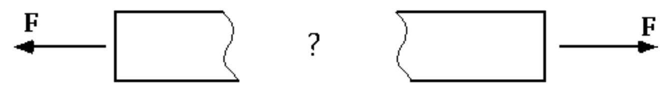


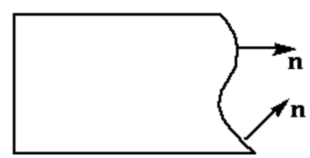


Let's take a closer look at one of the pieces

11 Hide Text

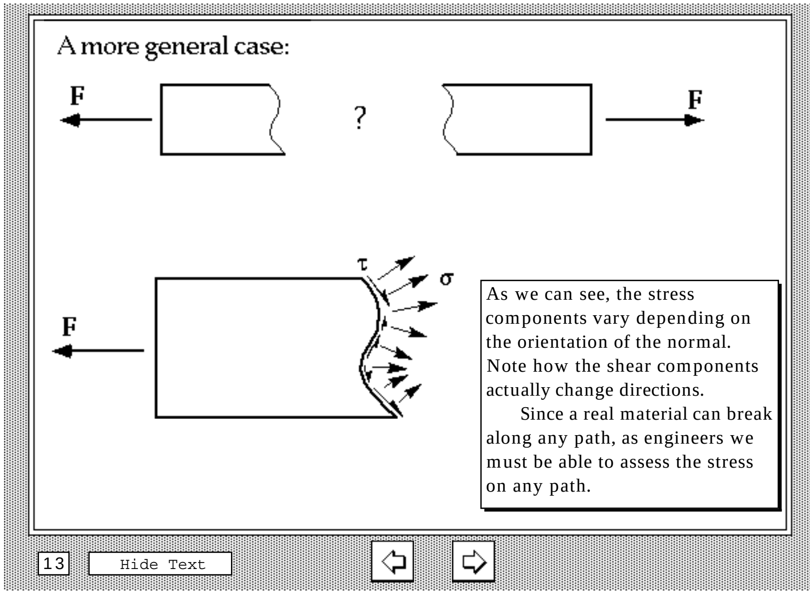
A more general case:





In this case, the normal vector varies along the cut. What do the stresses look like?

12 Hide Text


A more general case:
F
?
F
τ
σ
F
As we can see, the stress components vary depending on the orientation of the normal. Note how the shear components actually change directions.
Since a real material can break along any path, as engineers we must be able to assess the stress on any path.
13
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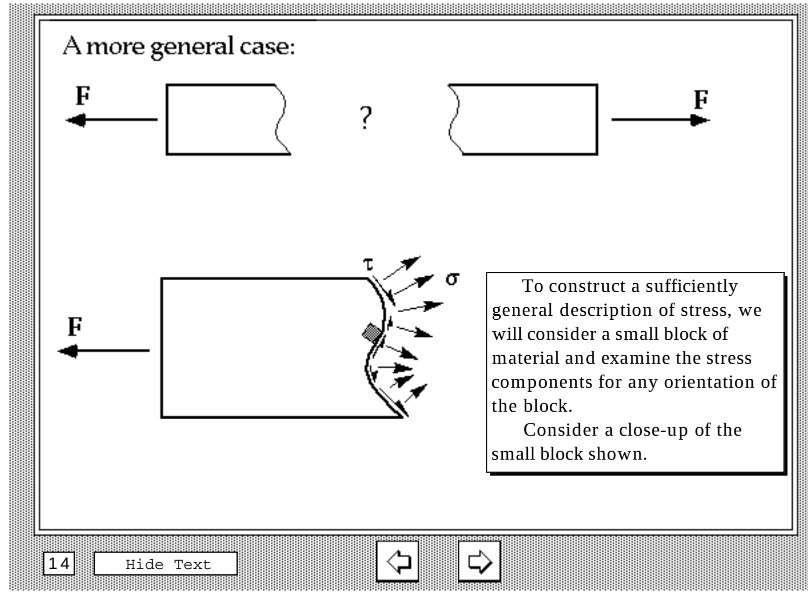
A more general case:
F
?
F
τ
σ
F
To construct a sufficiently general description of stress, we will consider a small block of material and examine the stress components for any orientation of the block.
Consider a close-up of the small block shown.
14
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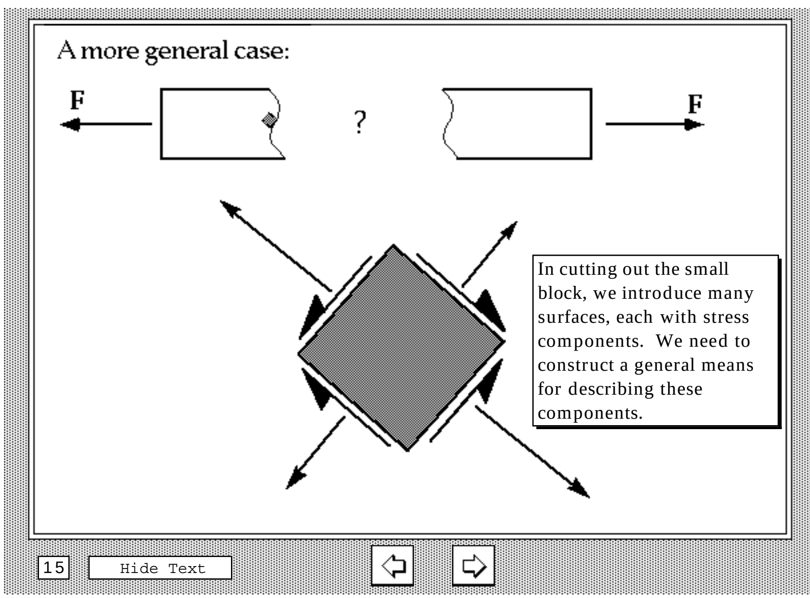
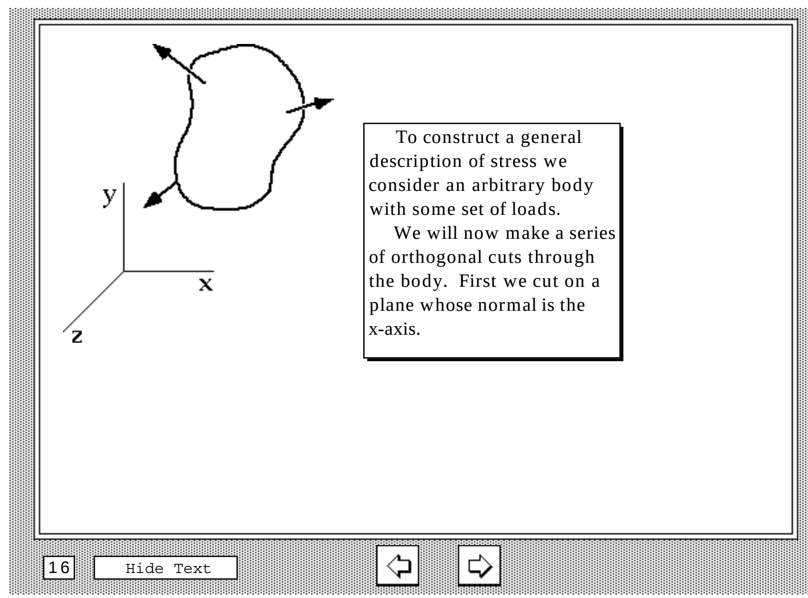

A more general case:
F
?
F
In cutting out the small block, we introduce many surfaces, each with stress components. We need to construct a general means for describing these components.
15
Hide Text


y
x
z
To construct a general description of stress we consider an arbitrary body with some set of loads.
We will now make a series of orthogonal cuts through the body. First we cut on a plane whose normal is the x-axis.
16
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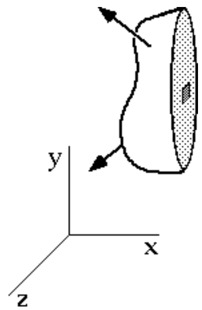
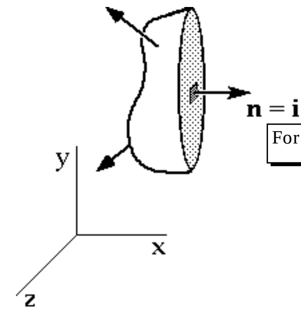


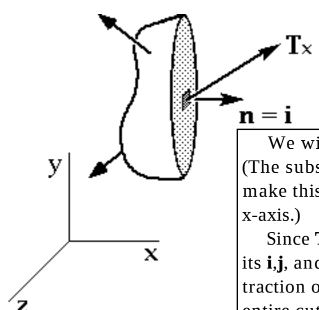
We will focus our interest on a particular block of material as shown.

17 Hide Text



For this plane, the normal vector is simply **i**.

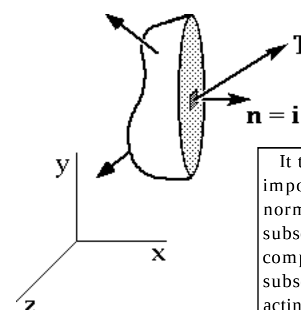
18 Hide Text



We will call the corresponding traction vector $\mathbf{T}_x$. (The subscript "x" refers to the plane we used to make this cut; i.e. the plane whose normal is the x-axis.)

Since $\mathbf{T}_x$ is a vector, we can express it in terms of its **i**,**j**, and **k** components. Remember, $\mathbf{T}_x$ is the traction on the small gray patch of material, not the entire cut.

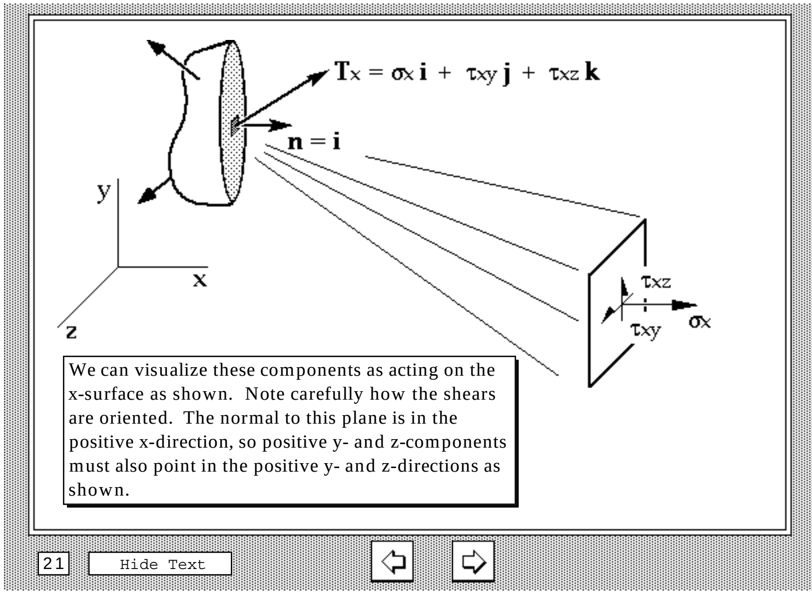
19 Hide Text



$$\mathbf{T}_x = \sigma_x\,\mathbf{i} + \tau_{xy}\,\mathbf{j} + \tau_{xz}\,\mathbf{k}$$

It turns out the components of this vector are very important, and so we give them special names. The normal component is denoted σ_x, where the "x" subscript again refers to the x-axis plane. The shear component in the y direction is denoted τ_{xy}. These subscripts can be interpreted as the shear stress acting on the x-axis plane in the y-direction. In similar fashion, τ_{xz} represents the shear component acting on the x-axis plane in the z-direction.

20 Hide Text



Tx = σx i + τxy j + τxz k
n = i
y
x
z
τxz
τxy
σx
We can visualize these components as acting on the x-surface as shown. Note carefully how the shears are oriented. The normal to this plane is in the positive x-direction, so positive y- and z-components must also point in the positive y- and z-directions as shown.
21
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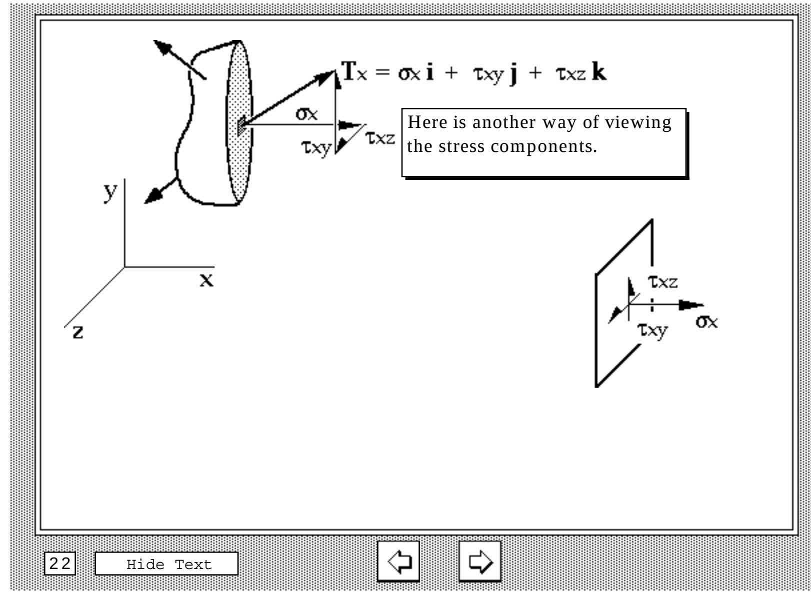
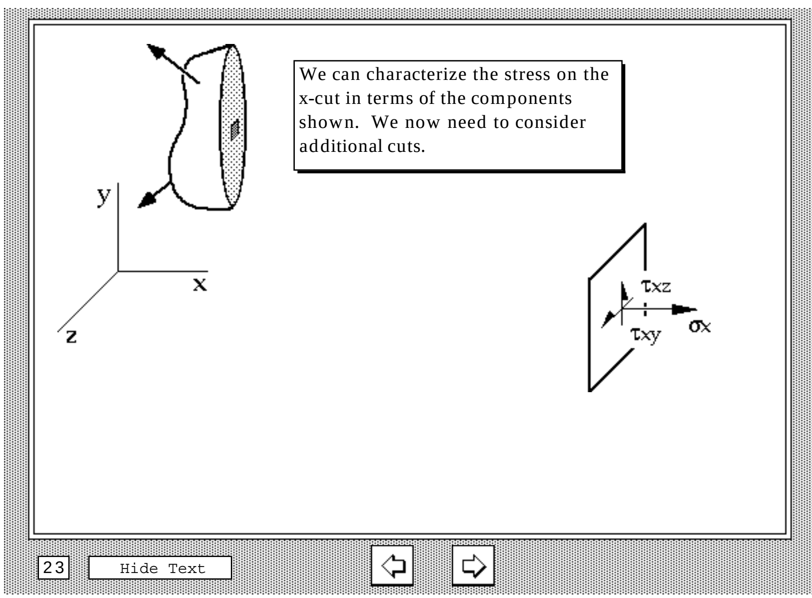
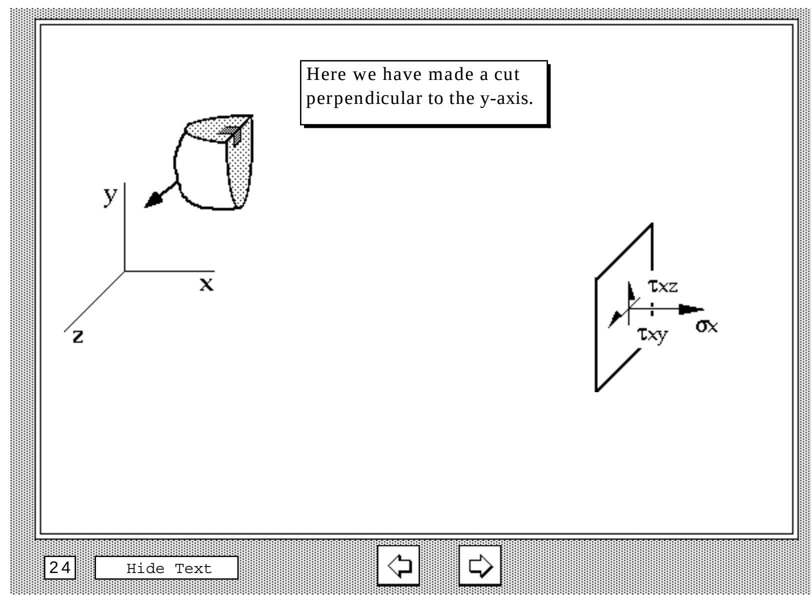


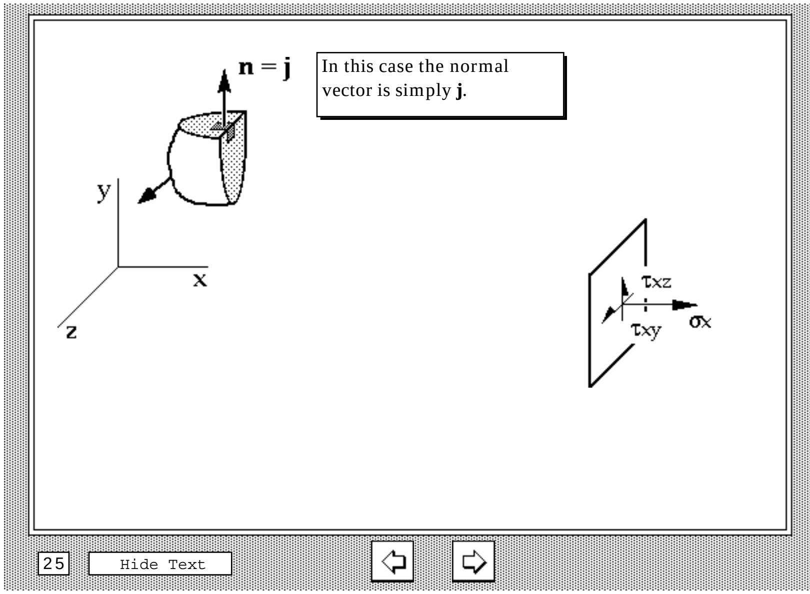
Tx = σx i + τxy j + τxz k
σx
τxy
τxz
Here is another way of viewing the stress components.
y
x
z
τxz
τxy
σx
22
Hide Text

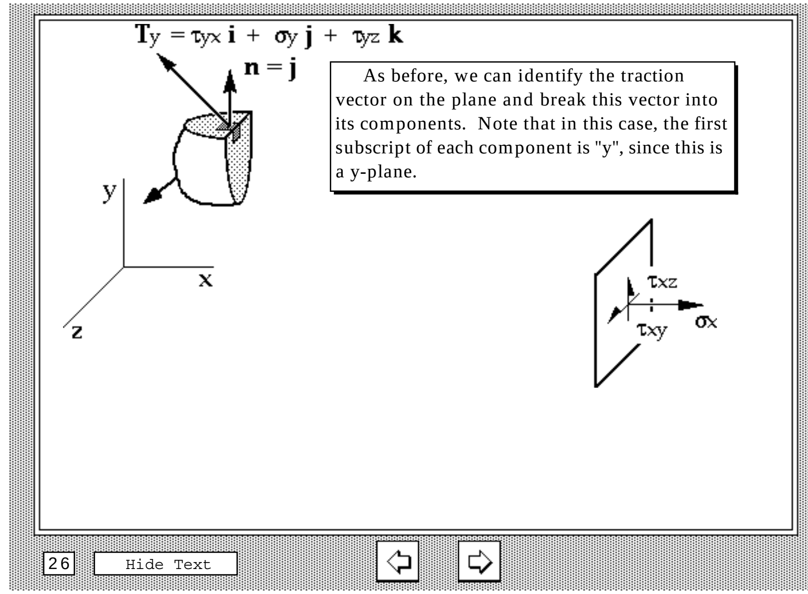


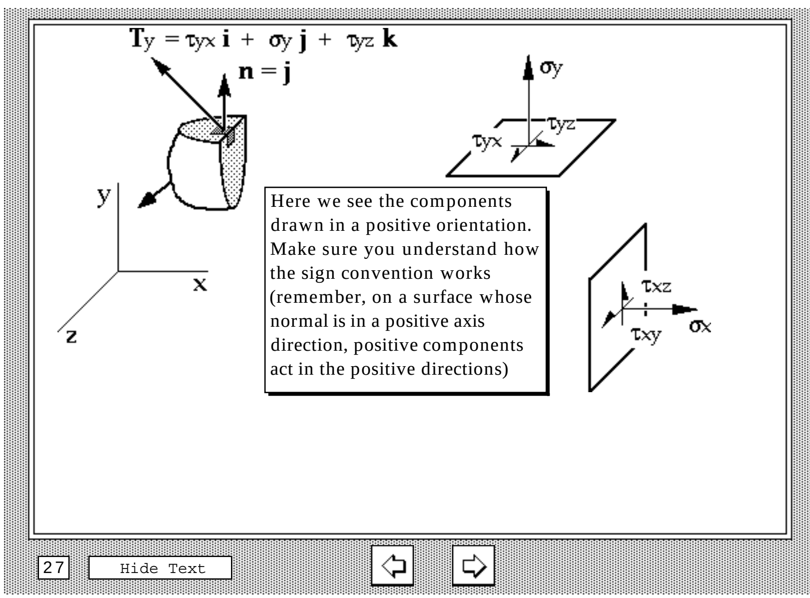
We can characterize the stress on the x-cut in terms of the components shown. We now need to consider additional cuts.
y
x
z
τxz
τxy
σx
23
Hide Text

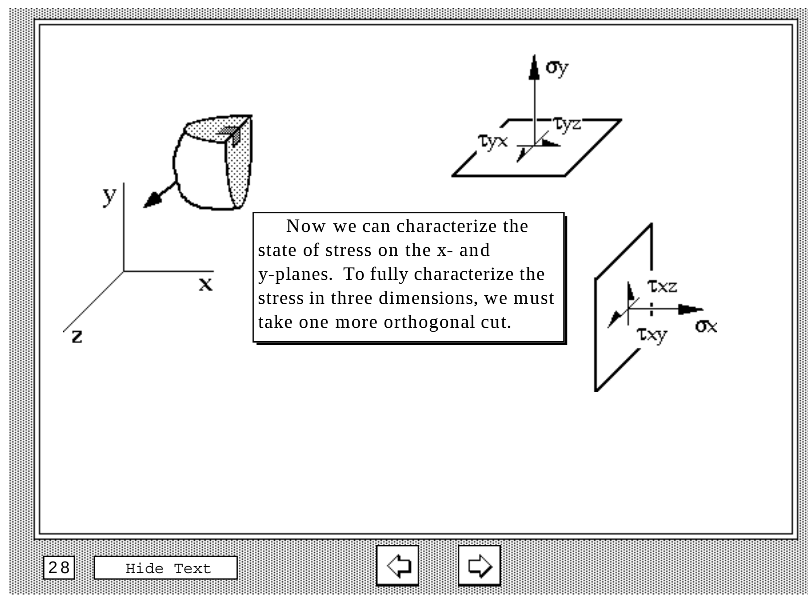


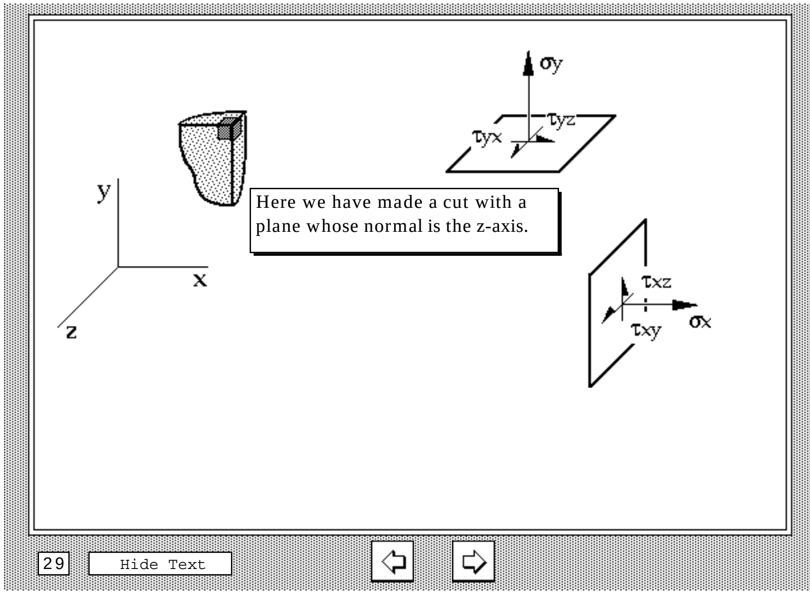
Here we have made a cut perpendicular to the y-axis.
y
x
z
τxz
τxy
σx
24
Hide Text

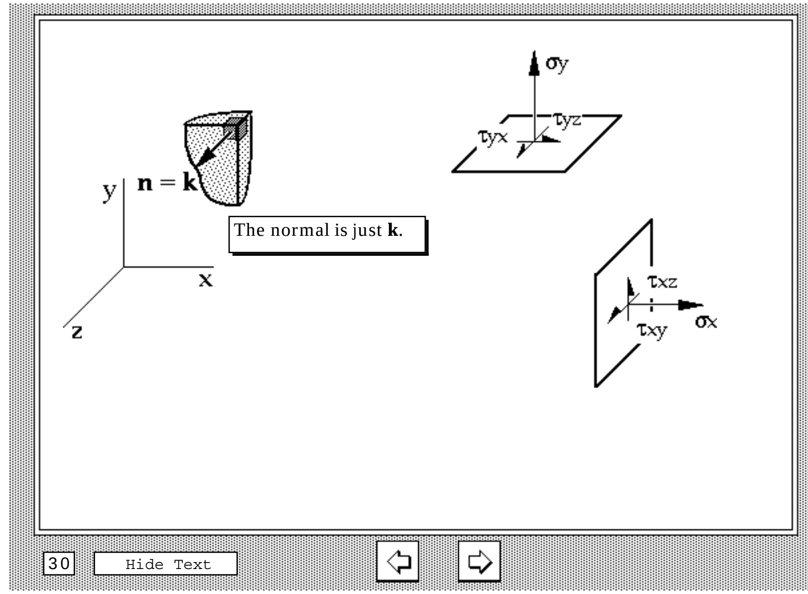

$\mathbf{n} = \mathbf{j}$
In this case the normal vector is simply **j**.
y
x
z
τ_{xz}
τ_{xy}
σ_x
25
Hide Text

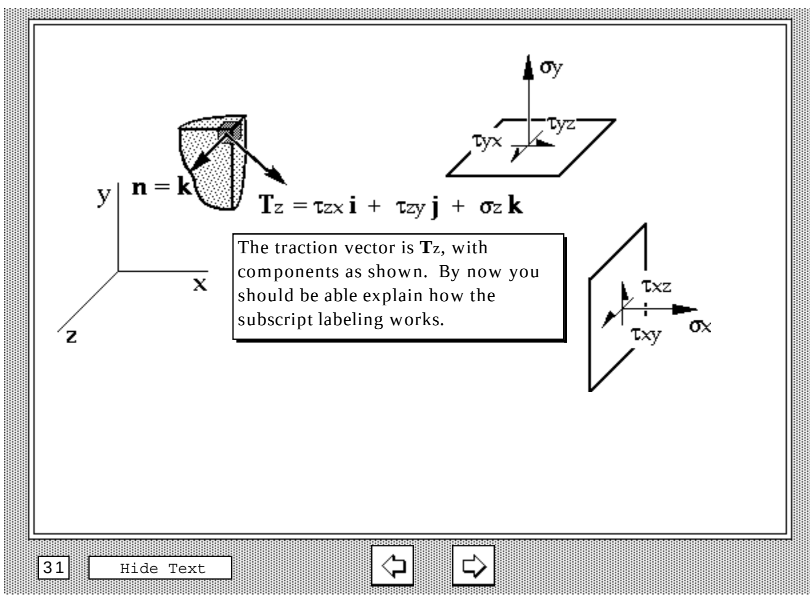

$\mathbf{T}_y = \tau_{yx}\,\mathbf{i} + \sigma_y\,\mathbf{j} + \tau_{yz}\,\mathbf{k}$
$\mathbf{n} = \mathbf{j}$
As before, we can identify the traction vector on the plane and break this vector into its components. Note that in this case, the first subscript of each component is "y", since this is a y-plane.
y
x
z
τ_{xz}
τ_{xy}
σ_x
26
Hide Text

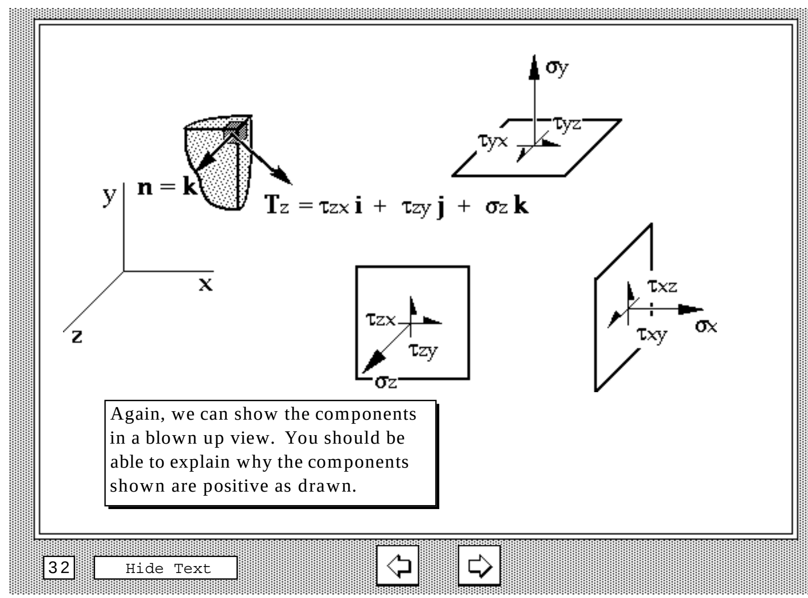

$\mathbf{T}_y = \tau_{yx}\,\mathbf{i} + \sigma_y\,\mathbf{j} + \tau_{yz}\,\mathbf{k}$
$\mathbf{n} = \mathbf{j}$
σ_y
τ_{yz}
τ_{yx}
Here we see the components drawn in a positive orientation. Make sure you understand how the sign convention works (remember, on a surface whose normal is in a positive axis direction, positive components act in the positive directions)
y
x
z
τ_{xz}
τ_{xy}
σ_x
27
Hide Text


σ_y
τ_{yz}
τ_{yx}
Now we can characterize the state of stress on the x- and y-planes. To fully characterize the stress in three dimensions, we must take one more orthogonal cut.
y
x
z
τ_{xz}
τ_{xy}
σ_x
28
Hide Text


σy
τyx
τyz
y
x
z
Here we have made a cut with a plane whose normal is the z-axis.
τxz
τxy
σx
29
Hide Text


σy
τyx
τyz
y
n = k
x
z
The normal is just k.
τxz
τxy
σx
30
Hide Text


σy
τyx
τyz
y
n = k
Tz = τzx i + τzy j + σz k
x
z
The traction vector is Tz, with components as shown. By now you should be able explain how the subscript labeling works.
τxz
τxy
σx
31
Hide Text


σy
τyx
τyz
y
n = k
Tz = τzx i + τzy j + σz k
x
z
τzx
τzy
σz
τxz
τxy
σx
Again, we can show the components in a blown up view. You should be able to explain why the components shown are positive as drawn.
32
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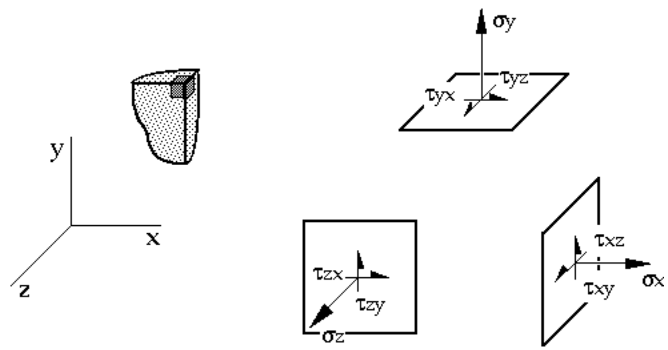
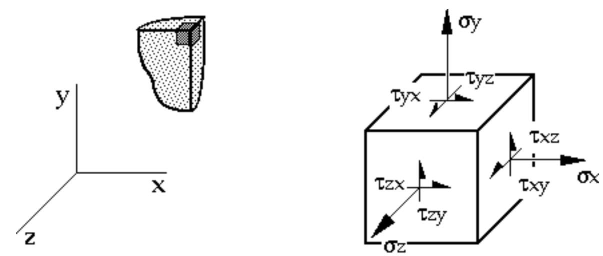


These nine stress components characterize the complete state of stress at the point in question. It is customary to show these components together on a single block of material.

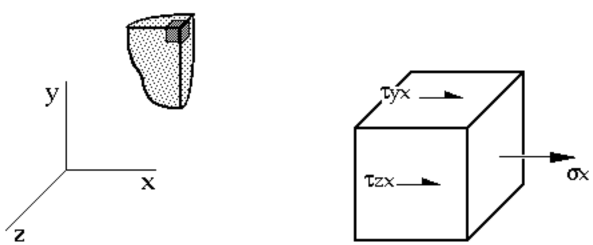
33 Hide Text



Consideration of equilibrium of the block will provide further information concerning both the stress components shown, and the components we can't see that must be acting on the back faces of the block. We will consider first x-direction equilibrium. To this end, we will clean up our figure by removing all the stress components that do not point in the x-direction....

34 Hide Text

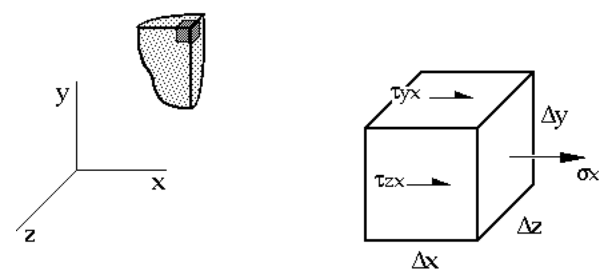
x-direction equilibrium



To check equilibrium, remember that stress components are not forces. They must be multiplied by the area they act on in order to obtain forces.

35 Hide Text

x-direction equilibrium

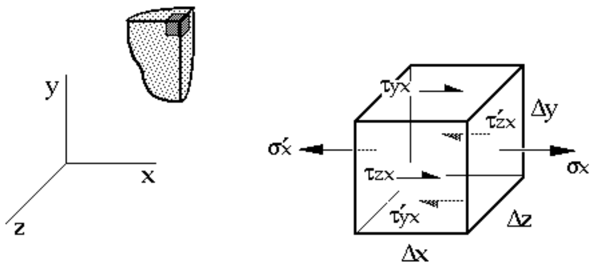


To compute the area of the block surfaces, we introduce the block dimensions as shown.

Clearly we have no chance of obtaining equilibrium unless we include the stress components acting on the back faces. So...

36 Hide Text

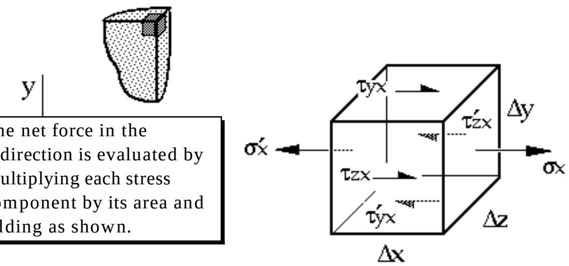
x-direction equilibrium



Here we show the entire set of stress components acting on the block in the x-direction. The components on the back faces are marked with primes, and are shown in positive orientations. Just as positive stress components on positive faces point in positive directions, positive stress components on negative faces point in negative

37 Hide Text

x-direction equilibrium



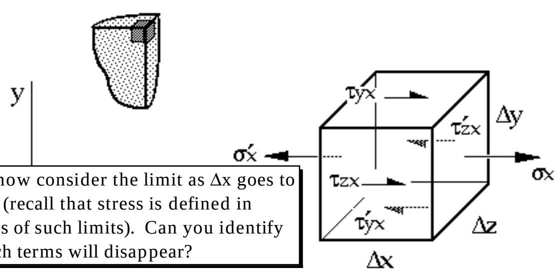
The net force in the x-direction is evaluated by multiplying each stress component by its area and adding as shown.

$$\Sigma F_x = 0$$

$$\sigma_x \Delta y \Delta z + \tau_{yx} \Delta x \Delta z + \tau_{zx} \Delta x \Delta y - [\sigma'_x \Delta y \Delta z + \tau'_{yx} \Delta x \Delta z + \tau'_{zx} \Delta x \Delta y] = 0$$

38 Hide Text

x-direction equilibrium



We now consider the limit as Δx goes to zero (recall that stress is defined in terms of such limits). Can you identify which terms will disappear?

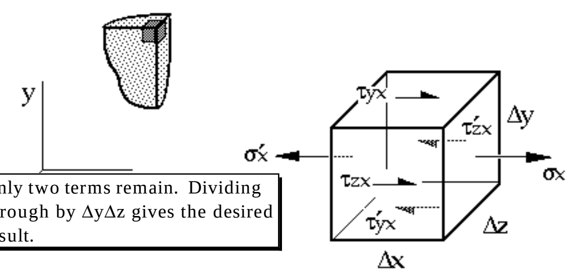
$$\Sigma F_x = 0$$

$$\sigma_x \Delta y \Delta z + \tau_{yx} \Delta x \Delta z + \tau_{zx} \Delta x \Delta y - [\sigma'_x \Delta y \Delta z + \tau'_{yx} \Delta x \Delta z + \tau'_{zx} \Delta x \Delta y] = 0$$

$$\Delta x \rightarrow 0$$

39 Hide Text

x-direction equilibrium



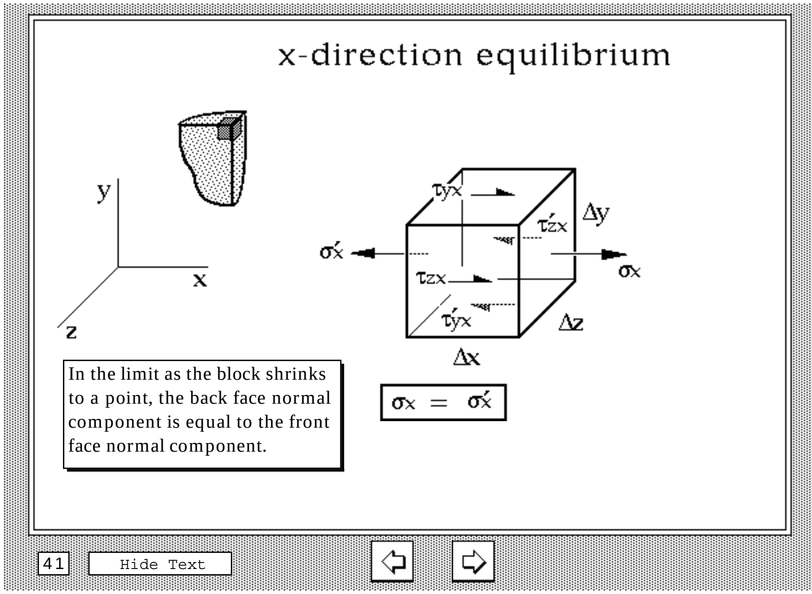
Only two terms remain. Dividing through by ΔyΔz gives the desired result.

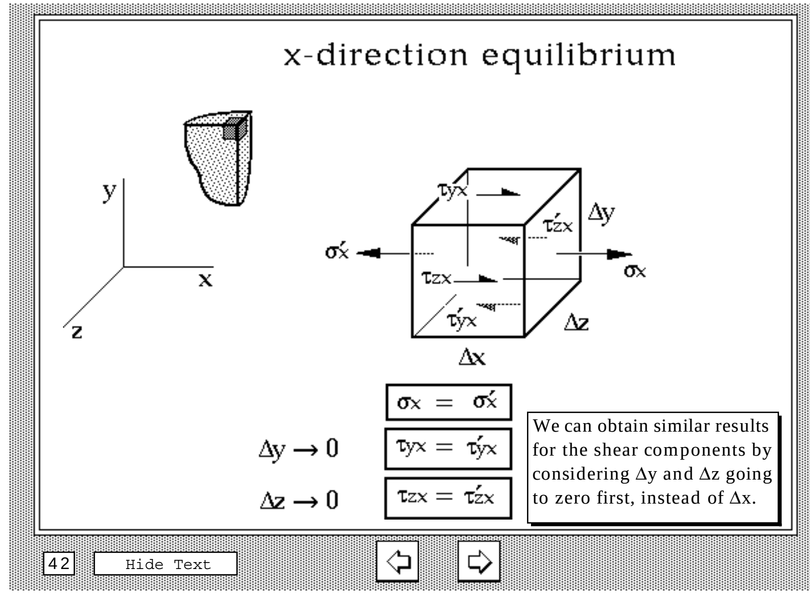
$$\Sigma F_x = 0$$

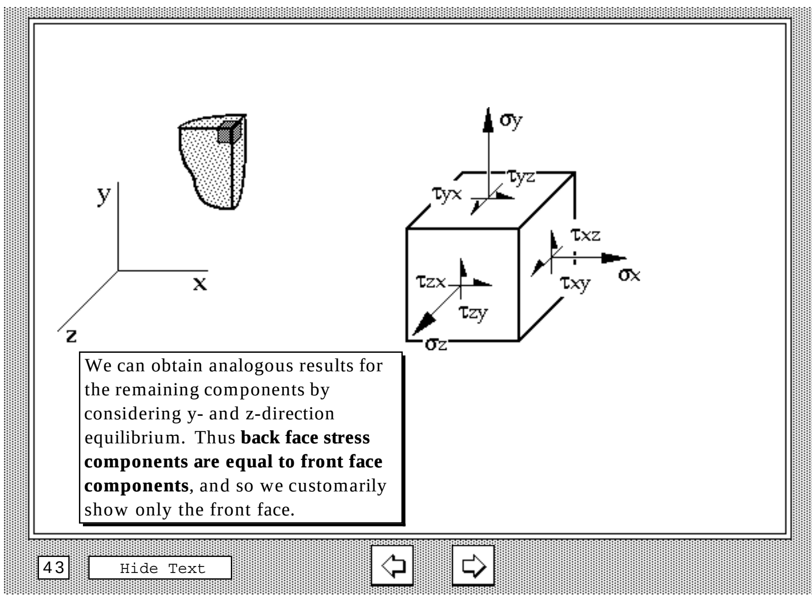
$$\sigma_x \Delta y \Delta z - [\sigma'_x \Delta y \Delta z] = 0$$

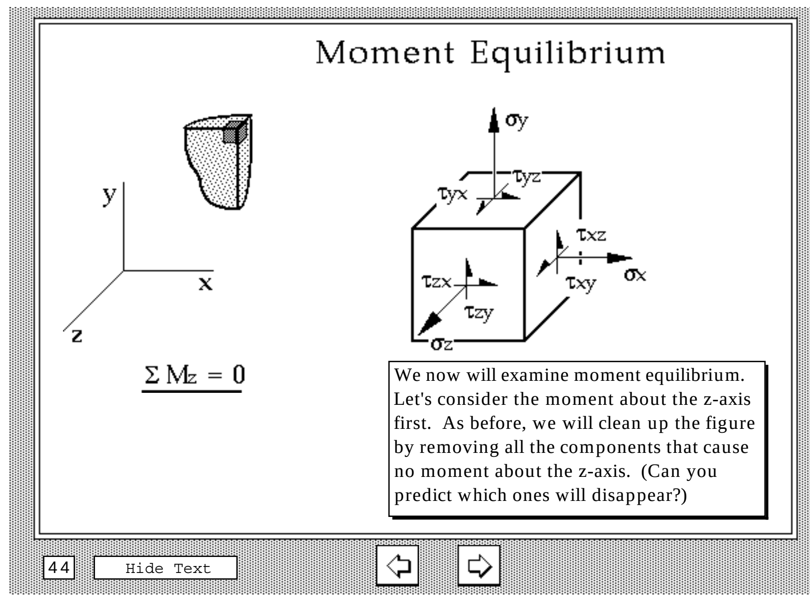
$$\Delta x \rightarrow 0$$

40 Hide Text

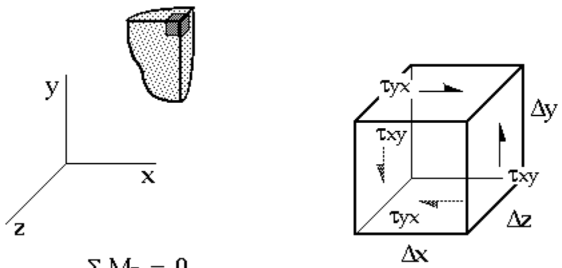

x-direction equilibrium
y
x
z
τyx
τ′zx
Δy
σ′x
σx
τzx
τ′yx
Δz
Δx
In the limit as the block shrinks to a point, the back face normal component is equal to the front face normal component.
σx = σ′x
41
Hide Text


x-direction equilibrium
y
x
z
τyx
τ′zx
Δy
σ′x
σx
τzx
τ′yx
Δz
Δx
σx = σ′x
Δy → 0
τyx = τ′yx
Δz → 0
τzx = τ′zx
We can obtain similar results for the shear components by considering Δy and Δz going to zero first, instead of Δx.
42
Hide Text


y
x
z
σy
τyz
τyx
τxz
σx
τzx
τxy
τzy
σz
We can obtain analogous results for the remaining components by considering y- and z-direction equilibrium. Thus back face stress components are equal to front face components, and so we customarily show only the front face.
43
Hide Text


Moment Equilibrium
y
x
z
σy
τyz
τyx
τxz
σx
τzx
τxy
τzy
σz
Σ Mz = 0
We now will examine moment equilibrium. Let's consider the moment about the z-axis first. As before, we will clean up the figure by removing all the components that cause no moment about the z-axis. (Can you predict which ones will disappear?)
44
Hide Text

Moment Equilibrium



$\Sigma M_z = 0$

Only the shear components shown cause moments about the z-axis. To compute these moments we must multiply the stress components by their area to obtain forces, and then multiply these forces by their moment arms to obtain moments.

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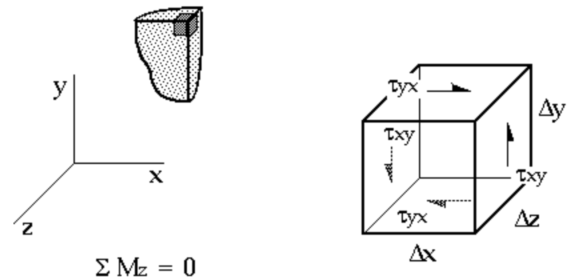
Moment Equilibrium



$\Sigma M_z = 0$

$$(\tau_{xy}\,\Delta y\,\Delta z)\,\Delta x - (\tau_{yx}\,\Delta x\,\Delta z)\,\Delta y = 0$$

Note the pattern of (stress x area) x moment arm.

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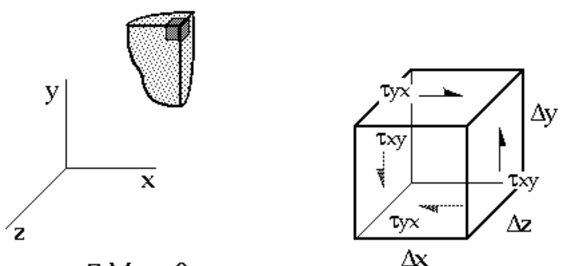
Moment Equilibrium



$\Sigma M_z = 0$

$$(\tau_{xy}\,\Delta y\,\Delta z)\,\Delta x - (\tau_{yx}\,\Delta x\,\Delta z)\,\Delta y = 0$$

$$(\tau_{xy} - \tau_{yx})\,\Delta x\,\Delta z\,\Delta y = 0$$

We can factor out the volume of the cube.

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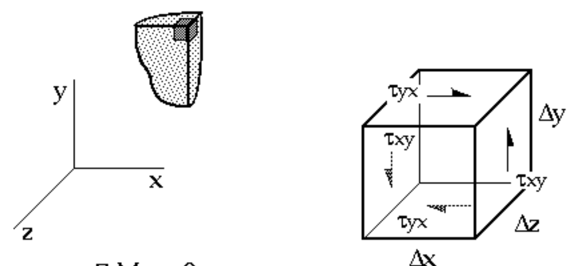
Moment Equilibrium



$\Sigma M_z = 0$

$$(\tau_{xy}\,\Delta y\,\Delta z)\,\Delta x - (\tau_{yx}\,\Delta x\,\Delta z)\,\Delta y = 0$$

$$(\tau_{xy} - \tau_{yx})\,\Delta x\,\Delta z\,\Delta y = 0$$

From which we obtain this important result.

$$\tau_{xy} = \tau_{yx}$$

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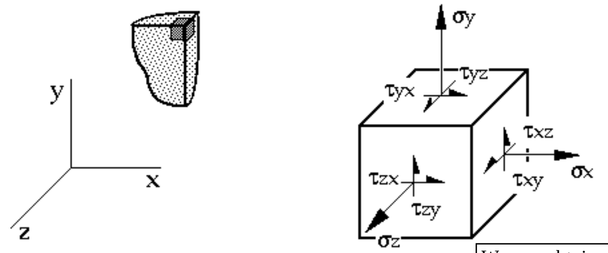
Moment Equilibrium



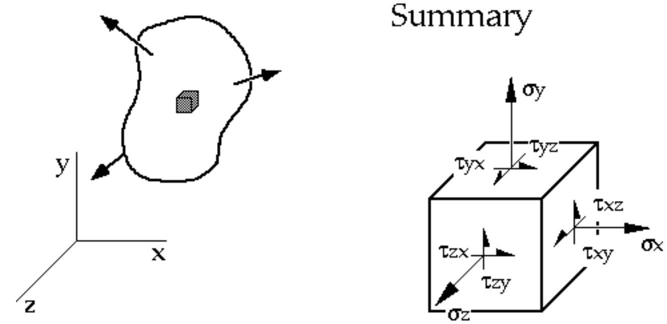
$\Sigma M_z = 0 \rightarrow$ $\tau_{xy} = \tau_{yx}$

$\Sigma M_y = 0 \rightarrow$ $\tau_{xz} = \tau_{zx}$

$\Sigma M_x = 0 \rightarrow$ $\tau_{zy} = \tau_{yz}$

We can obtain similar results by taking moments about the y- and x-axes. These are fundamental results. They show that the shear stress components are not independent.

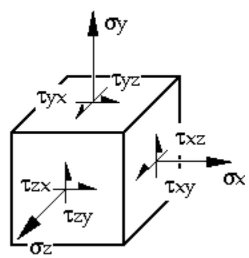
Summary



To describe the stress at a point, we can imagine an infinitesimal cube of material surrounding the point. The stress state is characterized by the stress components on the three orthogonal faces of the cube, and these stress components are given by six independent numbers.

Summary

Stress is a sufficiently complex animal that it takes many components to describe it. To describe a **scalar** quantity like temperature requires only one component. To describe a **vector** quantity like force requires three components. Stress is a **tensor** quantity, and we can see that it takes six components to describe it. These components are often written in matrix form as shown, and since $\tau_{xy} = \tau_{yx}$, etc. the resulting matrix is symmetric.



$$\text{Stress Tensor} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The End