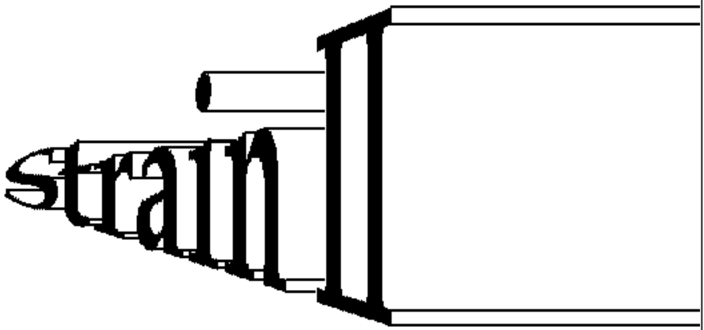
Just when you thought it was safe to go back in the water.......



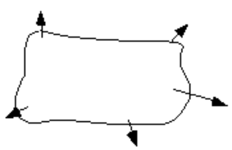
1 Hide Text

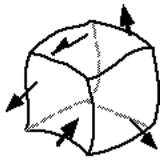
Stack Contents

- ☐ Strain at a Point
- ☐ 2D Strain
- ☐ 3D Strain
- ☐ Strain as an Ellipsoid

2 Hide Text

A General Description of Strain





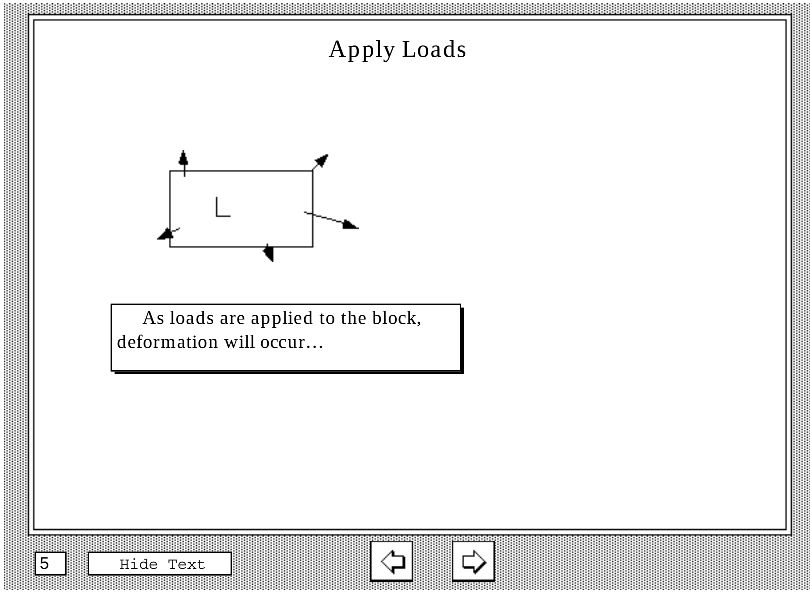
In this stack we will develop a general characterization of strain in two and three dimensions. As in the case of stress, we will find that to completely describe strain we use tensors.

3 Hide Text

Finite Reference Lines

Consider a block of material with two reference lines marked on it as shown.

4 Hide Text


Apply Loads
As loads are applied to the block, deformation will occur...
5
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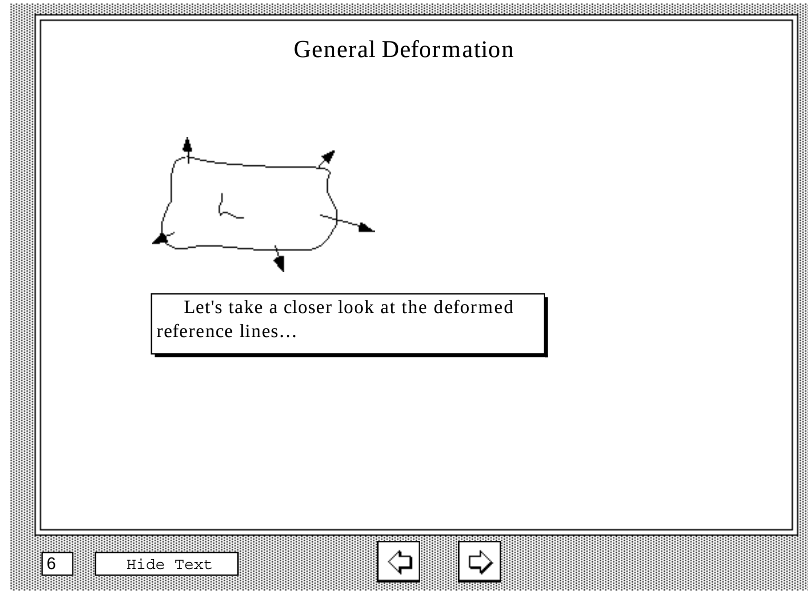
General Deformation
Let's take a closer look at the deformed reference lines...
6
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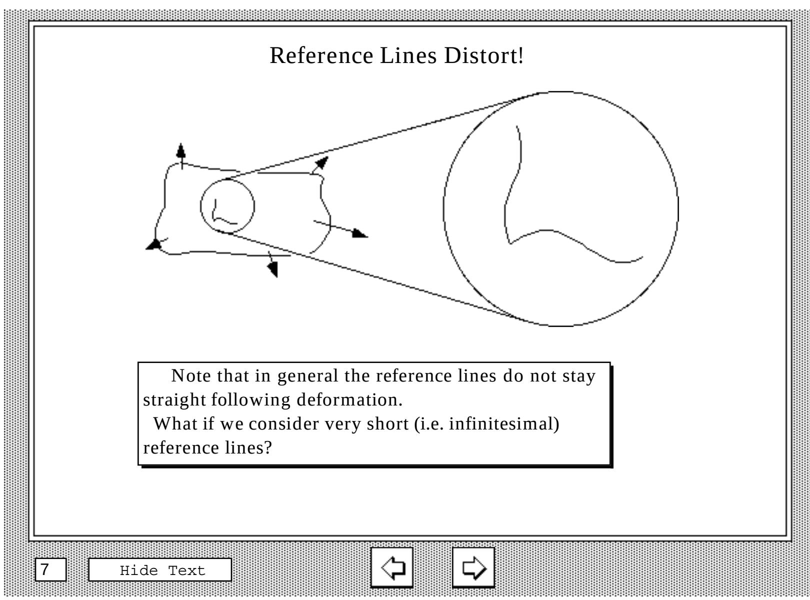
Reference Lines Distort!
Note that in general the reference lines do not stay straight following deformation.
What if we consider very short (i.e. infinitesimal) reference lines?
7
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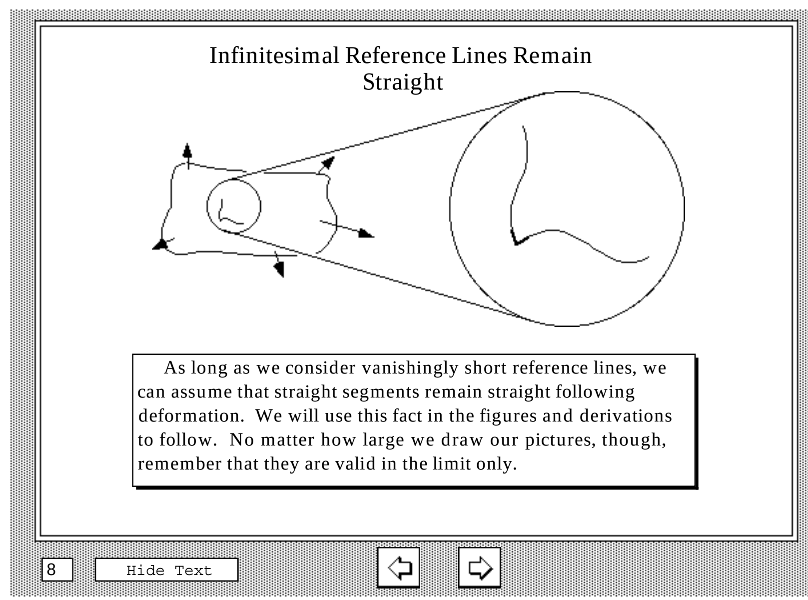
Infinitesimal Reference Lines Remain Straight
As long as we consider vanishingly short reference lines, we can assume that straight segments remain straight following deformation. We will use this fact in the figures and derivations to follow. No matter how large we draw our pictures, though, remember that they are valid in the limit only.
8
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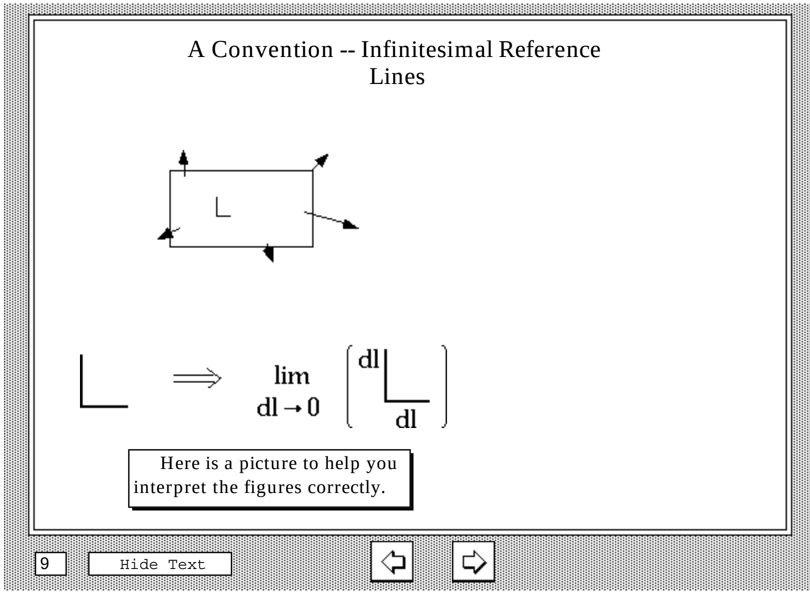
A Convention -- Infinitesimal Reference Lines
lim
dl → 0
dl
dl
Here is a picture to help you interpret the figures correctly.
9
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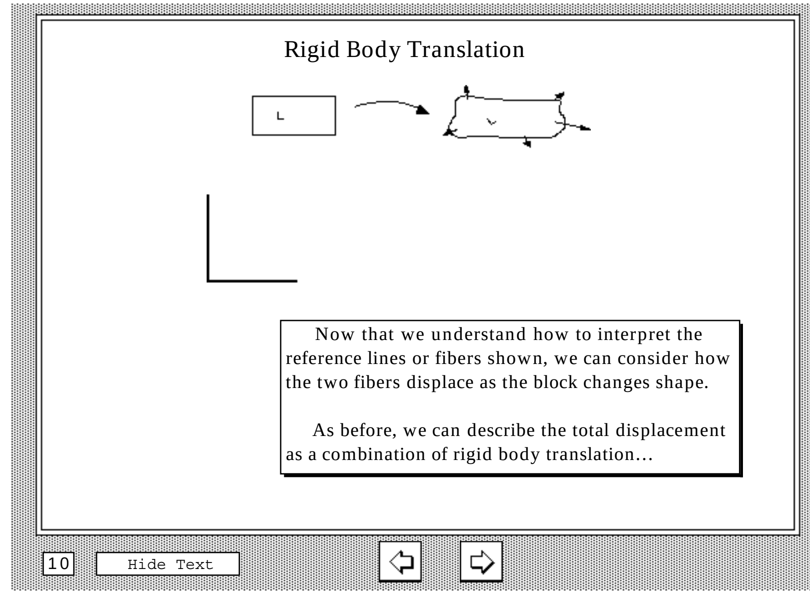
Rigid Body Translation
Now that we understand how to interpret the reference lines or fibers shown, we can consider how the two fibers displace as the block changes shape.
As before, we can describe the total displacement as a combination of rigid body translation…
10
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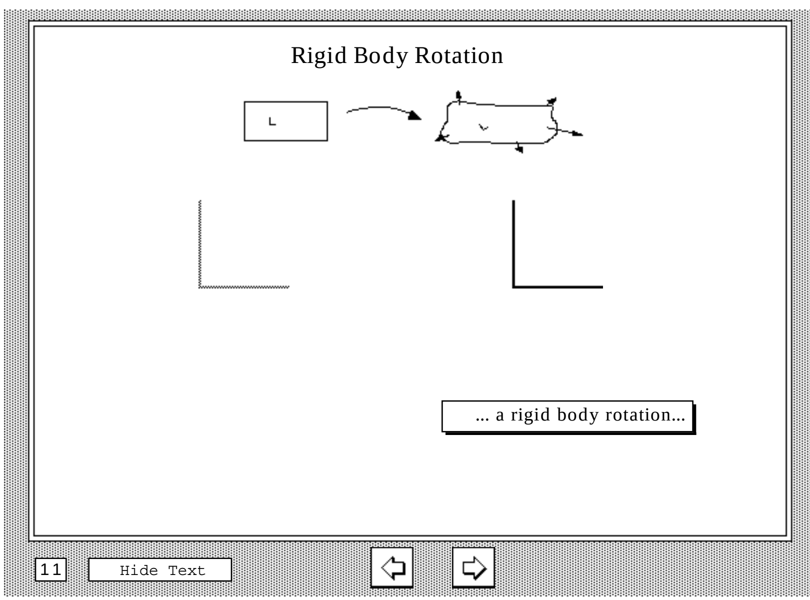
Rigid Body Rotation
... a rigid body rotation...
11
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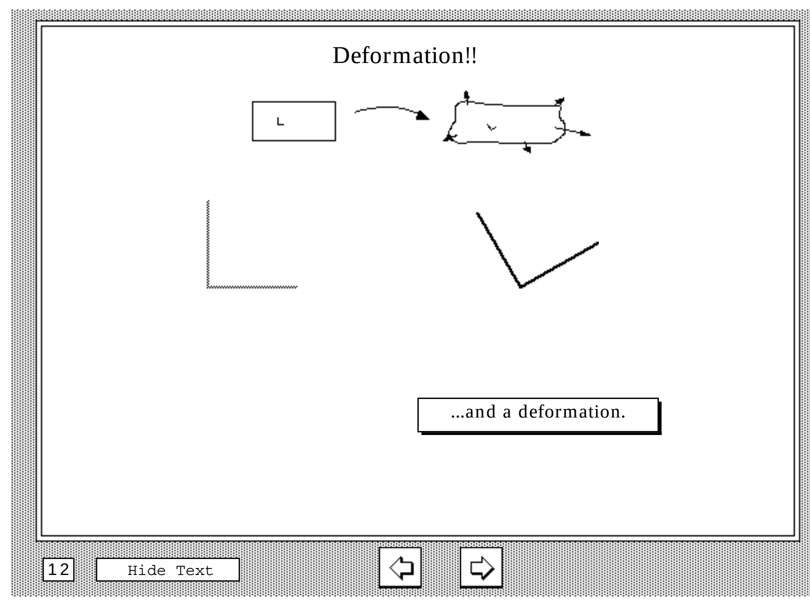
Deformation!!
...and a deformation.
12
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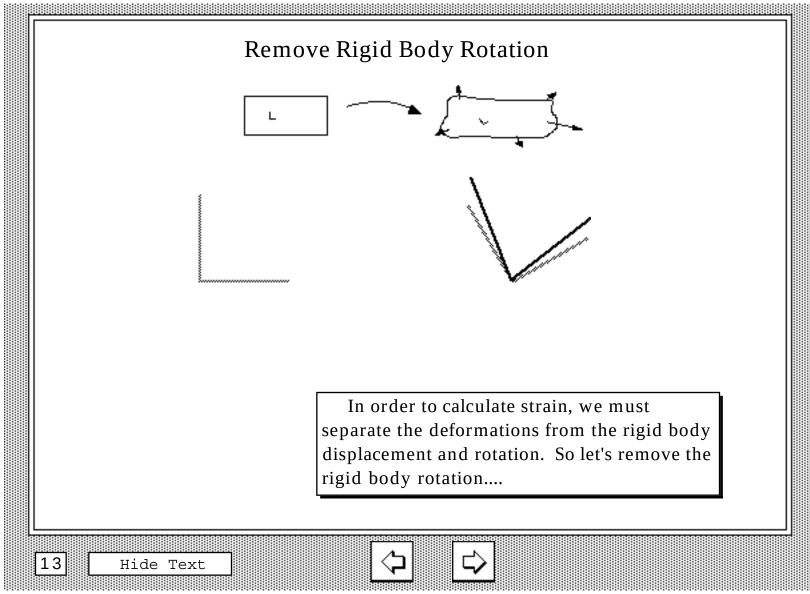
Remove Rigid Body Rotation
In order to calculate strain, we must separate the deformations from the rigid body displacement and rotation. So let's remove the rigid body rotation....
13
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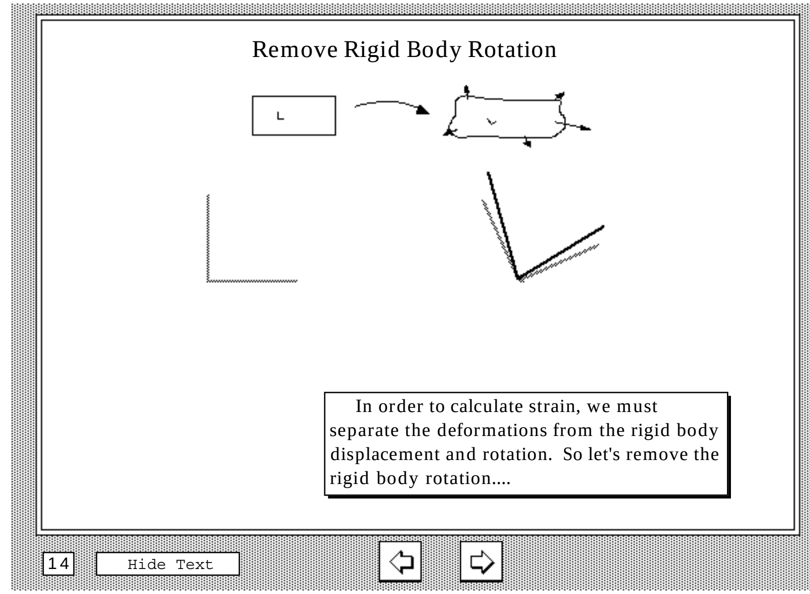
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14
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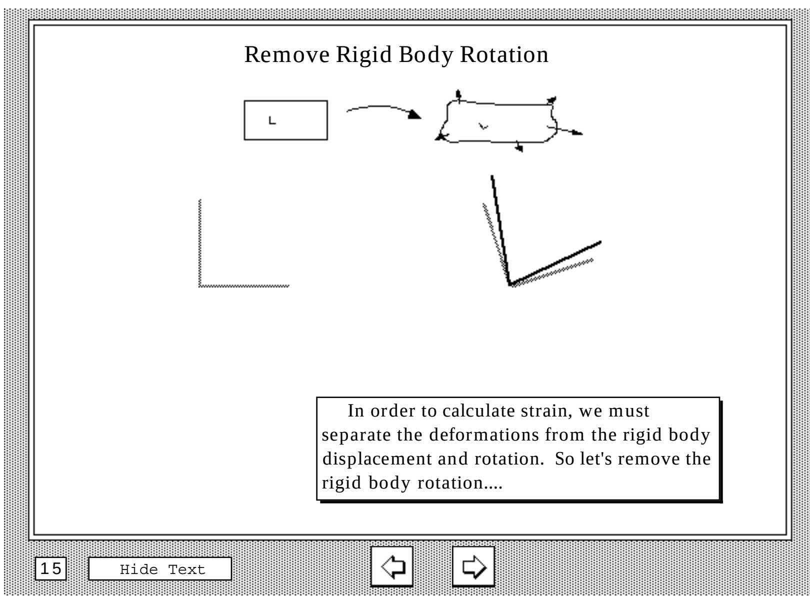
Remove Rigid Body Rotation
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15
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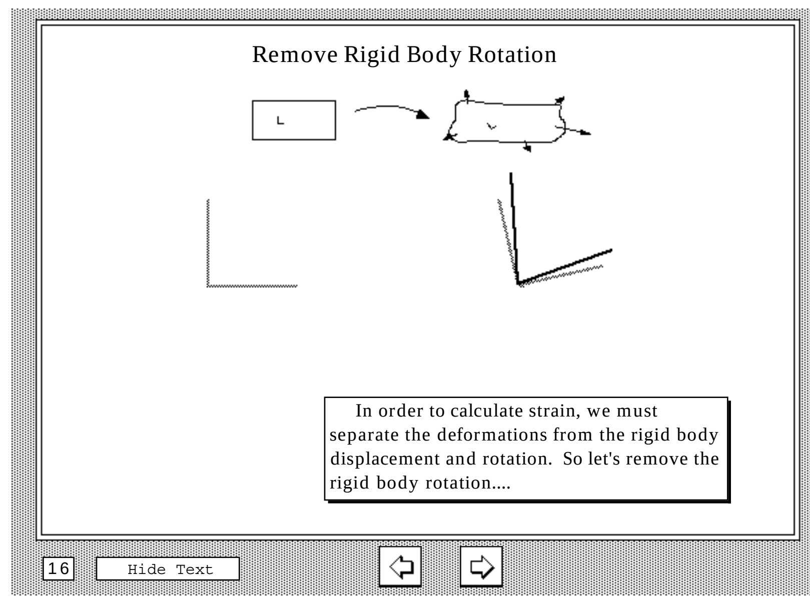
Remove Rigid Body Rotation
In order to calculate strain, we must separate the deformations from the rigid body displacement and rotation. So let's remove the rigid body rotation....
16
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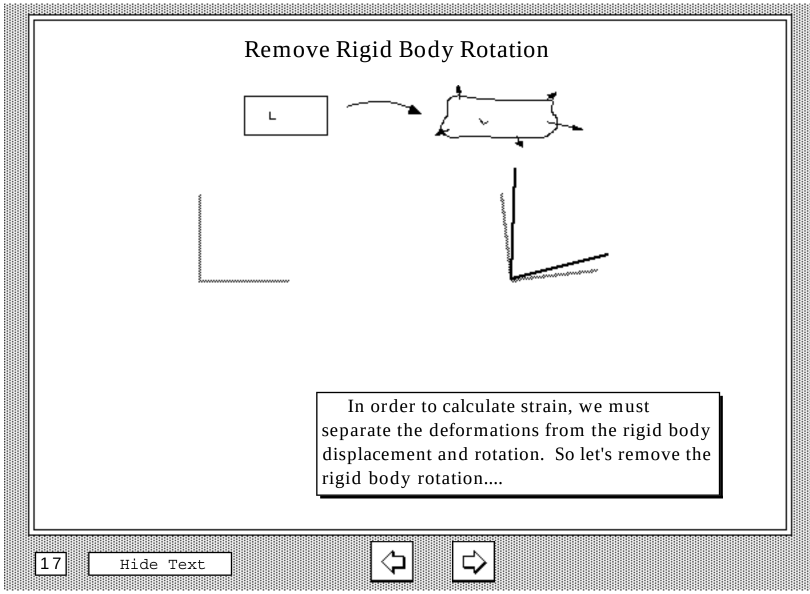
Remove Rigid Body Rotation
In order to calculate strain, we must separate the deformations from the rigid body displacement and rotation. So let's remove the rigid body rotation....
17
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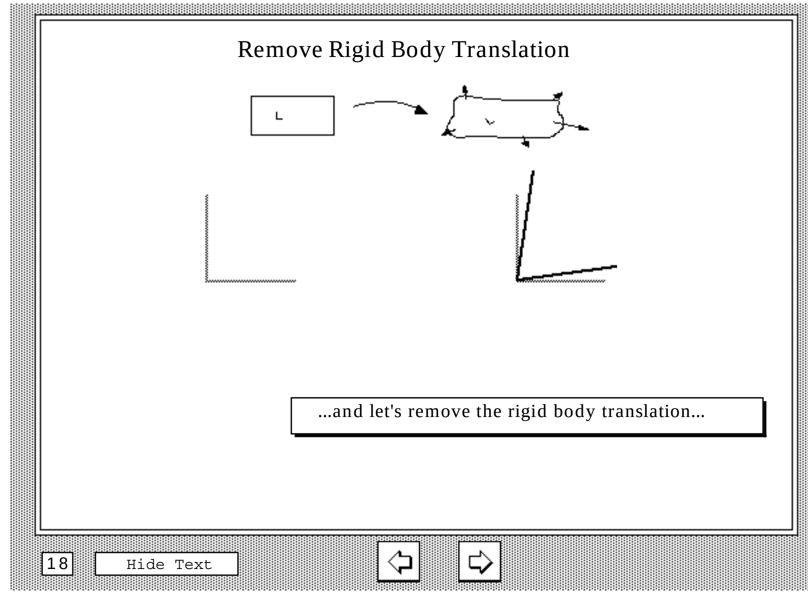
Remove Rigid Body Translation
...and let's remove the rigid body translation...
18
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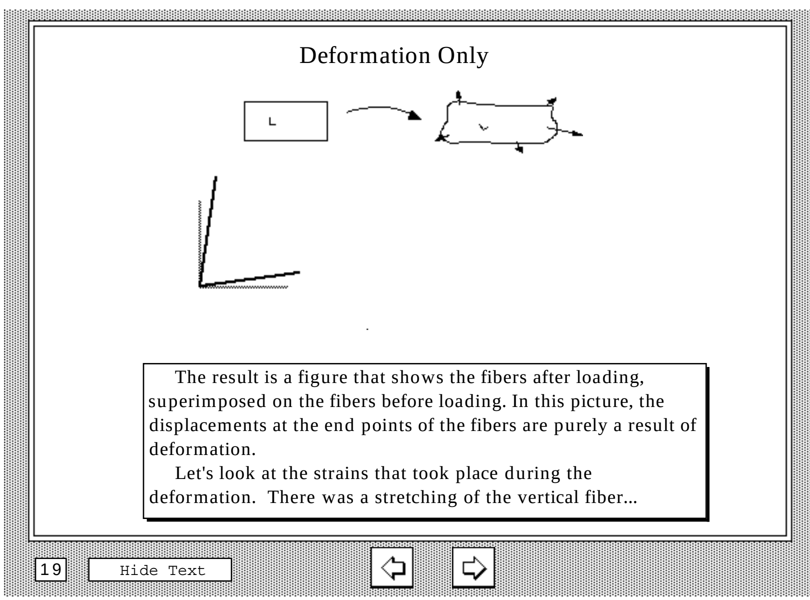
Deformation Only
The result is a figure that shows the fibers after loading, superimposed on the fibers before loading. In this picture, the displacements at the end points of the fibers are purely a result of deformation.
Let's look at the strains that took place during the deformation. There was a stretching of the vertical fiber...
19
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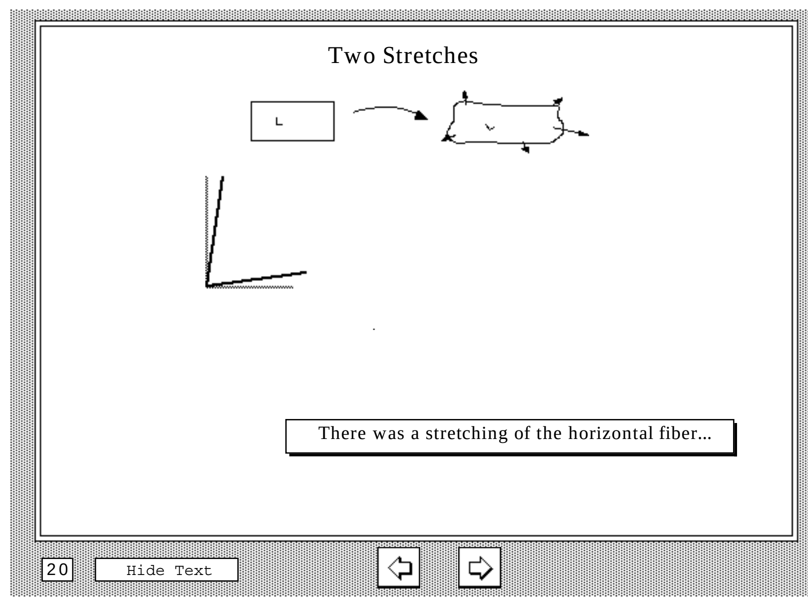
Two Stretches
There was a stretching of the horizontal fiber...
20
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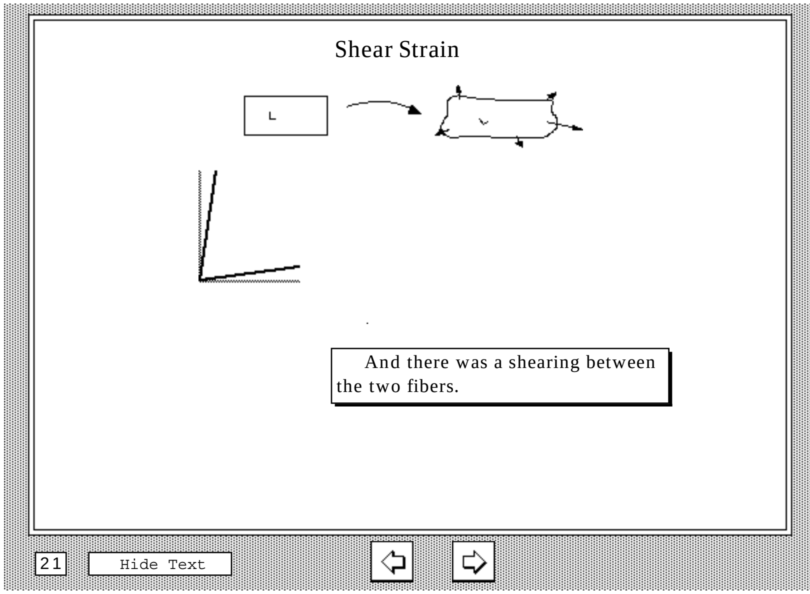
Shear Strain
And there was a shearing between the two fibers.
21
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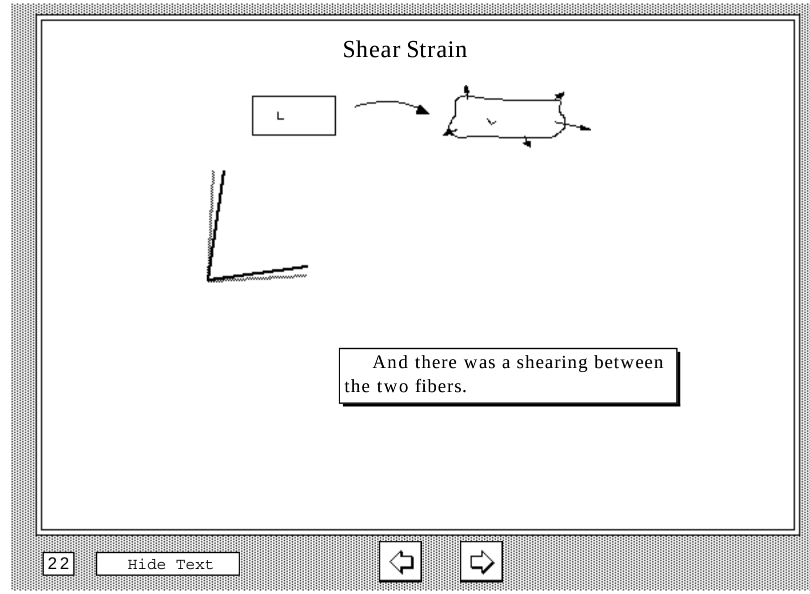
Shear Strain
And there was a shearing between the two fibers.
22
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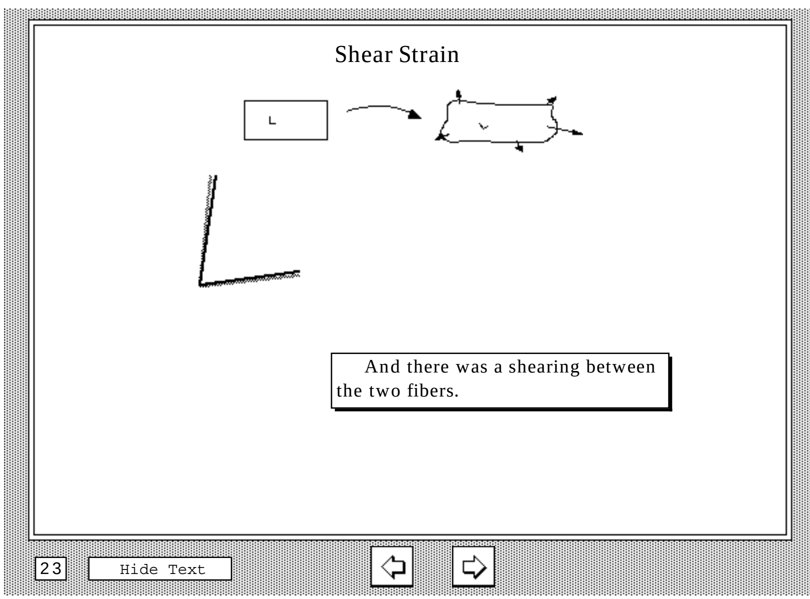
Shear Strain
And there was a shearing between the two fibers.
23
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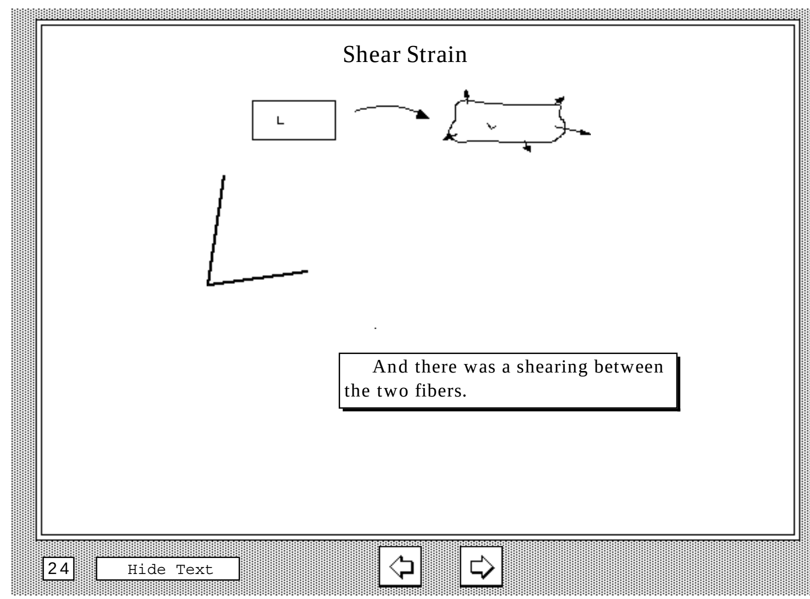
Shear Strain
And there was a shearing between the two fibers.
24
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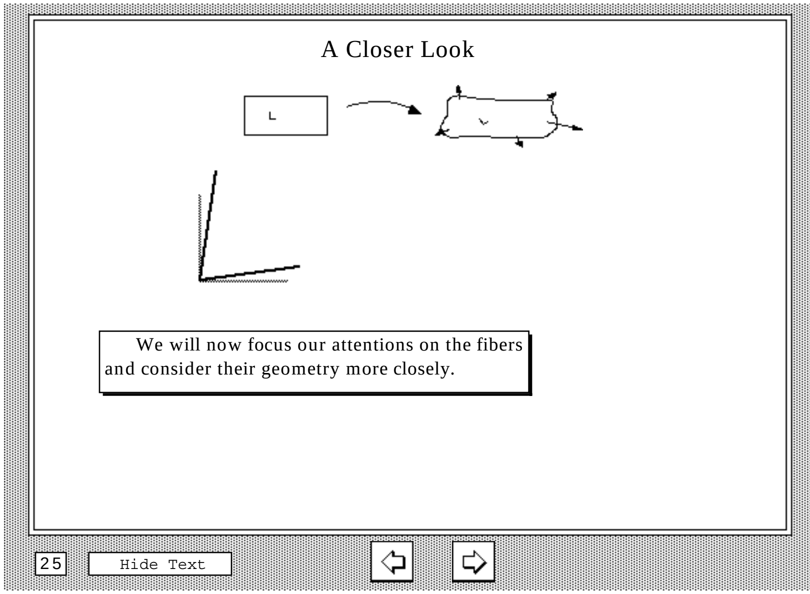
A Closer Look
We will now focus our attentions on the fibers and consider their geometry more closely.
25
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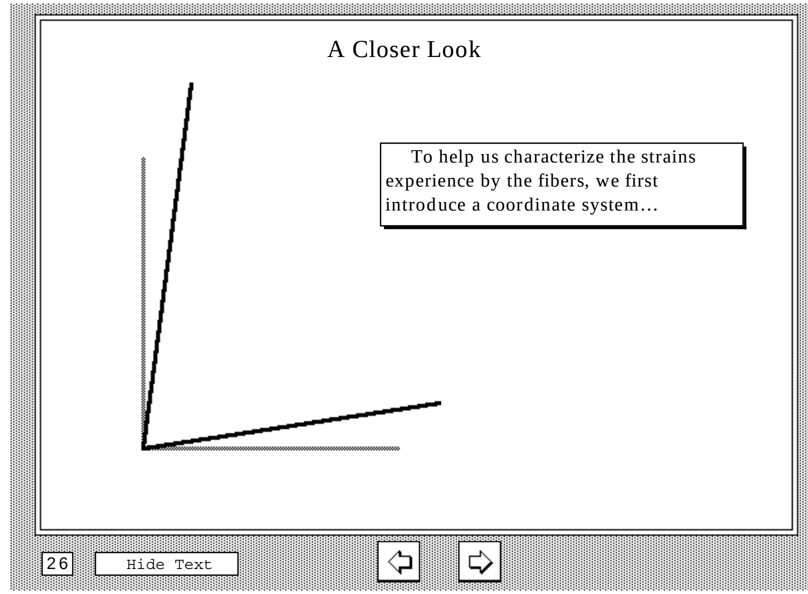
A Closer Look
To help us characterize the strains experience by the fibers, we first introduce a coordinate system…
26
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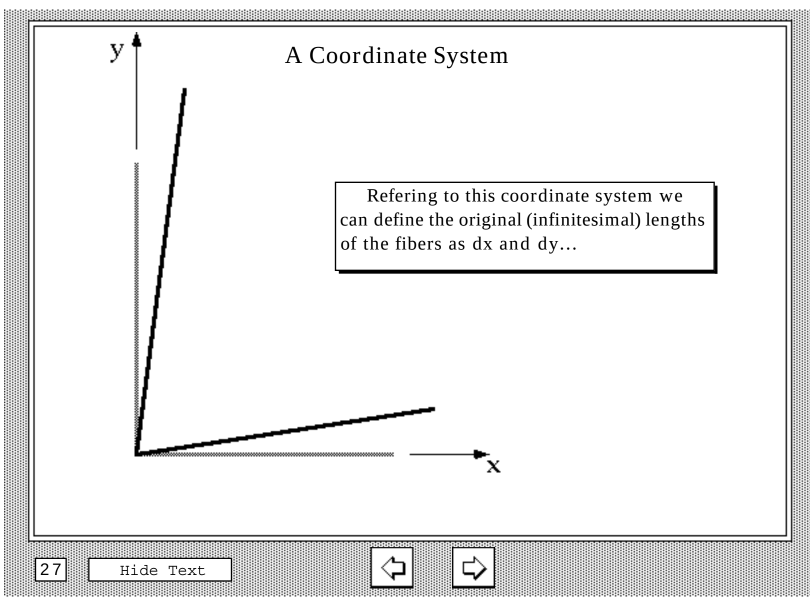
A Coordinate System
y
x
Refering to this coordinate system we can define the original (infinitesimal) lengths of the fibers as dx and dy…
27
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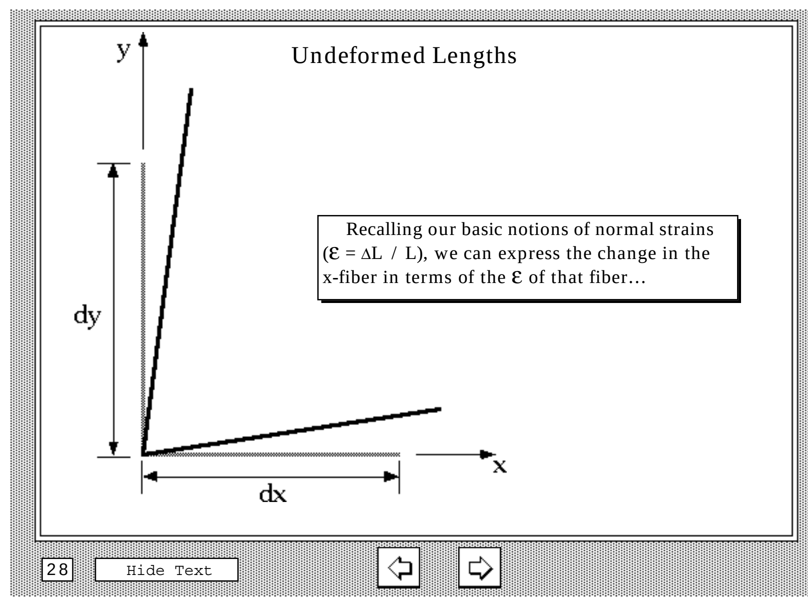
Undeformed Lengths
y
dy
x
dx
Recalling our basic notions of normal strains (ε = ΔL / L), we can express the change in the x-fiber in terms of the ε of that fiber…
28
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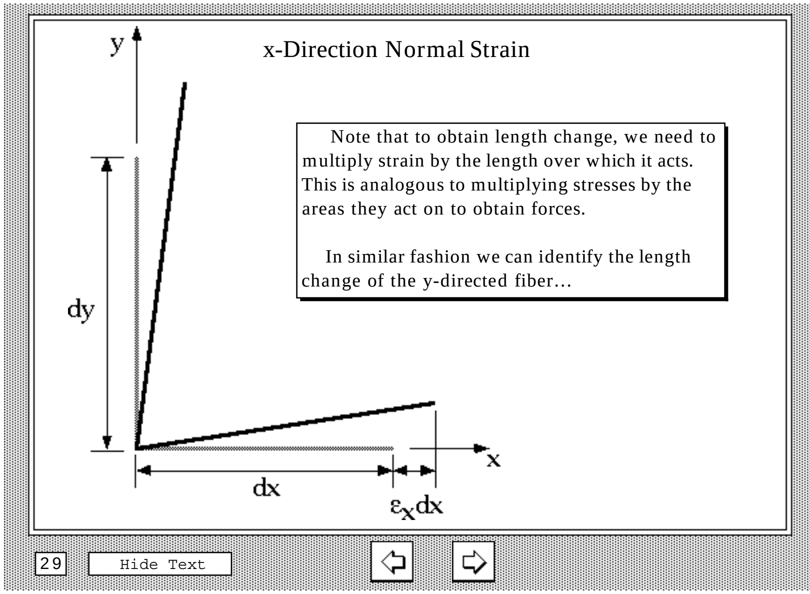
x-Direction Normal Strain
Note that to obtain length change, we need to multiply strain by the length over which it acts. This is analogous to multiplying stresses by the areas they act on to obtain forces.
In similar fashion we can identify the length change of the y-directed fiber...
y
x
dy
dx
$\varepsilon_x dx$
29
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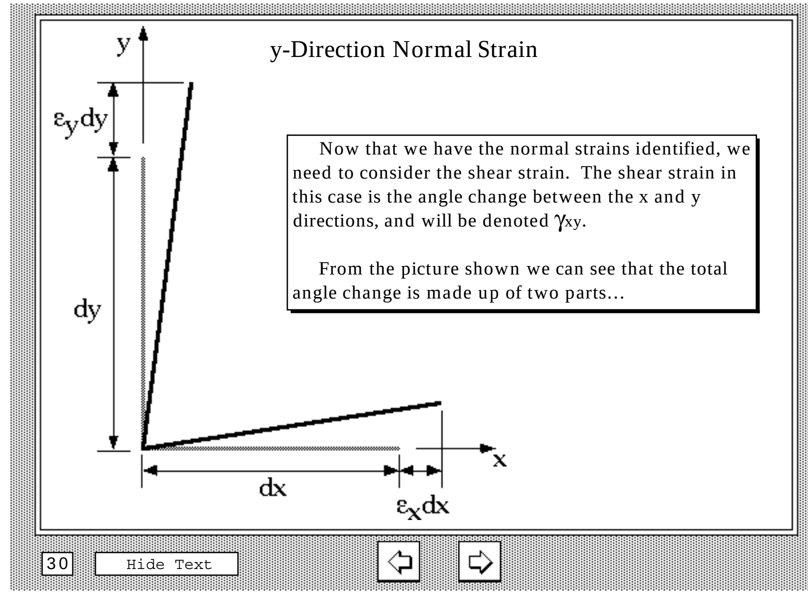
y-Direction Normal Strain
Now that we have the normal strains identified, we need to consider the shear strain. The shear strain in this case is the angle change between the x and y directions, and will be denoted γ_{xy}.
From the picture shown we can see that the total angle change is made up of two parts...
y
x
$\varepsilon_y dy$
dy
dx
$\varepsilon_x dx$
30
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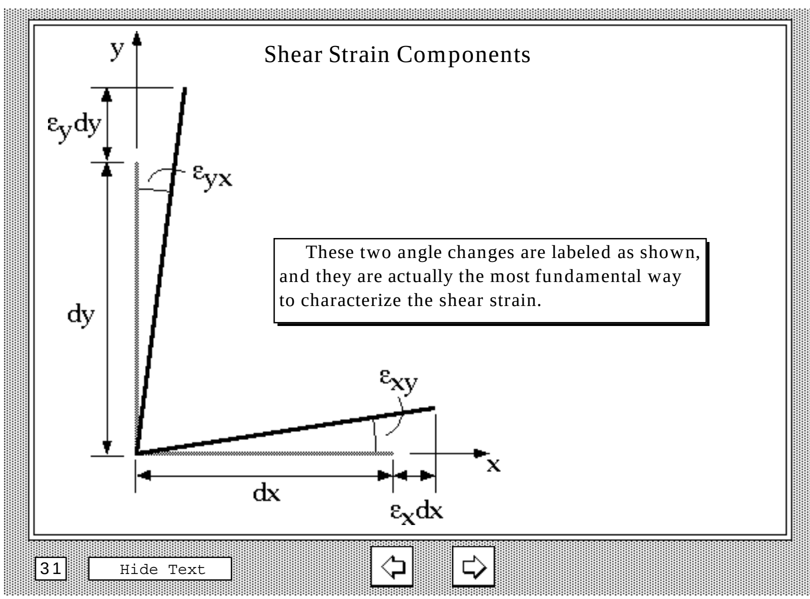
Shear Strain Components
These two angle changes are labeled as shown, and they are actually the most fundamental way to characterize the shear strain.
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
31
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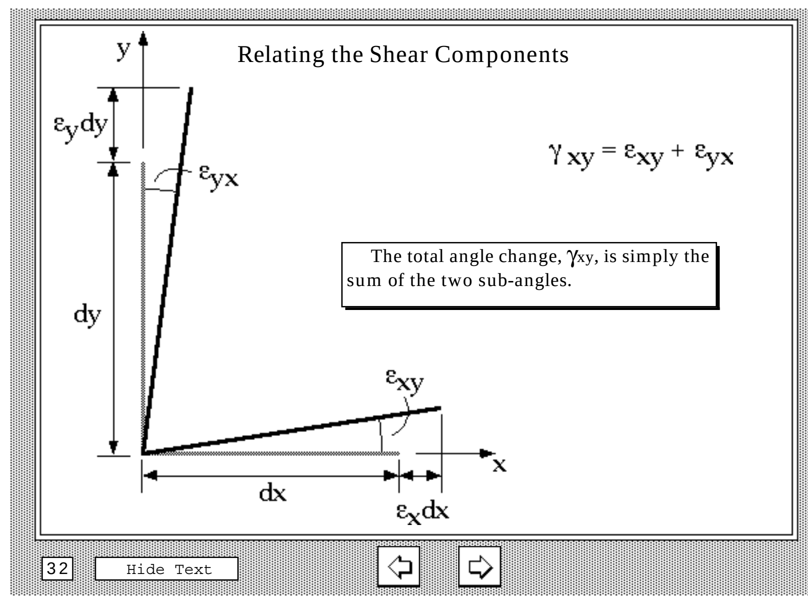
Relating the Shear Components
$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$
The total angle change, γ_{xy}, is simply the sum of the two sub-angles.
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
32
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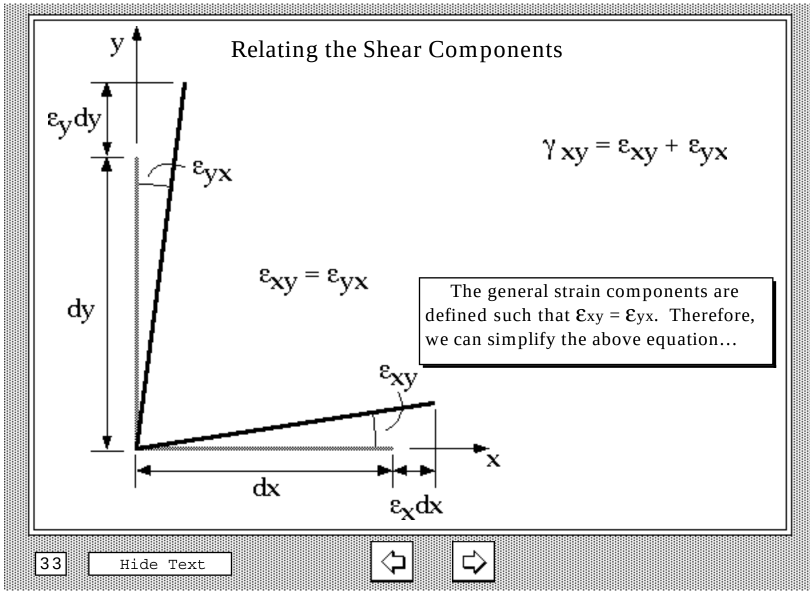
Relating the Shear Components
$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$
$\varepsilon_{xy} = \varepsilon_{yx}$
The general strain components are defined such that $\varepsilon_{xy} = \varepsilon_{yx}$. Therefore, we can simplify the above equation…
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
33
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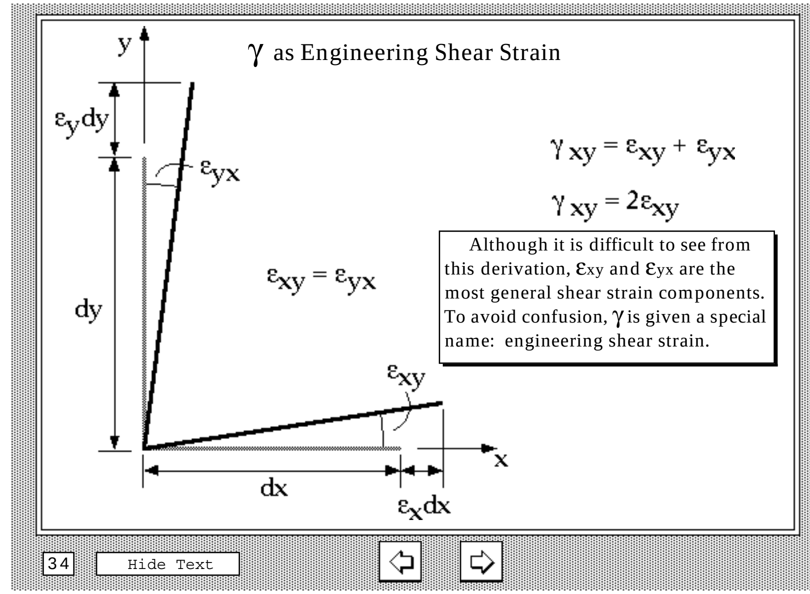
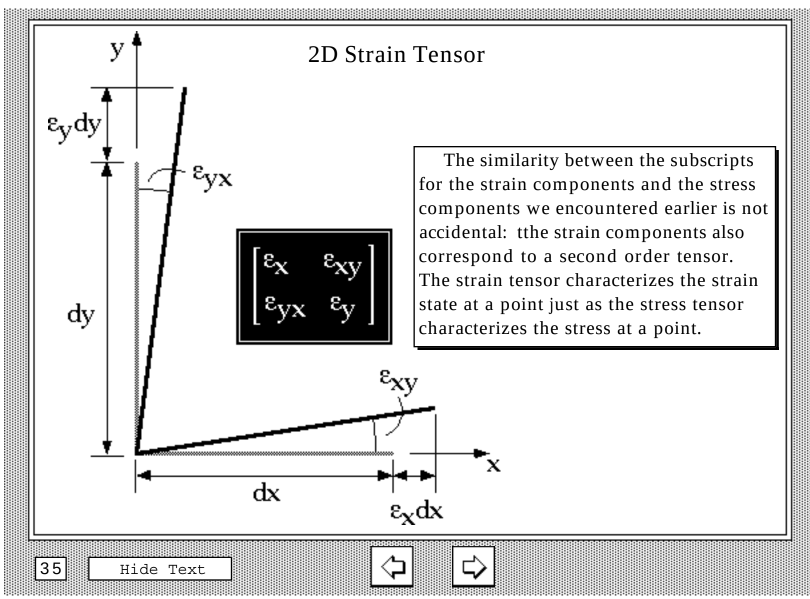
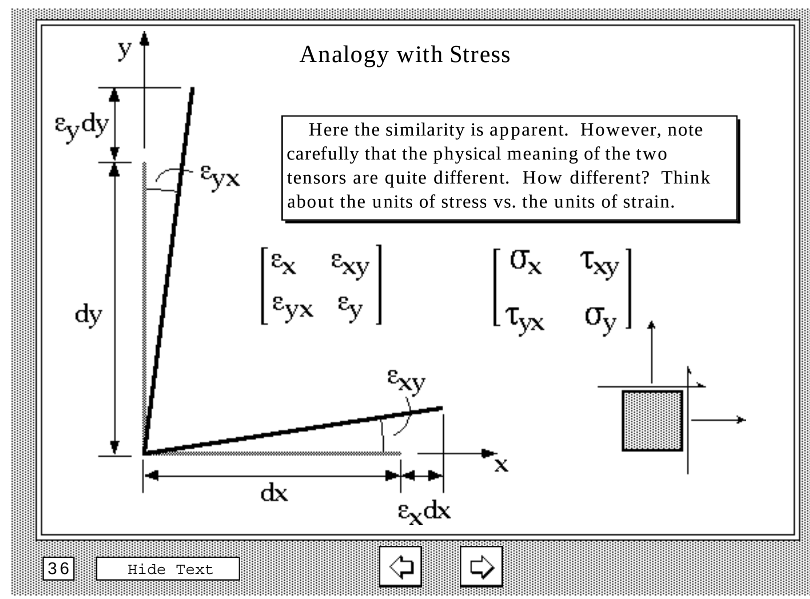

γ as Engineering Shear Strain
$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$
$\gamma_{xy} = 2\varepsilon_{xy}$
$\varepsilon_{xy} = \varepsilon_{yx}$
Although it is difficult to see from this derivation, ε_{xy} and ε_{yx} are the most general shear strain components. To avoid confusion, γ is given a special name: engineering shear strain.
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
34
Hide Text


2D Strain Tensor
$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$
The similarity between the subscripts for the strain components and the stress components we encountered earlier is not accidental: tthe strain components also correspond to a second order tensor. The strain tensor characterizes the strain state at a point just as the stress tensor characterizes the stress at a point.
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
35
Hide Text


Analogy with Stress
Here the similarity is apparent. However, note carefully that the physical meaning of the two tensors are quite different. How different? Think about the units of stress vs. the units of strain.
$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$
$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$
y
x
$\varepsilon_y dy$
ε_{yx}
dy
ε_{xy}
dx
$\varepsilon_x dx$
36
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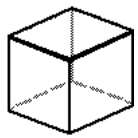
3D Strain

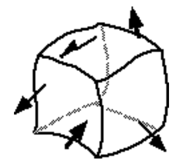


The previous discussion of strain was carried out for two dimensions. What happens in three dimensions? Let's see.

Imagine a cube of material which is loaded in many different directions.

37 Hide Text

3D Strain

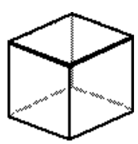


The loading in three dimensions will result in deformations in three dimensions, and, most likely, strain in three dimensions.

How can we characterize the strain at any point in the cube? We will certainly need a more complete description than we have developed for two dimensional strain states.

38 Hide Text

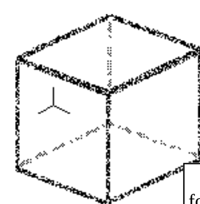
3D Strain



We begin our description of 3D Strain by focusing on three infinitesimally short fibers emanating from a single point in the cube.

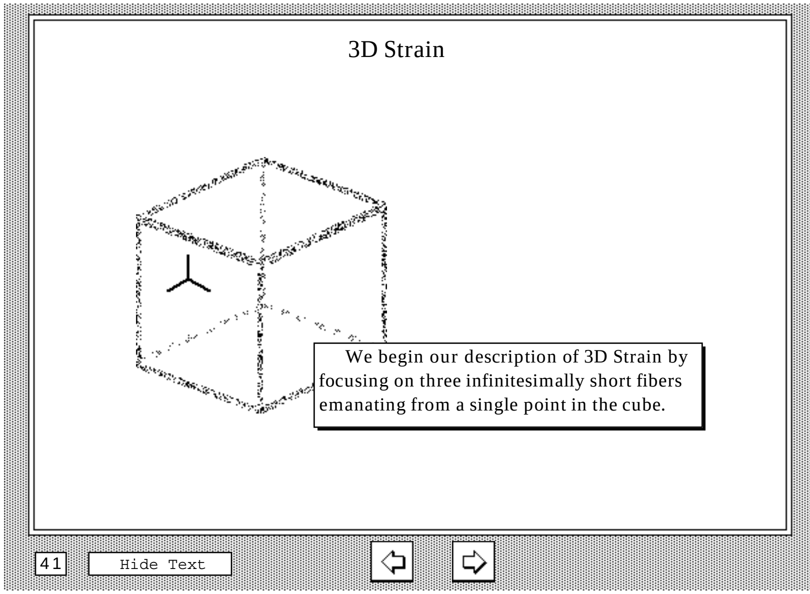
39 Hide Text

3D Strain



We begin our description of 3D Strain by focusing on three infinitesimally short fibers emanating from a single point in the cube.

40 Hide Text


3D Strain
We begin our description of 3D Strain by focusing on three infinitesimally short fibers emanating from a single point in the cube.
41
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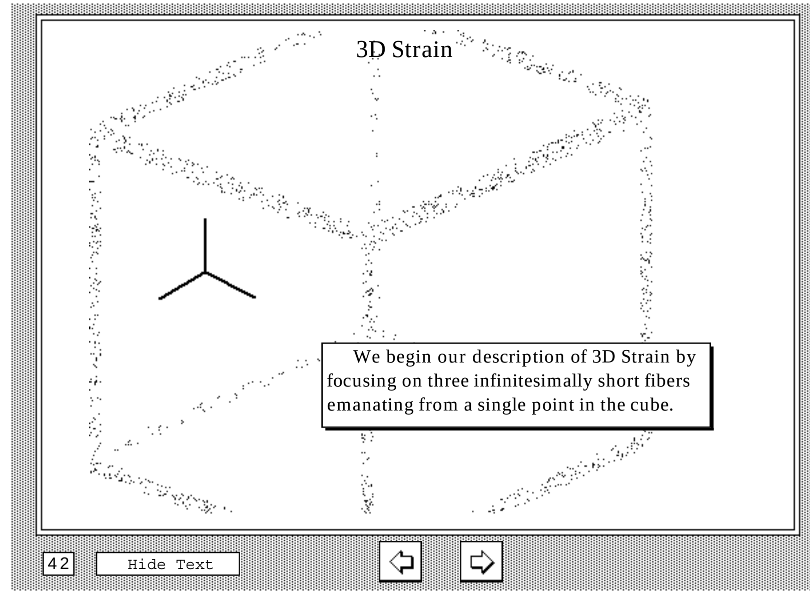
3D Strain
We begin our description of 3D Strain by focusing on three infinitesimally short fibers emanating from a single point in the cube.
42
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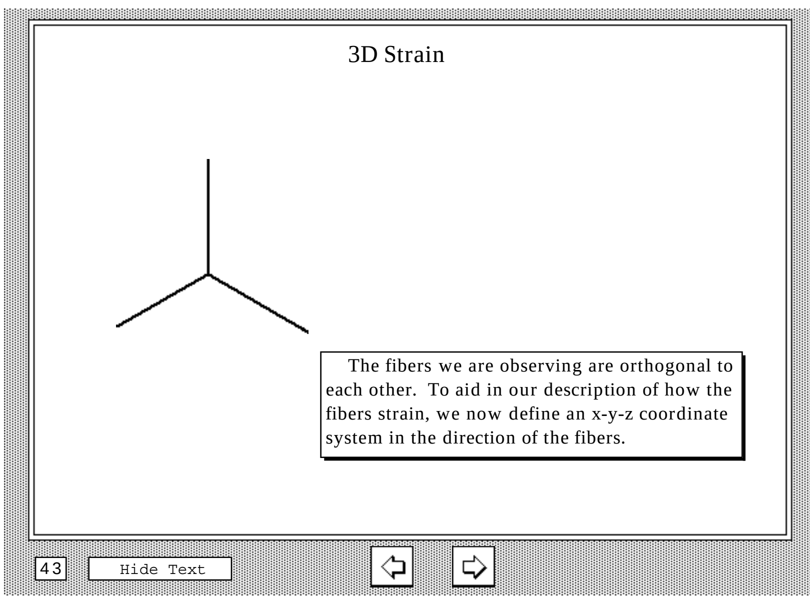
3D Strain
The fibers we are observing are orthogonal to each other. To aid in our description of how the fibers strain, we now define an x-y-z coordinate system in the direction of the fibers.
43
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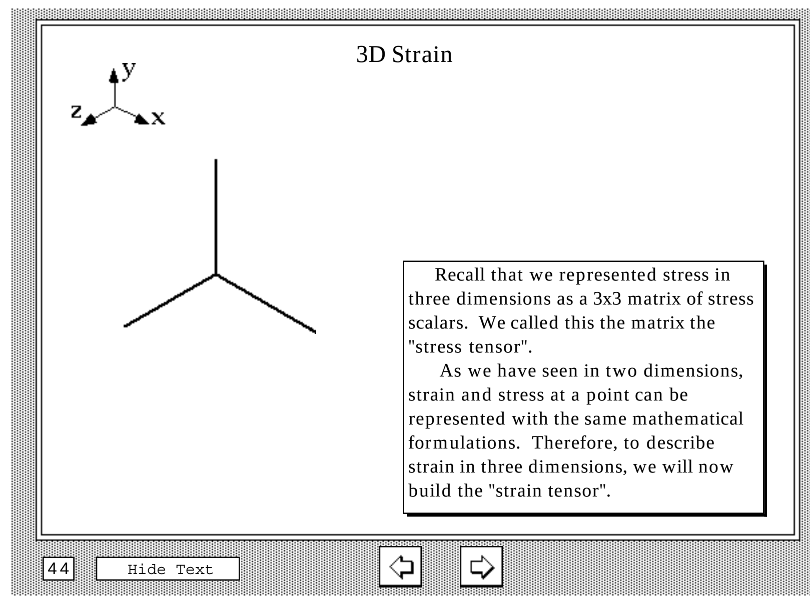
3D Strain
y
z
x
Recall that we represented stress in three dimensions as a 3x3 matrix of stress scalars. We called this the matrix the "stress tensor".
As we have seen in two dimensions, strain and stress at a point can be represented with the same mathematical formulations. Therefore, to describe strain in three dimensions, we will now build the "strain tensor".
44
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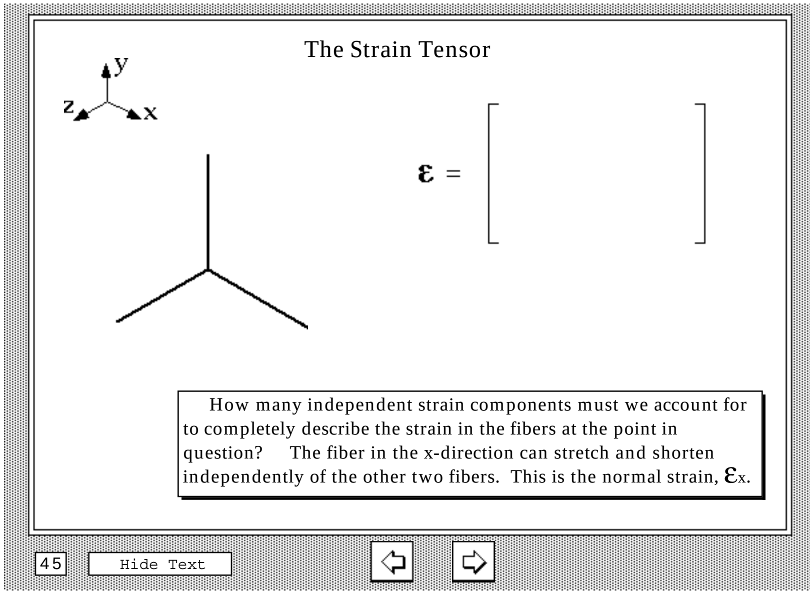
The Strain Tensor
y
z
x
ε =
How many independent strain components must we account for to completely describe the strain in the fibers at the point in question? The fiber in the x-direction can stretch and shorten independently of the other two fibers. This is the normal strain, ε_x.
45
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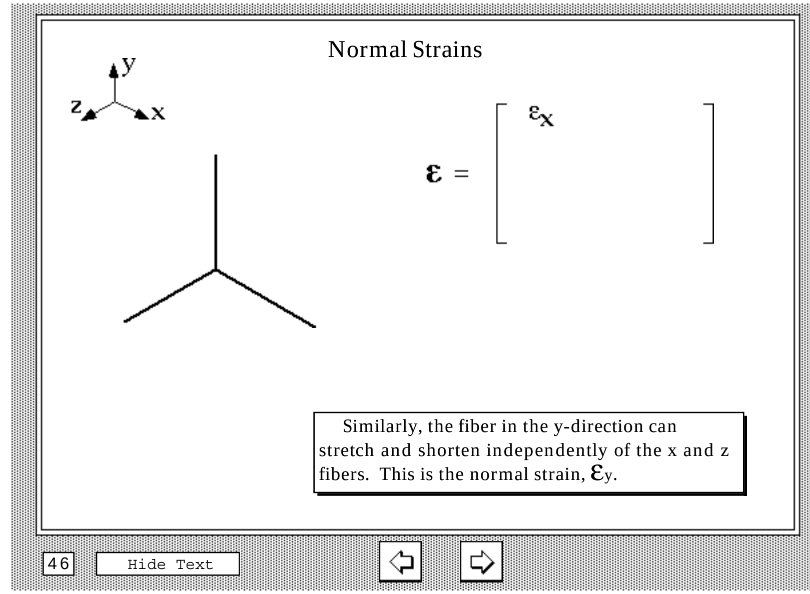
Normal Strains
y
z
x
$\varepsilon = \begin{bmatrix} \varepsilon_x & & \\ & & \\ & & \end{bmatrix}$
Similarly, the fiber in the y-direction can stretch and shorten independently of the x and z fibers. This is the normal strain, ε_y.
46
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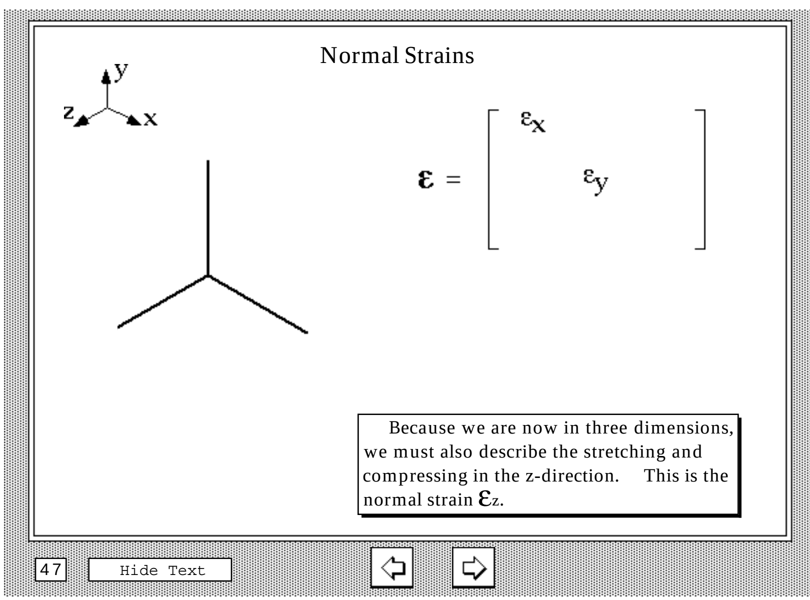
Normal Strains
y
z
x
$\varepsilon = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \end{bmatrix}$
Because we are now in three dimensions, we must also describe the stretching and compressing in the z-direction. This is the normal strain ε_z.
47
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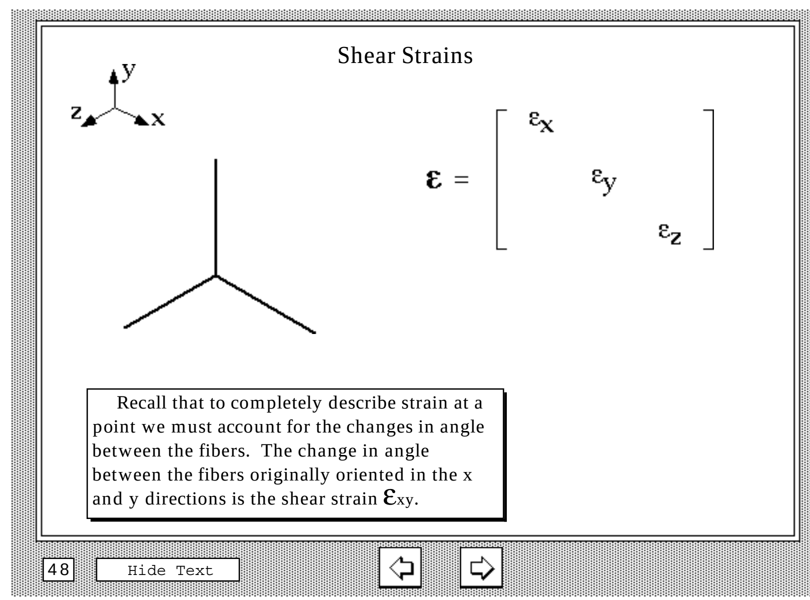
Shear Strains
y
z
x
$\varepsilon = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}$
Recall that to completely describe strain at a point we must account for the changes in angle between the fibers. The change in angle between the fibers originally oriented in the x and y directions is the shear strain ε_{xy}.
48
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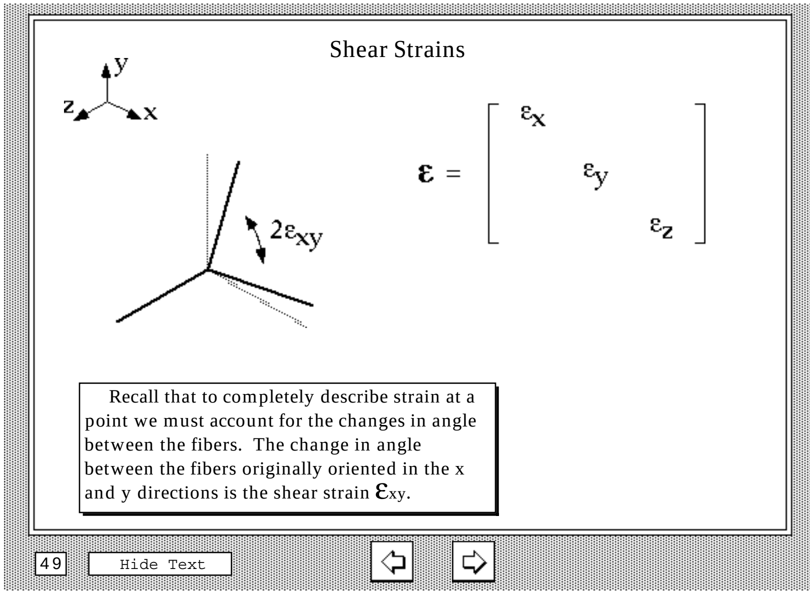
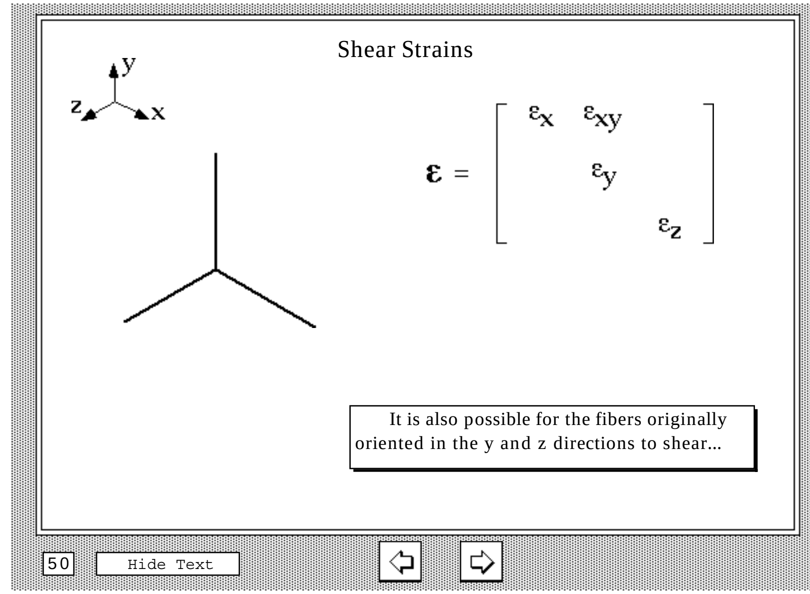

Shear Strains
y
z
x
$2\varepsilon_{xy}$
$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}$
Recall that to completely describe strain at a point we must account for the changes in angle between the fibers. The change in angle between the fibers originally oriented in the x and y directions is the shear strain ε_{xy}.
49
Hide Text


Shear Strains
y
z
x
$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}$
It is also possible for the fibers originally oriented in the y and z directions to shear...
50
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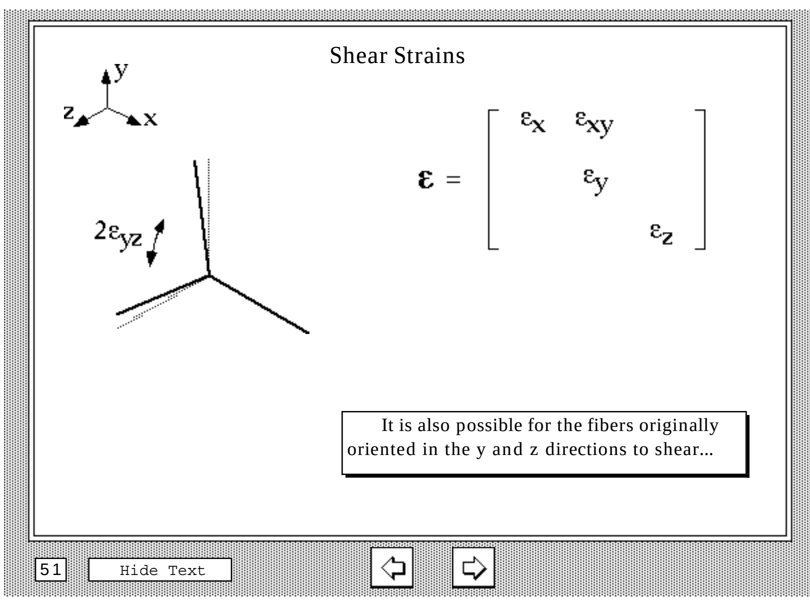
Shear Strains
y
z
x
$2\varepsilon_{yz}$
$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{bmatrix}$
It is also possible for the fibers originally oriented in the y and z directions to shear...
51
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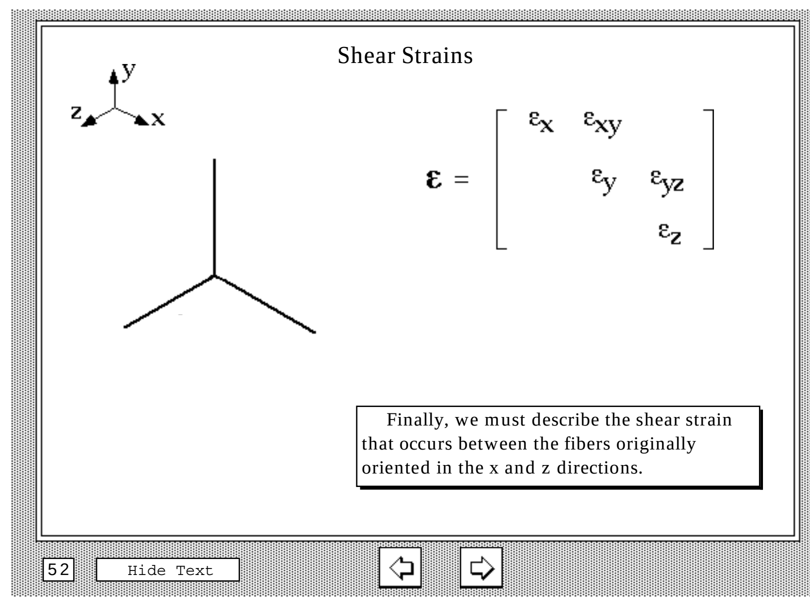
Shear Strains
y
z
x
$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \\ & \varepsilon_y & \varepsilon_{yz} \\ & & \varepsilon_z \end{bmatrix}$
Finally, we must describe the shear strain that occurs between the fibers originally oriented in the x and z directions.
52
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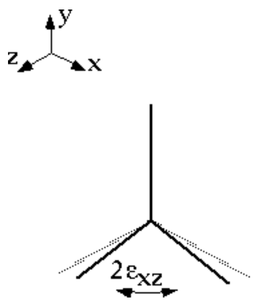
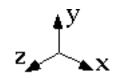
Shear Strains



$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \\ & \varepsilon_y & \varepsilon_{yz} \\ & & \varepsilon_z \end{bmatrix}$$

Finally, we must describe the shear strain that occurs between the fibers originally oriented in the x and z directions.

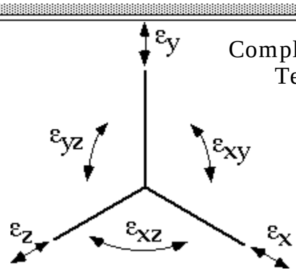
Shear Strains



$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ & & \varepsilon_z \end{bmatrix}$$

It turns out that like the 2D strain tensor, the 3D strain tensor is symmetric.

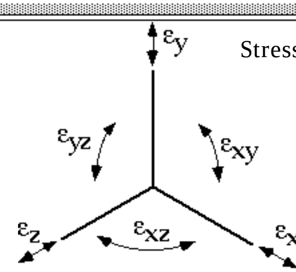
Complete Strain Tensor



$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

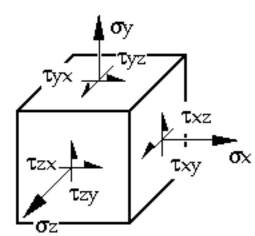
These nine components completely characterize the strain at a point. As we just have seen, the diagonal terms are stretches of the material (normal strains), and the off-diagonal terms are relative changes of angle (shear strains).

Stress Analogy

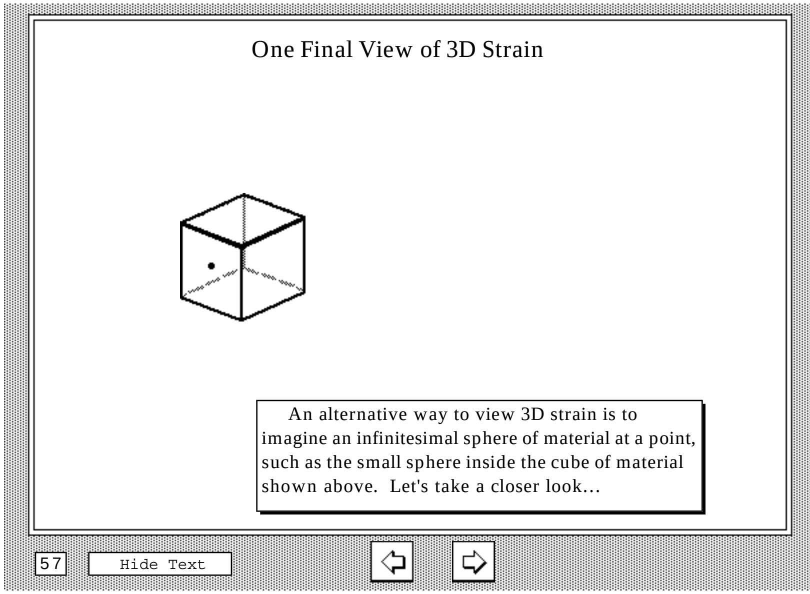


$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

It reminds me of the stress tensor. Both strain and stress are represented as tensor quantities.



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$


One Final View of 3D Strain
An alternative way to view 3D strain is to imagine an infinitesimal sphere of material at a point, such as the small sphere inside the cube of material shown above. Let's take a closer look...
57
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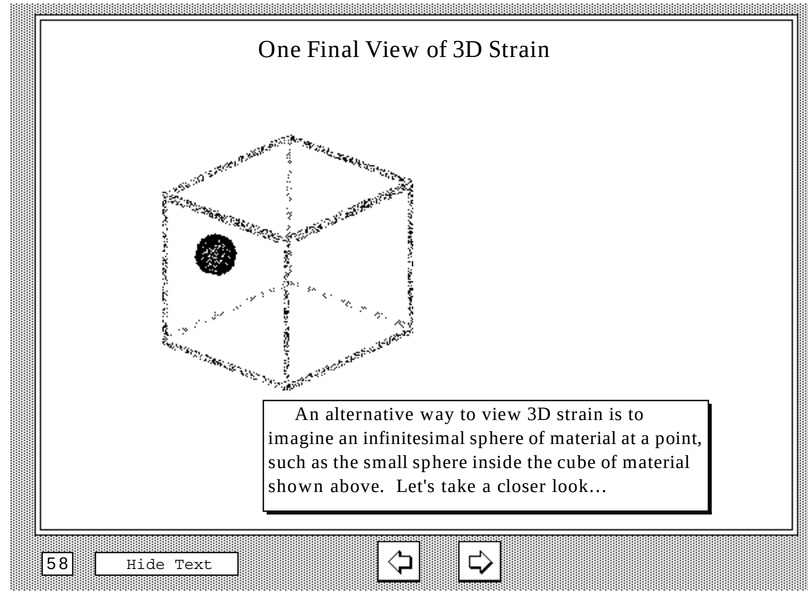
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58
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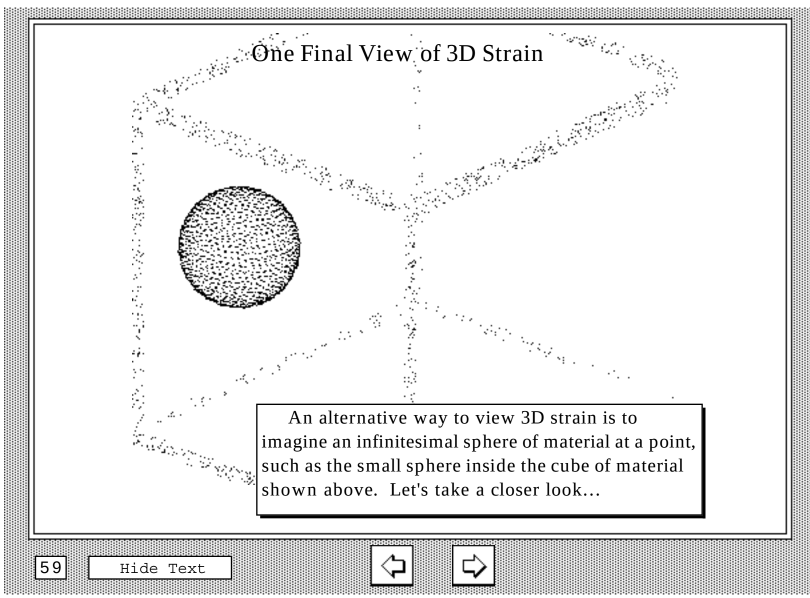
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59
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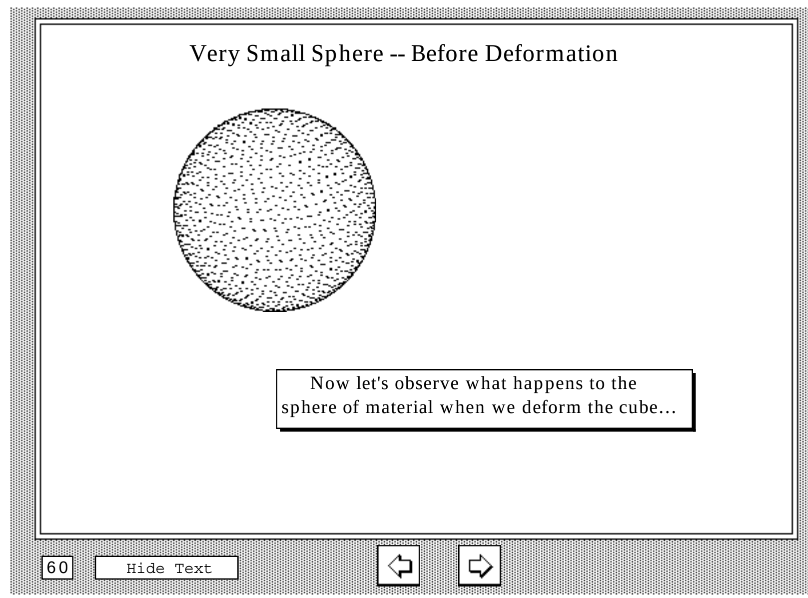
Very Small Sphere -- Before Deformation
Now let's observe what happens to the sphere of material when we deform the cube...
60
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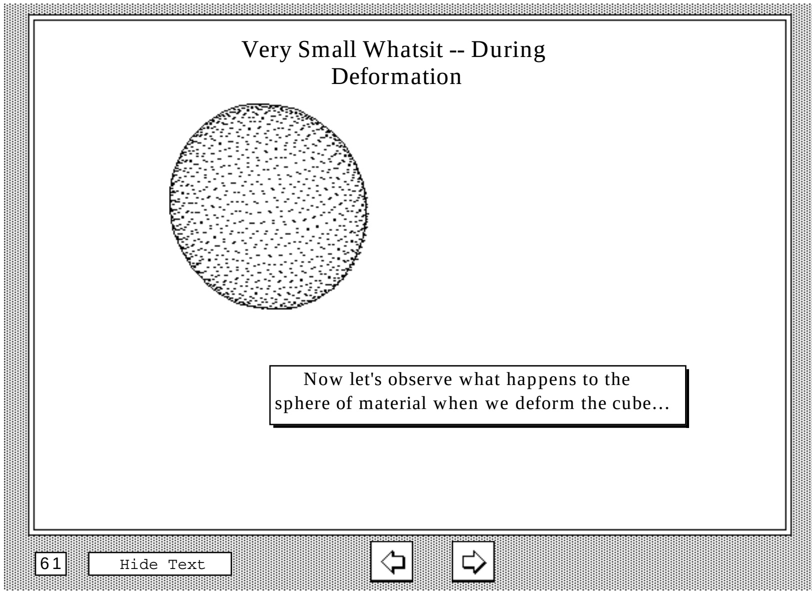
Very Small Whatsit -- During Deformation
Now let's observe what happens to the sphere of material when we deform the cube...
61
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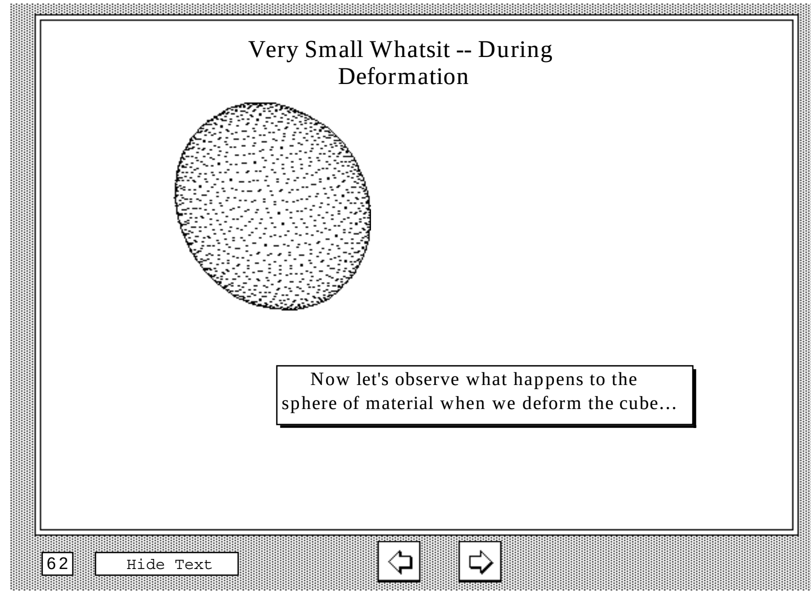
Very Small Whatsit -- During Deformation
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62
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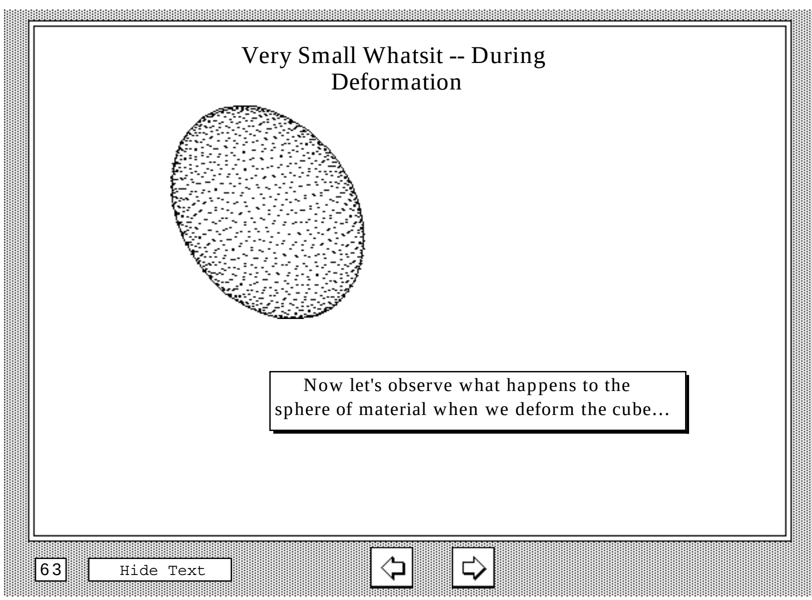
Very Small Whatsit -- During Deformation
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63
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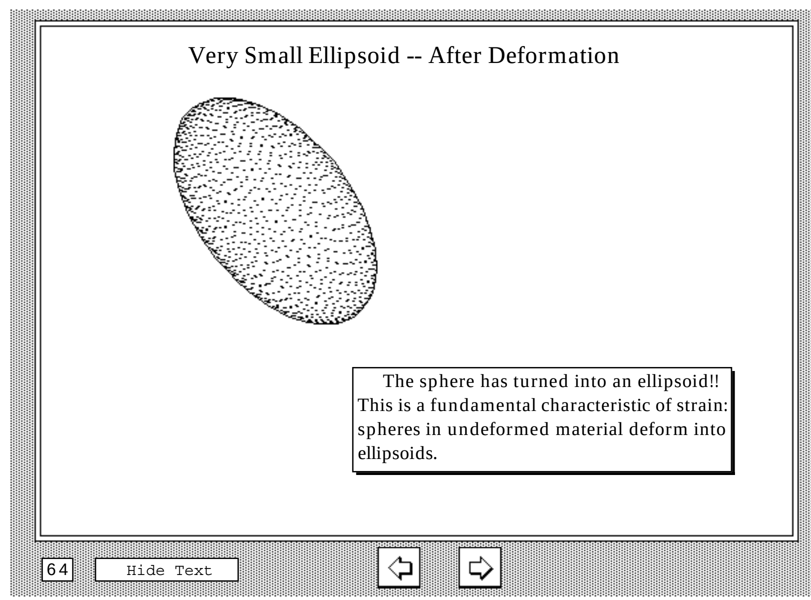
Very Small Ellipsoid -- After Deformation
The sphere has turned into an ellipsoid!! This is a fundamental characteristic of strain: spheres in undeformed material deform into ellipsoids.
64
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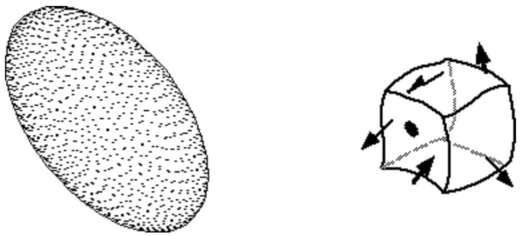
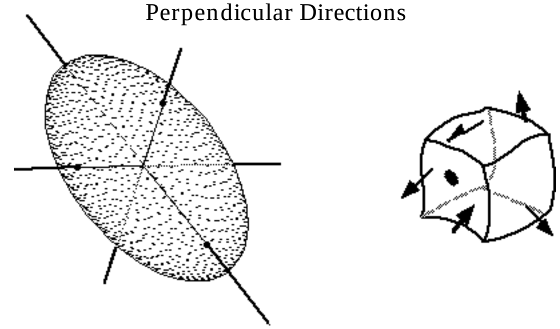
Very Small Ellipsoid -- After Deformation



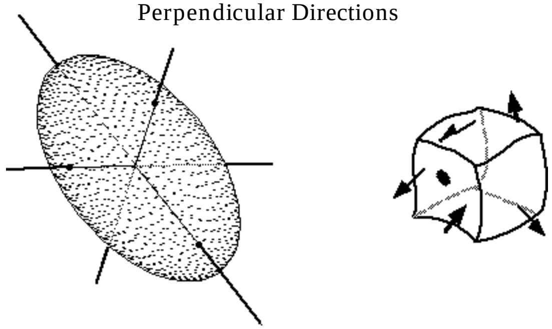
The orientation and shape of the ellipsoid depend on the particular deformation in the material.
As you may recall from high school, an ellipsoid is simply a sphere which has been stretched along three axes. Let's identify these three axes on our

Perpendicular Directions



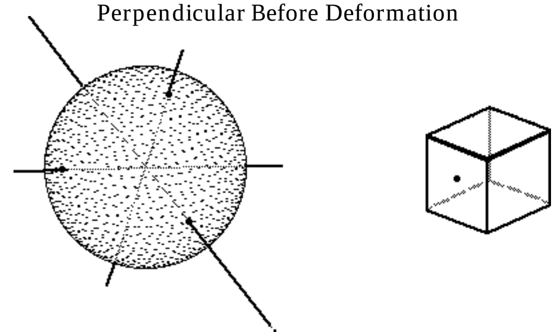
The important thing about these axes is that they are mutually perpendicular.

Perpendicular Directions



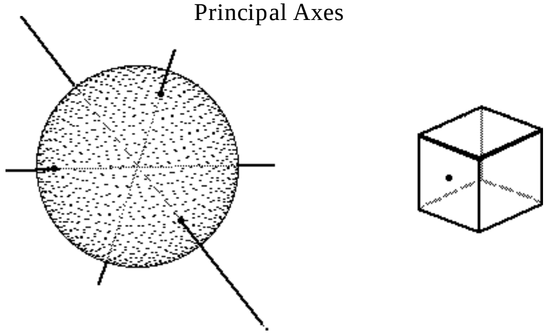
If we look at the same axes before deformation...

Perpendicular Before Deformation



...we see that they were perpendicular to start with. These axes have not experienced any shear strain!

Principal Axes



The directions between which no shear strains occur are principal directions. We can visualize the principal strains -- and the entire strain state -- by superposing the deformed ellipsoid on the sphere…

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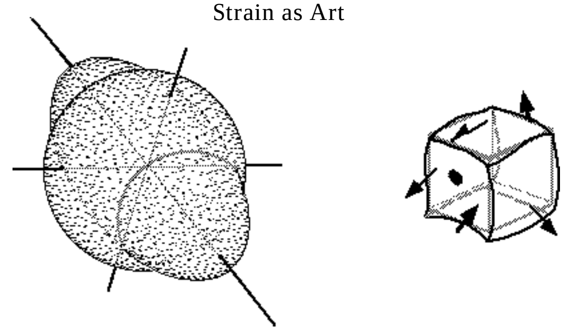
Strain as Art



As seen in the figure above, the principal directions experience only stretching or shortening. In fact, the deformed ellipsoid is completely defined by the orientation of the axes and the magnitude of the three stretches (principal strains). This means that the strain at a point is completely defined by three normal strains in three directions.

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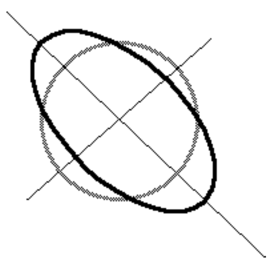
Strain in the Plane



In two dimensions, a similar picture of strain can be drawn using circles and ellipses. There are now only two principal directions and two principal strains. This is consistent with what we observed previously.

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