Let's consider the general case of strain energy in a bit more detail.

We have seen that any stress state can be decomposed into a pressure part...

\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z \\
\end{bmatrix}
\]

\[p = (\sigma_x + \sigma_y + \sigma_z)/3\]

...and a shape changing part.
The shape changing part is given a special name "Deviatoric".

We can rearrange this relation to determine a means for determining the deviatoric stresses from a general stress state.

The deviatoric part of the stress is simply the total stress minus the pressure part.

\[
\tilde{s} = \tilde{\sigma} - \tilde{p}
\]

Written out in terms of components, we have this relation.

\[
p = \frac{(\sigma_x + \sigma_y + \sigma_z)}{3}
\]
The normal and shear components of the deviatoric stress can thus be expressed by the above relations. (Note that only the normal components are different.)

The strain can be decomposed in similar fashion. The normal deviatoric strain components are as shown here, while the deviatoric shear strain components are the same as the regular shear strain components.

We can simplify these relations by using shorter notation for the pressure and volumetric strain.
Recall strain energy

For a linear material, the strain energy density is as indicated above. Note that if all the stress and strain components are set to zero except for $\sigma_x$ and $\varepsilon_x$, this simplifies to the simple uniaxial result we had before.

We now wish to express the strain energy density in terms of the deviatoric components of stress and strain, along with the volumetric components of stress and strain. To this end, consider the term in the box.

These terms can be written in terms of deviatoric and volumetric components as shown.

These relations can be substituted into the original strain energy equation.
We can perform a similar substitution for the y- and z-components.

We will now expand these terms.

The result is given below. Note that the terms on the second line involve the sums of the deviatoric stress and strain components. By definition, these sums are

This leads to a simpler result...
This result allows us to express the strain energy density in terms of a deviatoric and volumetric part.

Our next task is to determine the appropriate form of Hooke's law for deviatoric stress and strain.

Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \]

Hooke's law for the regular stress and strain components is of the form above.

This can be written in the alternative form shown by adding and subtracting \( \nabla \sigma_x \).
Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \]

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x(1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z) \right] \]

\[ \varepsilon'_x + \frac{e}{3} = \frac{1}{E} \left[ s_x(1 + \nu) + p(1 + \nu) - \nu(3p) \right] \]

\[ \varepsilon'_x = \frac{s_x(1 + \nu)}{E} + \frac{p(1 - 2\nu)}{E} - \frac{e}{3} \]

Combining terms leads to this relation.

We now make the substitutions

\[ \sigma_x = s_x + p \]

\[ \sigma_x + \sigma_y + \sigma_z = 3p \]

This expression involves the deviatoric components and the volumetric components. It can be simplified further.

Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \]

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x(1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z) \right] \]

\[ \varepsilon'_x + \frac{e}{3} = \frac{1}{E} \left[ s_x(1 + \nu) + p(1 + \nu) - \nu(3p) \right] \]

If we recall the definition of the bulk modulus, we can substitute as shown.

\[ k = \frac{E}{3(1 - 2\nu)} \]
Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \]
\[ \varepsilon_x = \frac{1}{E} [\sigma_x (1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z)] \]
\[ \varepsilon_x' + \frac{\varepsilon}{3} = \frac{1}{E} [s_x(1 + \nu) + p(1 + \nu) - \nu(3p)] \]
\[ \varepsilon_x' = \frac{s_x(1 + \nu)}{E} + \frac{p}{3k} - \frac{\varepsilon}{3} \]

By definition, we have \( p/k = e \). Thus...

\[ k = \frac{E}{3(1 - 2\nu)} \]

Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \]
\[ \varepsilon_x = \frac{1}{E} [\sigma_x (1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z)] \]
\[ \varepsilon_x' + \frac{\varepsilon}{3} = \frac{1}{E} [s_x(1 + \nu) + p(1 + \nu) - \nu(3p)] \]
\[ \varepsilon_x' = \frac{s_x(1 + \nu)}{E} + \frac{p}{3k} - \frac{\varepsilon}{3} \]

These terms cancel. This leaves us with the simple result...

Recall Hooke's Law

This can be written in even simpler form by means of the shear modulus.

\[ \varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \]
\[ \varepsilon_x = \frac{1}{E} [\sigma_x (1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z)] \]
\[ \varepsilon_x' + \frac{\varepsilon}{3} = \frac{1}{E} [s_x(1 + \nu) + p(1 + \nu) - \nu(3p)] \]
\[ \varepsilon_x' = \frac{s_x(1 + \nu)}{E} \]

\[ G = E/(2(1+\nu)) \]
Recall Hooke's Law

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu(\sigma_y + \sigma_z) \right] \]
\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x (1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z) \right] \]
\[ \varepsilon'_x + \frac{\varepsilon}{3} = \frac{1}{E} \left[ \sigma_x (1 + \nu) + p(1 + \nu) - \nu(3p) \right] \]

This is the relation between the normal components of the deviatoric stress and strain.

\[ G = \frac{E}{2(1 + \nu)} \]

Strain Energy

\[ u = \frac{1}{2} S_x \varepsilon'_x + \frac{1}{2} S_y \varepsilon'_y + \frac{1}{2} S_z \varepsilon'_z + \frac{1}{2} p e \]
+ \[ \tau_{xy} \varepsilon_{xy} + \tau_{xz} \varepsilon_{xz} + \tau_{yz} \varepsilon_{yz} \]

We can use this result to write the strain energy density completely in terms of the stress components.

\[ \varepsilon'_x = \frac{S_x}{2G} \]

Strain Energy

\[ u = \frac{S_x^2}{4G} + \frac{S_y^2}{4G} + \frac{S_z^2}{4G} + \frac{1}{2} p e \]
+ \[ \tau_{xy}^2 \varepsilon_{xy}^2 + \tau_{xz}^2 \varepsilon_{xz}^2 + \tau_{yz}^2 \varepsilon_{yz}^2 \]

Here we have substituted for all the components. At last we are ready to state our final conclusion.
For an isotropic material, the strain energy density can be expressed in two parts as shown: a volumetric part and a deviatoric part.