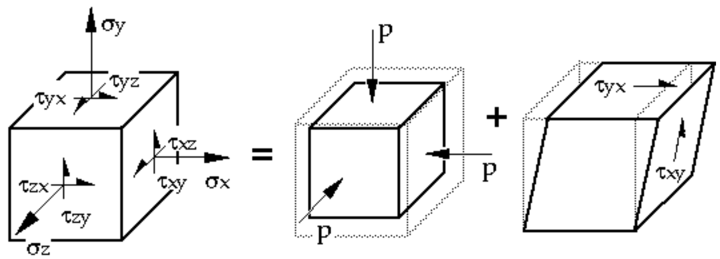
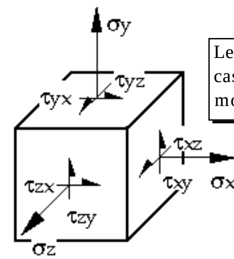
More on Strain Energy

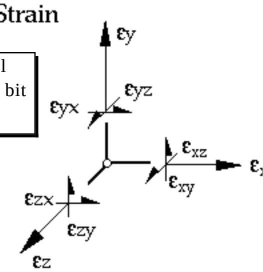


Stress



Let's consider the general case of strain energy in a bit more detail.

Strain



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

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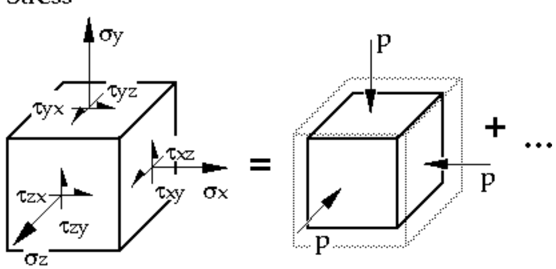
Stress



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$p = (\sigma_x + \sigma_y + \sigma_z)/3$$

We have seen that any stress state can be decomposed into a pressure part...

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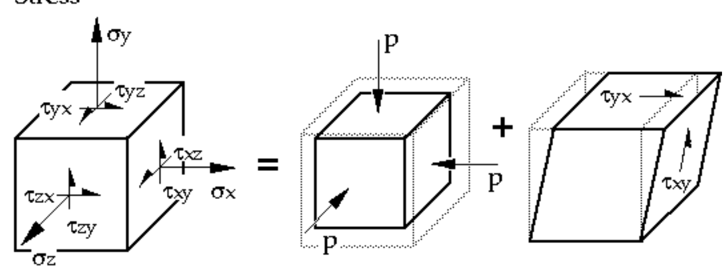
Stress



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$p = (\sigma_x + \sigma_y + \sigma_z)/3$$

...and a shape changing part.

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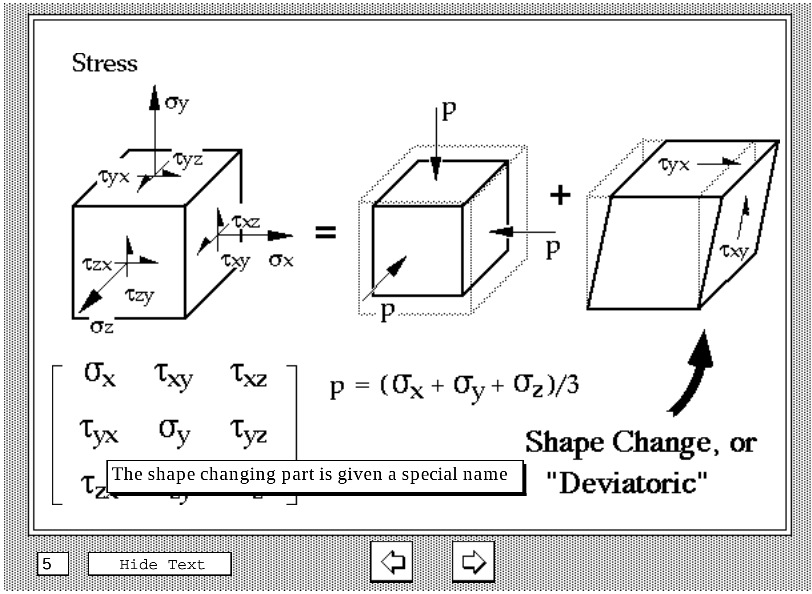
Stress
$p = (\sigma_x + \sigma_y + \sigma_z)/3$
Shape Change, or "Deviatoric"
The shape changing part is given a special name
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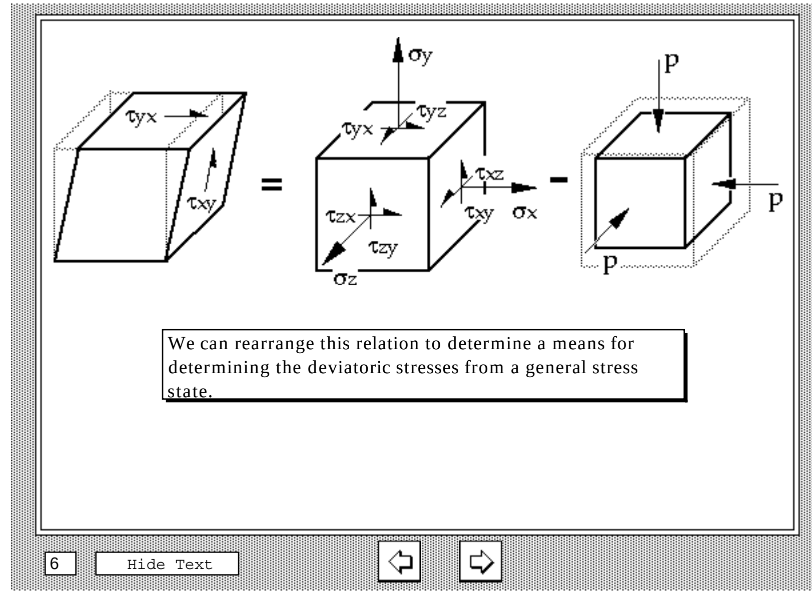
We can rearrange this relation to determine a means for determining the deviatoric stresses from a general stress state.
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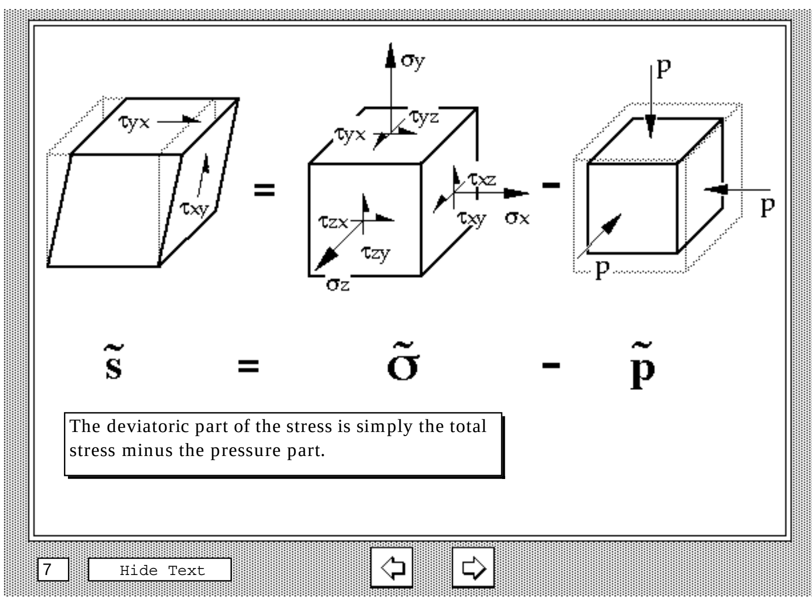
$\tilde{s} = \tilde{\sigma} - \tilde{p}$
The deviatoric part of the stress is simply the total stress minus the pressure part.
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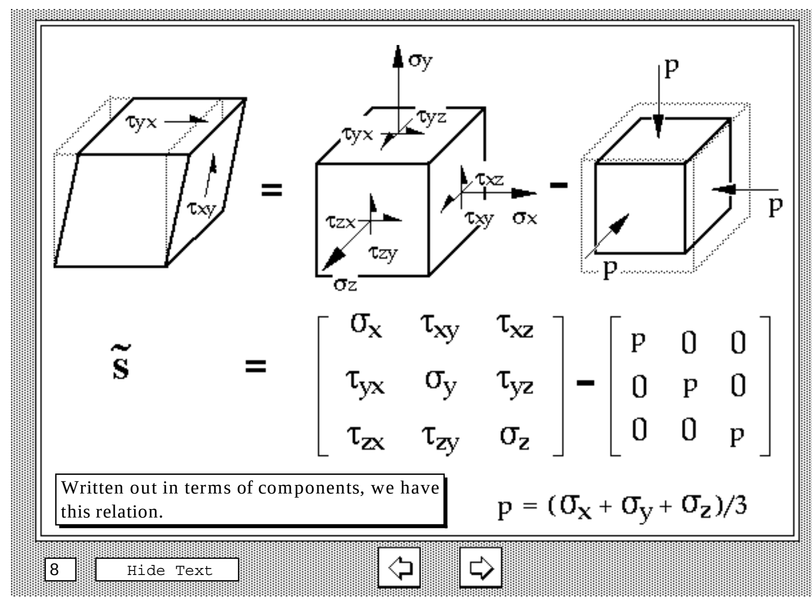
$\tilde{s} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$
Written out in terms of components, we have this relation.
$p = (\sigma_x + \sigma_y + \sigma_z)/3$
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$$s_x = \sigma_x - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \qquad s_{xy} = \tau_{xy}$$

$$s_y = \sigma_y - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \qquad \cdots = \cdots$$

$$\cdots = \cdots \quad \cdots$$

The normal and shear components of the deviatoric stress can thus be expressed by the above relations. (Note that only the normal components are different.)

Similar For Strain...

$$s_x = \sigma_x - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\varepsilon_x' = \varepsilon_x - \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

The strain can be decomposed in similar fashion. The normal deviatoric strain components are as shown here, while the deviatoric shear strain components are the same as the regular shear strain components.

Similar For Strain...

$$s_x = \sigma_x - \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\varepsilon_x' = \varepsilon_x - \frac{1}{3}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

Pressure

$$\boxed{p = (\sigma_x + \sigma_y + \sigma_z)/3}$$

Volumetric Strain

$$\boxed{e = (\varepsilon_x + \varepsilon_y + \varepsilon_z)}$$

We can simplify these relations by using shorter notation for the pressure and volumetric strain.

Similar For Strain...

$$s_x = \sigma_x - p$$

$$\varepsilon_x' = \varepsilon_x - \frac{1}{3}e$$

We thus have these simple relations between deviatoric and total stress components. We now consider the strain energy.

Pressure

$$\boxed{p = (\sigma_x + \sigma_y + \sigma_z)/3}$$

Volumetric Strain

$$\boxed{e = (\varepsilon_x + \varepsilon_y + \varepsilon_z)}$$

Recall Strain Energy

$$u = \frac{1}{2}\sigma_x \varepsilon_x + \frac{1}{2}\sigma_y \varepsilon_y + \frac{1}{2}\sigma_z \varepsilon_z + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

For a linear material, the strain energy density is as indicated above. Note that if all the stress and strain components are set to zero except for σ_x and ε_x, this simplifies to the simple uniaxial result we had before.

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Strain Energy

$$u = \boxed{\frac{1}{2}\sigma_x \varepsilon_x} + \frac{1}{2}\sigma_y \varepsilon_y + \frac{1}{2}\sigma_z \varepsilon_z + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

We now wish to express the strain energy density in terms of the deviatoric components of stress and strain, along with the volumetric components of stress and strain. To this end, consider the term in the box.

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Strain Energy

$$u = \boxed{\frac{1}{2}\sigma_x \varepsilon_x} + \frac{1}{2}\sigma_y \varepsilon_y + \frac{1}{2}\sigma_z \varepsilon_z + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

$$\sigma_x = s_x + p$$

$$\varepsilon_x = \varepsilon_x' + \frac{1}{3}e$$

These terms can be written in terms of deviatoric and volumetric components as shown.

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Strain Energy

$$u = \frac{1}{2}(s_x + p)\cdot\left(\varepsilon_x' + \frac{e}{3}\right) + \frac{1}{2}\sigma_y \varepsilon_y + \frac{1}{2}\sigma_z \varepsilon_z + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

$$\sigma_x = s_x + p$$

$$\varepsilon_x = \varepsilon_x' + \frac{1}{3}e$$

These relations can be substituted into the original strain energy equation.

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Strain Energy

$$u = \frac{1}{2}(s_x + p)\cdot\left(\varepsilon_x' + \frac{e}{3}\right) + \boxed{\frac{1}{2}\sigma_y\,\varepsilon_y} + \boxed{\frac{1}{2}\sigma_z\,\varepsilon_z}$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

We can perform a similar substitution for the y- and z-components.

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Strain Energy

$$u = \frac{1}{2}(s_x + p)\cdot\left(\varepsilon_x' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_y + p)\cdot\left(\varepsilon_y' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_z + p)\cdot\left(\varepsilon_z' + \frac{e}{3}\right)$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

We will now expand these terms.

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Strain Energy

The result is given below. Note that the terms on the second line involve the sums of the deviatoric stress and strain components. By definition, these sums are

$$u = \frac{1}{2}(s_x + p)\cdot\left(\varepsilon_x' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_y + p)\cdot\left(\varepsilon_y' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_z + p)\cdot\left(\varepsilon_z' + \frac{e}{3}\right)$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

$$u = \frac{1}{2}s_x\,\varepsilon_x' + \frac{1}{2}s_y\,\varepsilon_y' + \frac{1}{2}s_z\,\varepsilon_z' + \frac{1}{2}p\,e$$
$$+ \frac{p}{2}(\varepsilon_x' + \varepsilon_y' + \varepsilon_z') + \frac{e}{6}(s_x + s_y + s_z)$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

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Strain Energy

$$u = \frac{1}{2}(s_x + p)\cdot\left(\varepsilon_x' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_y + p)\cdot\left(\varepsilon_y' + \frac{e}{3}\right)$$
$$+ \frac{1}{2}(s_z + p)\cdot\left(\varepsilon_z' + \frac{e}{3}\right)$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

$$u = \frac{1}{2}s_x\,\varepsilon_x' + \frac{1}{2}s_y\,\varepsilon_y' + \frac{1}{2}s_z\,\varepsilon_z' + \frac{1}{2}p\,e$$
$$+ \frac{p}{2}(\cancel{\varepsilon_x' + \varepsilon_y' + \varepsilon_z'}) + \frac{e}{6}(\cancel{s_x + s_y + s_z})$$
$$+ \tau_{xy}\,\varepsilon_{xy} + \tau_{xz}\,\varepsilon_{xz} + \tau_{yz}\,\varepsilon_{yz}$$

This leads to a simpler result…

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Strain Energy

$$u = \tfrac{1}{2}(s_x + p)\cdot(\varepsilon_x' + \tfrac{e}{3}) + \tfrac{1}{2}(s_y + p)\cdot(\varepsilon_y' + \tfrac{e}{3}) + \tfrac{1}{2}(s_z + p)\cdot(\varepsilon_z' + \tfrac{e}{3}) + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

$$u = \tfrac{1}{2}s_x\varepsilon_x' + \tfrac{1}{2}s_y\varepsilon_y' + \tfrac{1}{2}s_z\varepsilon_z' + \tfrac{1}{2}p\,e + \tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

This result allows us to express the strain energy density in terms of a deviatoric and volumetric part.

Deviatoric Stress vs Deviatoric Strain?

$$s_x \;=?\; \varepsilon_x'$$

Our next task is to determine the appropriate form of Hooke's law for deviatoric stress and strain.

Recall Hooke's Law

$$\varepsilon_x = \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

Hooke's law for the regular stress and strain components is of the form above.

Recall Hooke's Law

$$\varepsilon_x = \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \tfrac{1}{E}\left[\sigma_x(1 + \nu) - \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

This can be written in the alternative form shown by adding and subtracting $\nu\,\sigma_x$.

Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) \;-\nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\boxed{\varepsilon_x = \varepsilon_x' + \frac{1}{3}e}$$

$$\boxed{\sigma_x = s_x + p}$$

$$\boxed{\sigma_x + \sigma_y + \sigma_z = 3p}$$

We now make the substitutions

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Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) \;-\nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) \;+p(1+\nu) - \nu(3p)\right]$$

This expression involves the deviatoric components and the volumetric components. It can be simplified further.

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Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) \;-\nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) \;+p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E} + \frac{p(1-2\nu)}{E} - \frac{e}{3}$$

Combining terms leads to this relation.

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Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) \;-\nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) \;+p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E} + \frac{p(1-2\nu)}{E} - \frac{e}{3}$$

If we recall the definition of the bulk modulus, we can substitute as shown.

$$\boxed{\mathbf{k} = \frac{E}{3(1-2\nu)}}$$

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Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) - \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) + p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E} + \frac{p}{3k} - \frac{e}{3}$$

By definition, we have $p/k = e$. Thus...

$$k = \frac{E}{3(1-2\nu)}$$

Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) - \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) + p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E} + \cancel{\frac{p}{3k}} \cancel{- \frac{e}{3}}$$

...these terms cancel. This leaves us with the simple result...

Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) - \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) + p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E}$$

This can be written in even simpler form by means of the shear modulus.

Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) - \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) + p(1+\nu) - \nu(3p)\right]$$

$$\varepsilon'_x = \frac{s_x(1+\nu)}{E}$$

$$\boxed{G = E/2(1+\nu)}$$

Recall Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1+\nu) \;- \nu(\sigma_x + \sigma_y + \sigma_z)\right]$$

$$\varepsilon'_x + \frac{e}{3} = \frac{1}{E}\left[s_x(1+\nu) \;+ p(1+\nu) - \nu(3p)\right]$$

$$\boxed{\varepsilon'_x = \frac{s_x}{2G}}$$

This is the relation between the normal components of the deviatoric stress and strain.

$$G = E/2(1+\nu)$$

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Strain Energy

$$u = \tfrac{1}{2}s_x\varepsilon_x' + \tfrac{1}{2}s_y\varepsilon_y' + \tfrac{1}{2}s_z\varepsilon_z' + \tfrac{1}{2}p\,e$$
$$+ \;\tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

$$\boxed{\varepsilon'_x = \frac{s_x}{2G}}$$

We can use this result to write the strain energy density completely in terms of the stress components.

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Strain Energy

$$u = \frac{s_x^{\,2}}{4G} + \tfrac{1}{2}s_y\varepsilon_y' + \tfrac{1}{2}s_z\varepsilon_z' + \tfrac{1}{2}p\,e$$
$$+ \;\tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz} + \tau_{yz}\varepsilon_{yz}$$

Here we have substituted for the x-component.

$$\varepsilon'_x = \frac{s_x}{2G}$$

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Strain Energy

$$u = \frac{s_x^{\,2}}{4G} + \frac{s_y^{\,2}}{4G} + \frac{s_z^{\,2}}{4G} + \tfrac{1}{2}p\,e$$
$$+ \frac{\tau_{xy}^{\,2}}{2G} + \frac{\tau_{xz}^{\,2}}{2G} + \frac{\tau_{yz}^{\,2}}{2G}$$

Here we have substituted for all the components. At last we are ready to state our final conclusion.

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Strain Energy

$$u = \boxed{\frac{s_x^{\,2}}{4\,G} + \frac{s_y^{\,2}}{4\,G} + \frac{s_z^{\,2}}{4\,G} + \frac{\tau_{xy}^{\,2}}{2\,G} + \frac{\tau_{xz}^{\,2}}{2\,G} + \frac{\tau_{yz}^{\,2}}{2\,G}} + \boxed{\tfrac{1}{2}\,p\,e}$$

Deviatoric Strain Energy

Volumetric Strain Energy

For an isotropic material, the strain energy density can be expressed in two parts as shown: a volumetric part and a deviatoric part.

The End