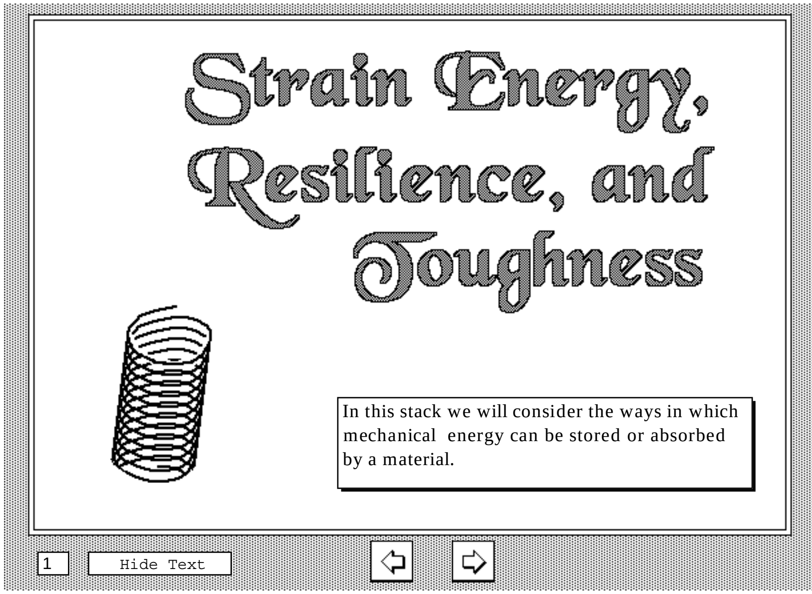
Strain Energy,
Resilience, and
Toughness
In this stack we will consider the ways in which mechanical energy can be stored or absorbed by a material.
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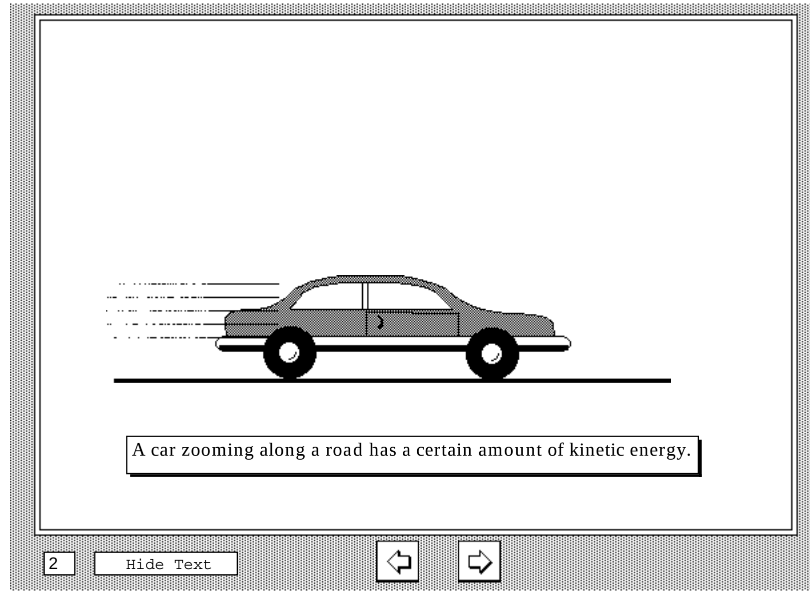
A car zooming along a road has a certain amount of kinetic energy.
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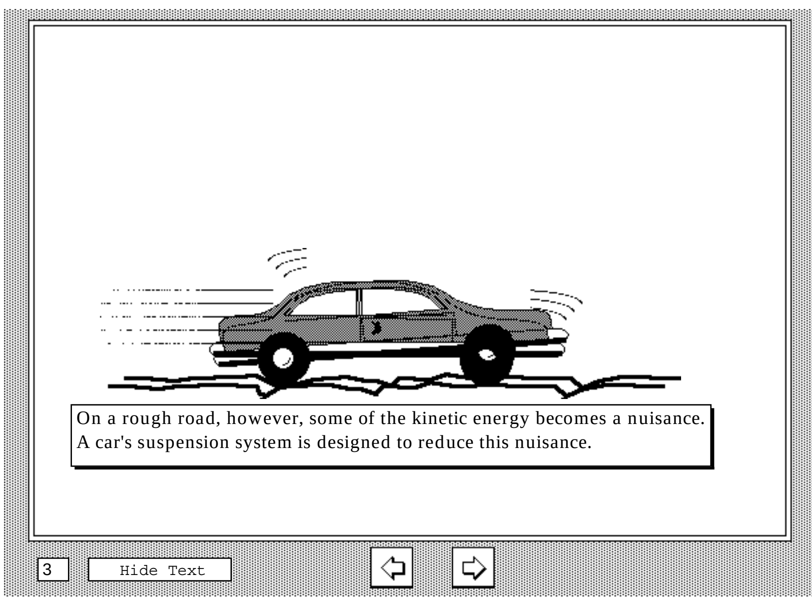
On a rough road, however, some of the kinetic energy becomes a nuisance. A car's suspension system is designed to reduce this nuisance.
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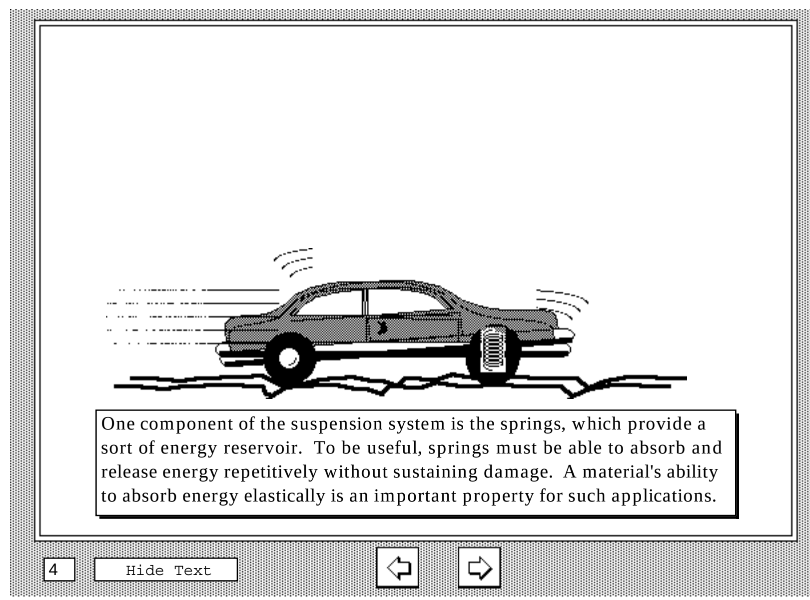
One component of the suspension system is the springs, which provide a sort of energy reservoir. To be useful, springs must be able to absorb and release energy repetitively without sustaining damage. A material's ability to absorb energy elastically is an important property for such applications.
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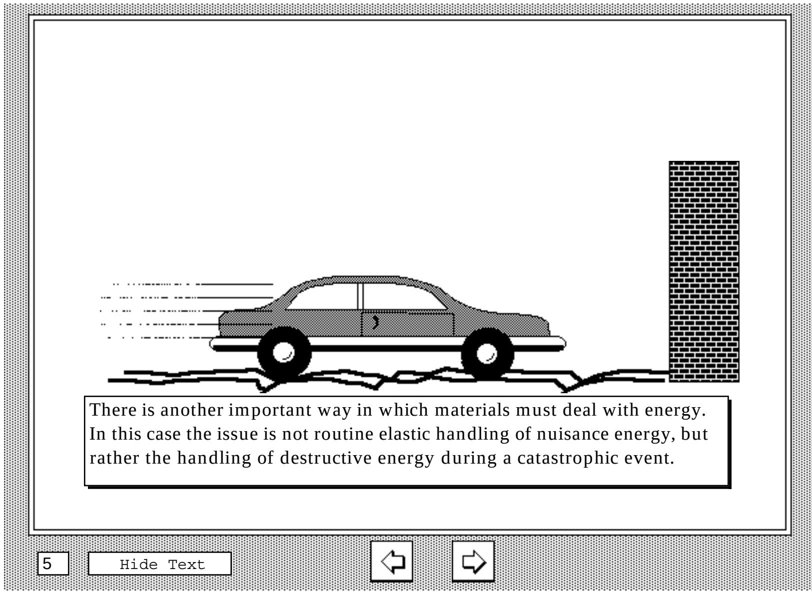
There is another important way in which materials must deal with energy. In this case the issue is not routine elastic handling of nuisance energy, but rather the handling of destructive energy during a catastrophic event.
5
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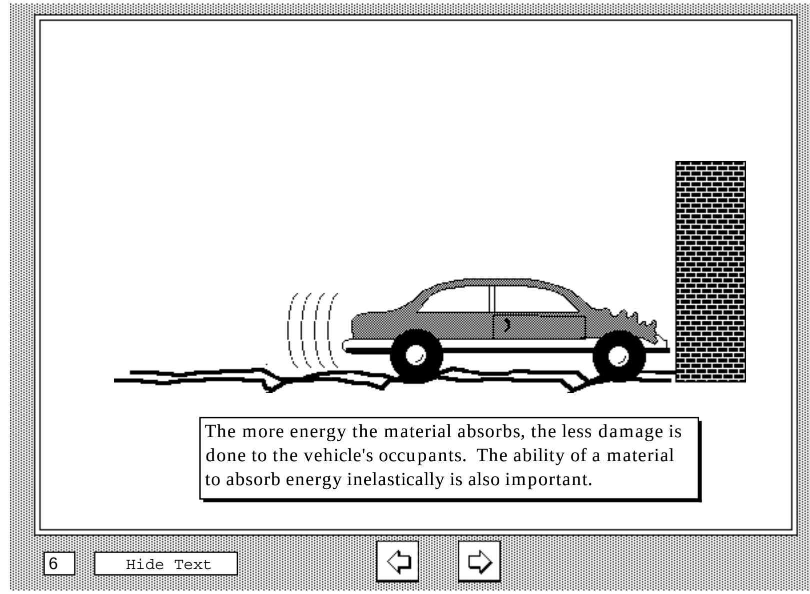
The more energy the material absorbs, the less damage is done to the vehicle's occupants. The ability of a material to absorb energy inelastically is also important.
6
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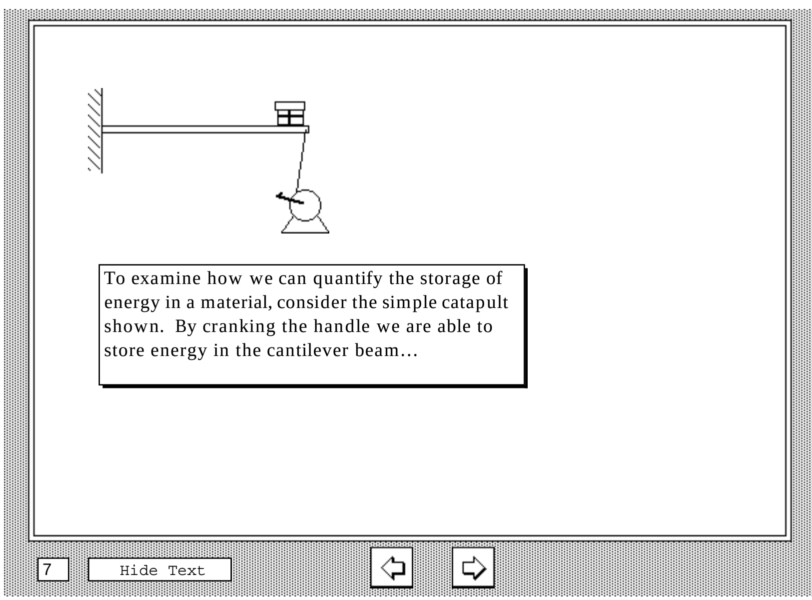
To examine how we can quantify the storage of energy in a material, consider the simple catapult shown. By cranking the handle we are able to store energy in the cantilever beam…
7
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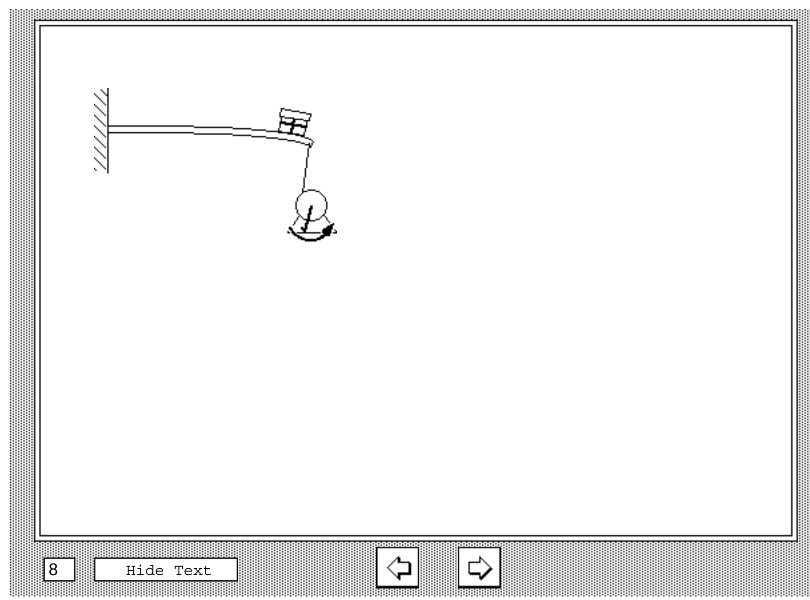
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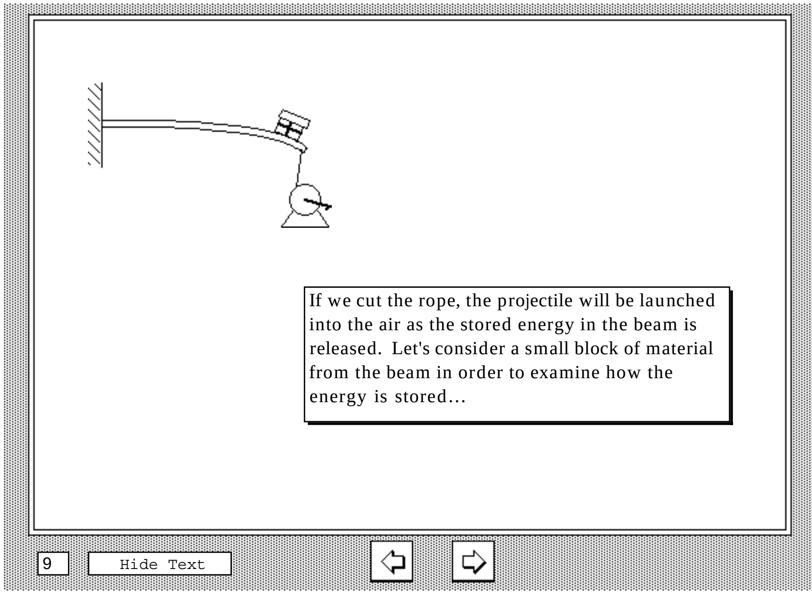
If we cut the rope, the projectile will be launched into the air as the stored energy in the beam is released. Let's consider a small block of material from the beam in order to examine how the energy is stored...
9
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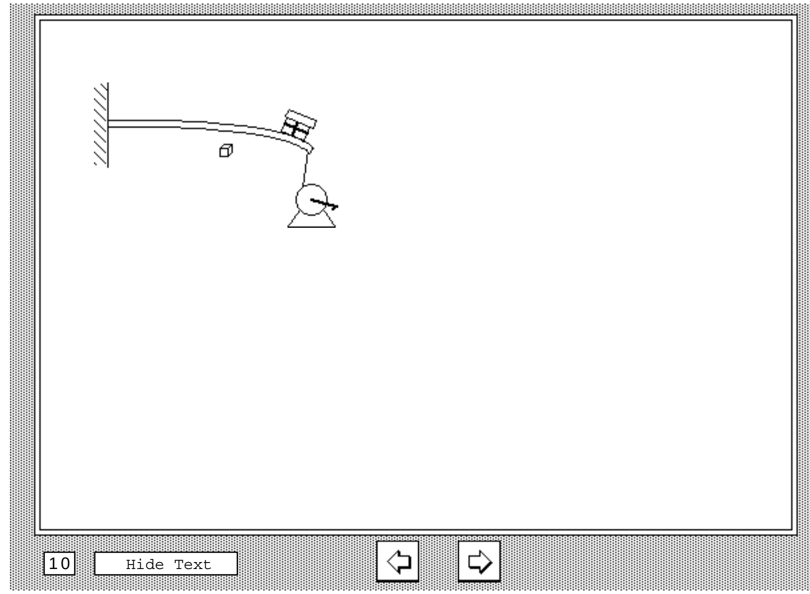
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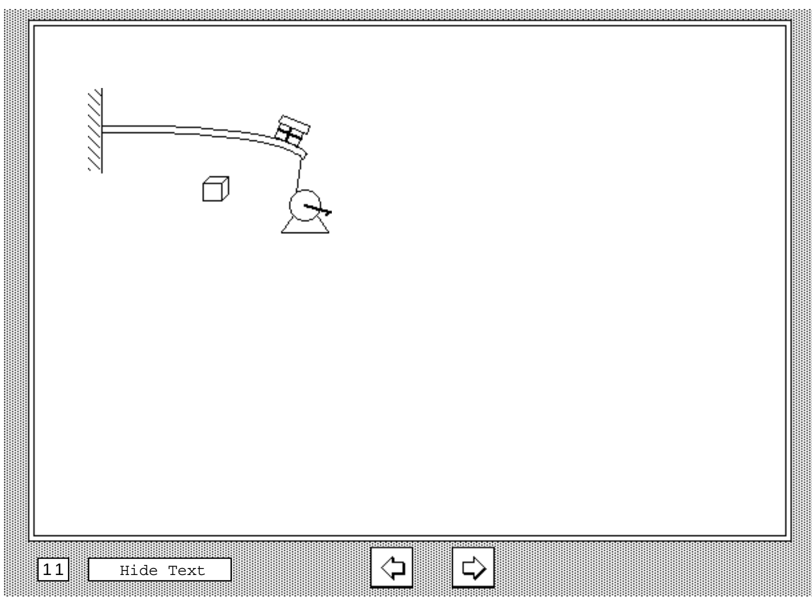
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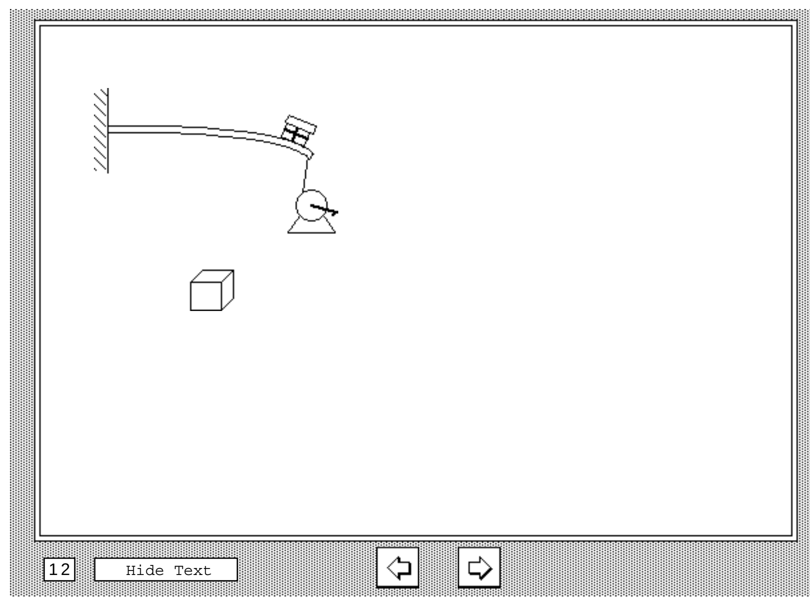
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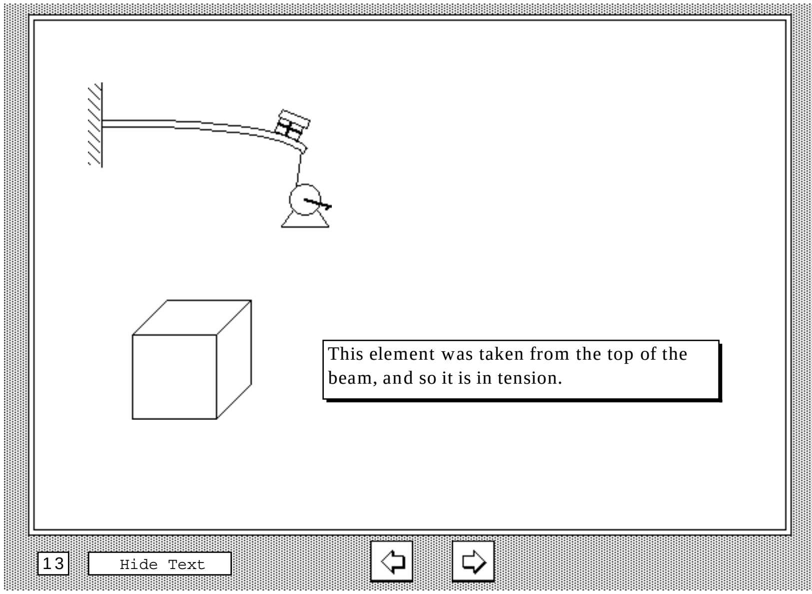
This element was taken from the top of the beam, and so it is in tension.
13
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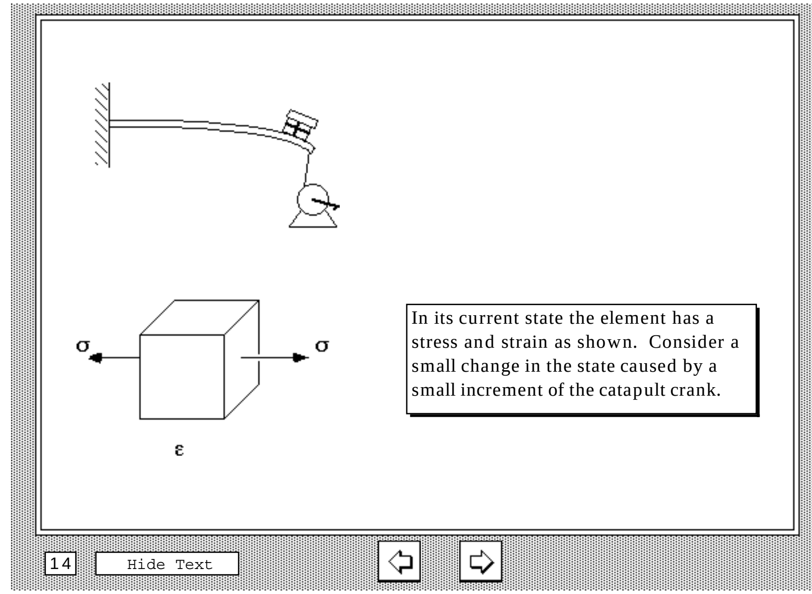
σ
σ
ε
In its current state the element has a stress and strain as shown. Consider a small change in the state caused by a small increment of the catapult crank.
14
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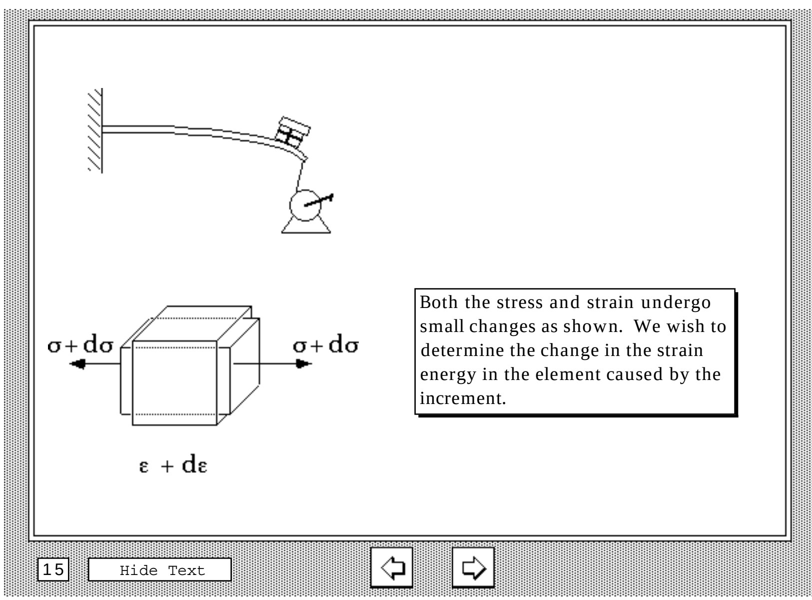
σ+dσ
σ+dσ
ε + dε
Both the stress and strain undergo small changes as shown. We wish to determine the change in the strain energy in the element caused by the increment.
15
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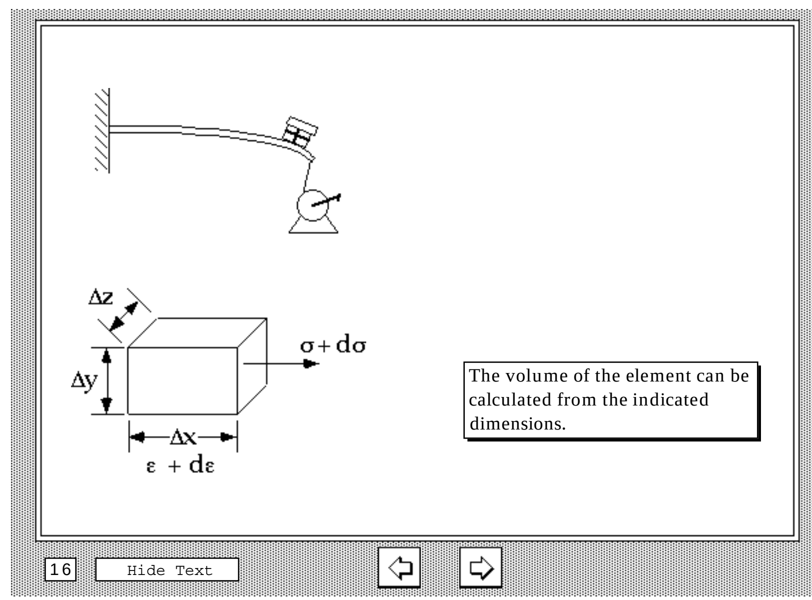
Δz
Δy
Δx
σ+dσ
ε + dε
The volume of the element can be calculated from the indicated dimensions.
16
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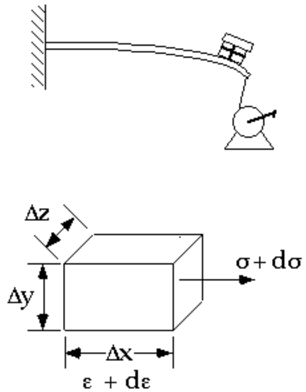

$$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$$

The increment in the stored (strain) energy can be calculated by multiplying the average stress by its area to obtain force, and then multiplying by the length change, given by the strain increment times the length.

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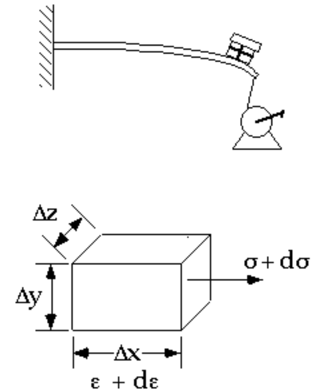
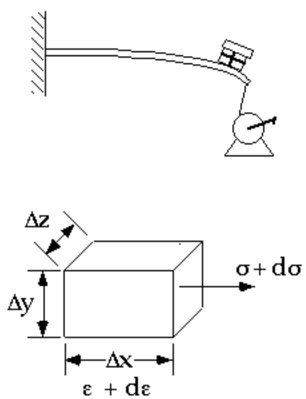


$$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$$

$$dU = \left[\sigma + \frac{d\sigma}{2}\right] d\varepsilon (\Delta x\, \Delta y\, \Delta z)$$

This can be simplified and rearranged as shown.

18 Hide Text



$$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$$

$$dU = \left[\sigma + \frac{d\sigma}{2}\right] d\varepsilon (\Delta x\, \Delta y\, \Delta z)$$

$$\lim_{d\sigma \to 0} dU = \sigma\, d\varepsilon\, \Delta V$$

If we then consider the limit as the stress increment approaches zero, we obtain the above result.

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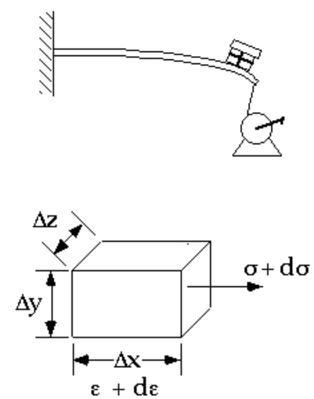

$$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$$

$$dU = \left[\sigma + \frac{d\sigma}{2}\right] d\varepsilon (\Delta x\, \Delta y\, \Delta z)$$

$$\lim_{d\sigma \to 0} dU = \sigma\, d\varepsilon\, \Delta V$$

$$\Delta U = \left\{\int_{\varepsilon_1}^{\varepsilon_2} \sigma\, d\varepsilon\right\} \Delta V$$

To obtain the change in stored energy for a finite change in state, we simply integrate from one strain state to the other.

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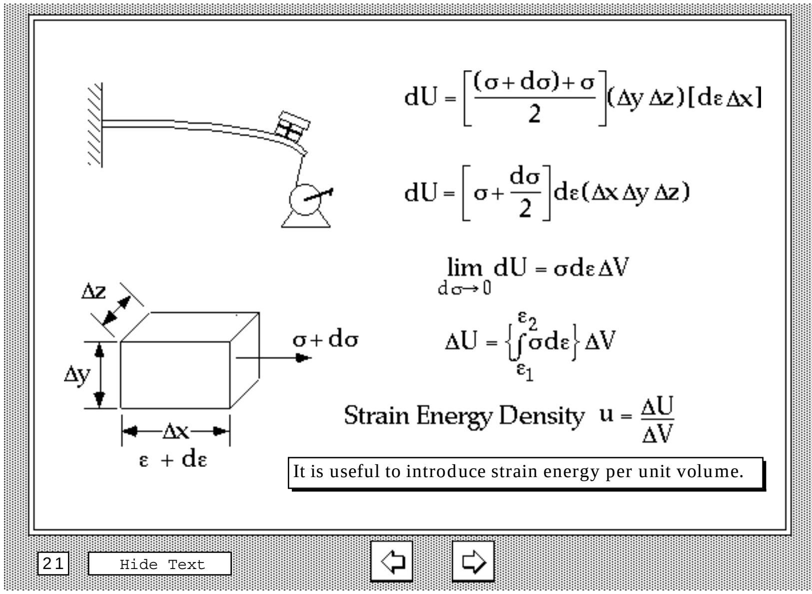
$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$
$dU = \left[\sigma + \frac{d\sigma}{2}\right] d\varepsilon(\Delta x\, \Delta y\, \Delta z)$
$\lim_{d\sigma \to 0} dU = \sigma d\varepsilon \Delta V$
$\Delta U = \left\{\int_{\varepsilon_1}^{\varepsilon_2} \sigma d\varepsilon\right\} \Delta V$
Strain Energy Density $u = \frac{\Delta U}{\Delta V}$
Δz
Δy
Δx
$\sigma + d\sigma$
$\varepsilon + d\varepsilon$
It is useful to introduce strain energy per unit volume.
21
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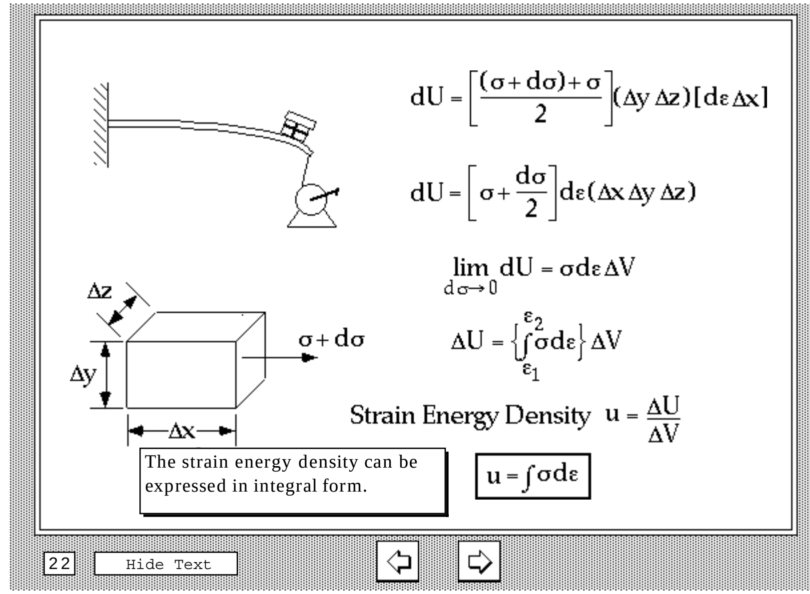
$dU = \left[\frac{(\sigma + d\sigma) + \sigma}{2}\right](\Delta y\, \Delta z)[d\varepsilon\, \Delta x]$
$dU = \left[\sigma + \frac{d\sigma}{2}\right] d\varepsilon(\Delta x\, \Delta y\, \Delta z)$
$\lim_{d\sigma \to 0} dU = \sigma d\varepsilon \Delta V$
$\Delta U = \left\{\int_{\varepsilon_1}^{\varepsilon_2} \sigma d\varepsilon\right\} \Delta V$
Strain Energy Density $u = \frac{\Delta U}{\Delta V}$
Δz
Δy
Δx
$\sigma + d\sigma$
The strain energy density can be expressed in integral form.
$u = \int \sigma d\varepsilon$
22
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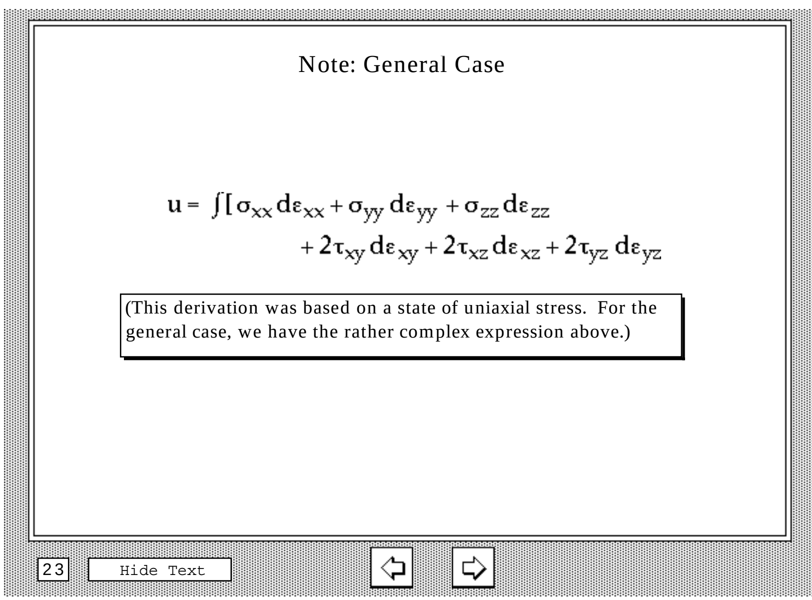
Note: General Case
$u = \int [\sigma_{xx} d\varepsilon_{xx} + \sigma_{yy} d\varepsilon_{yy} + \sigma_{zz} d\varepsilon_{zz} + 2\tau_{xy} d\varepsilon_{xy} + 2\tau_{xz} d\varepsilon_{xz} + 2\tau_{yz} d\varepsilon_{yz}$
(This derivation was based on a state of uniaxial stress. For the general case, we have the rather complex expression above.)
23
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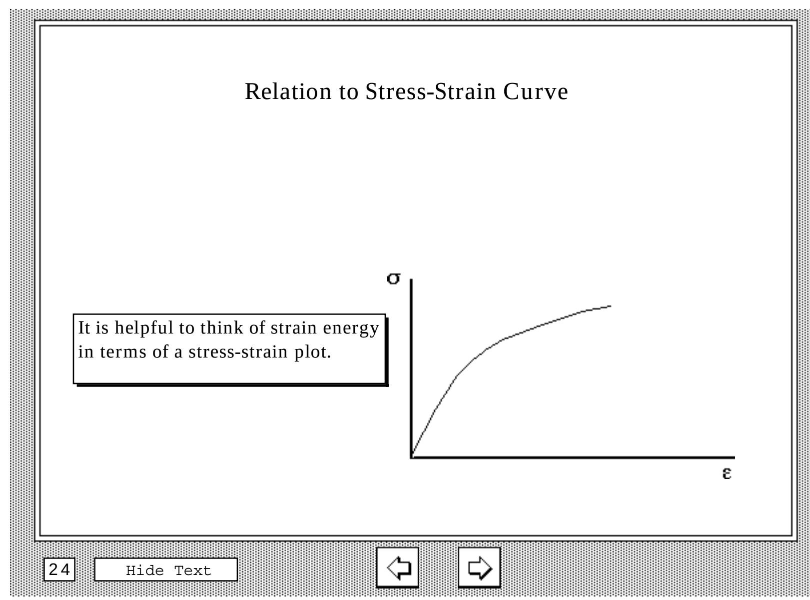
Relation to Stress-Strain Curve
It is helpful to think of strain energy in terms of a stress-strain plot.
σ
ε
24
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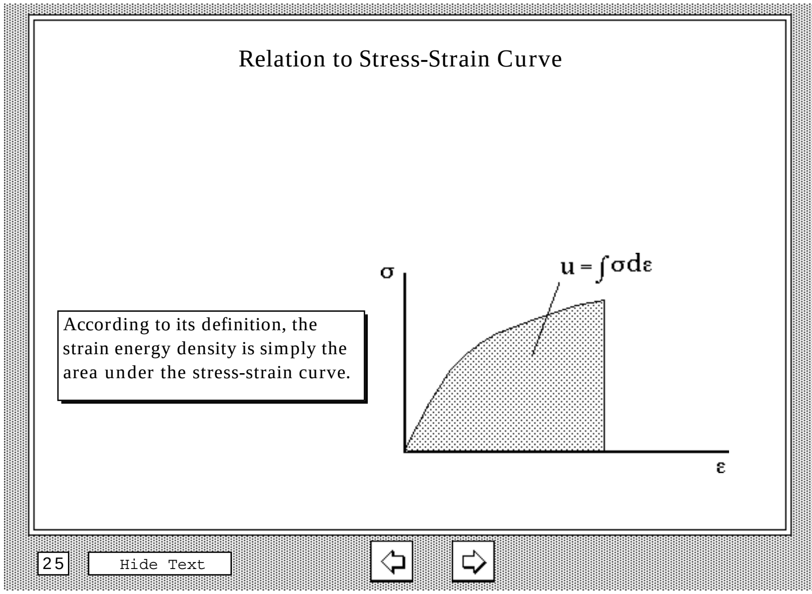
Relation to Stress-Strain Curve
According to its definition, the strain energy density is simply the area under the stress-strain curve.
$u = \int \sigma d\varepsilon$
σ
ε
25
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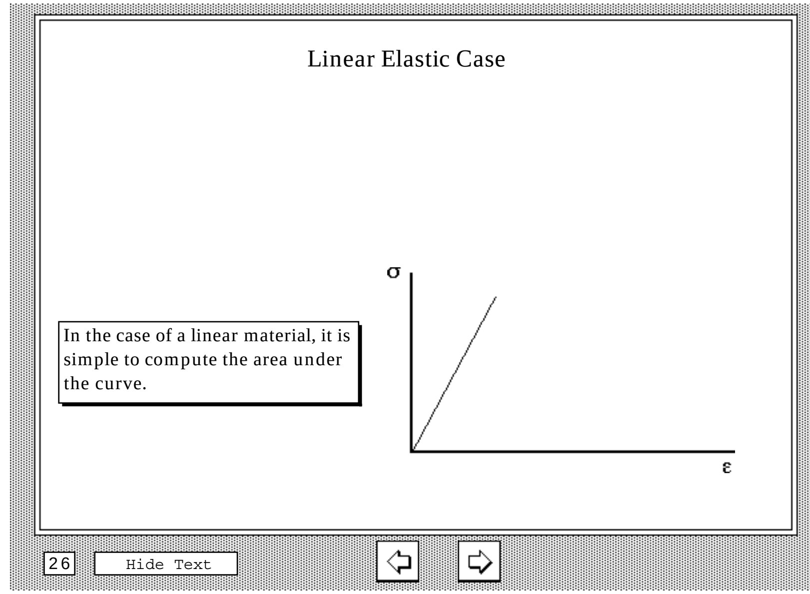
Linear Elastic Case
In the case of a linear material, it is simple to compute the area under the curve.
σ
ε
26
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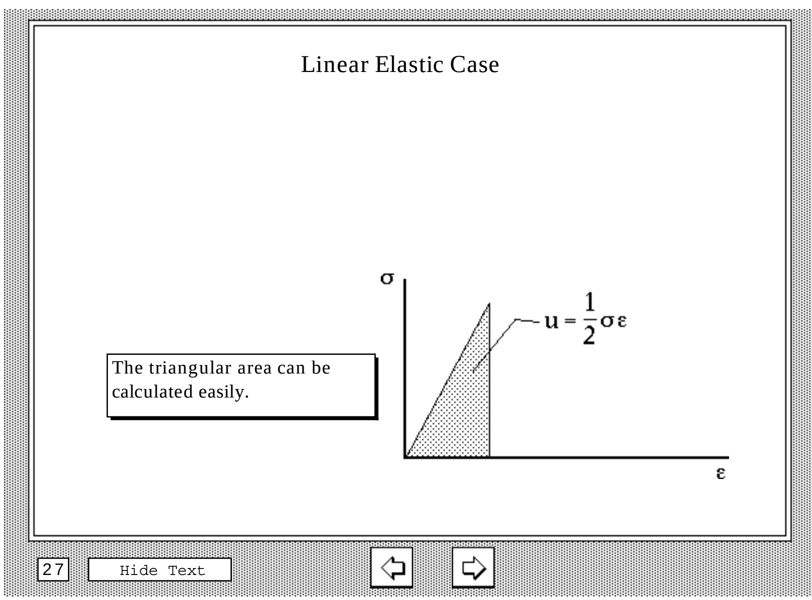
Linear Elastic Case
The triangular area can be calculated easily.
$u = \frac{1}{2}\sigma\varepsilon$
σ
ε
27
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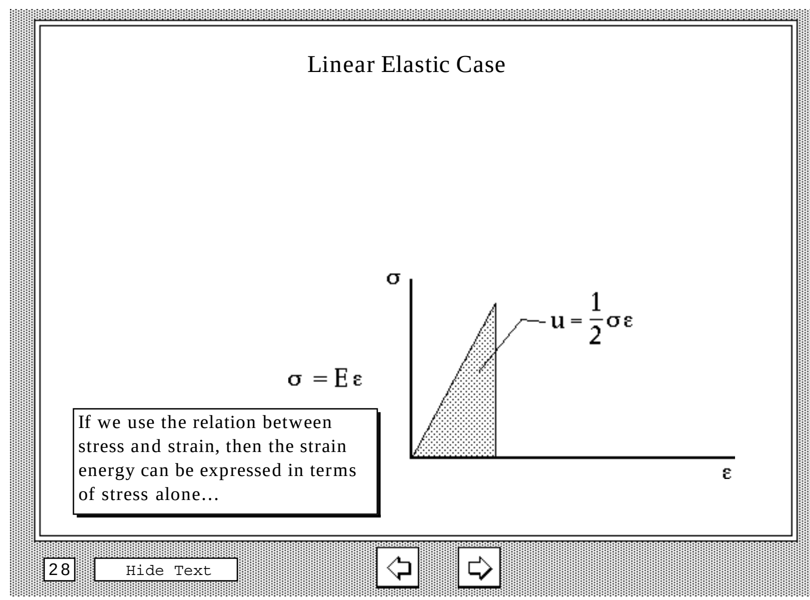
Linear Elastic Case
$\sigma = E\varepsilon$
If we use the relation between stress and strain, then the strain energy can be expressed in terms of stress alone…
$u = \frac{1}{2}\sigma\varepsilon$
σ
ε
28
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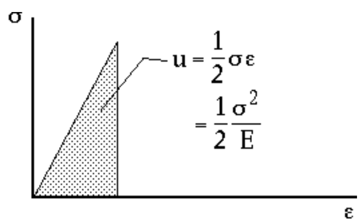
Linear Elastic Case

$\sigma = E\varepsilon$



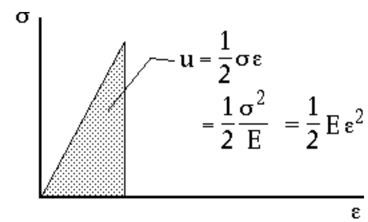
$u = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\frac{\sigma^2}{E}$

...or strain alone...

29 Hide Text

Linear Elastic Case

$\sigma = E\varepsilon$



$u = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\frac{\sigma^2}{E} = \frac{1}{2}E\varepsilon^2$

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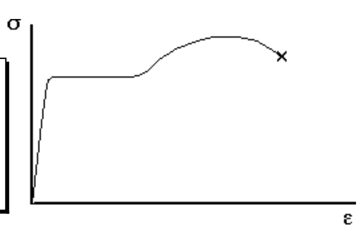
Elastic and Plastic Strain Energy

Earlier we made the distinction between elastic strain energy capacity and total strain energy capacity. In terms of strain energy density we define two additional material measures/moduli.



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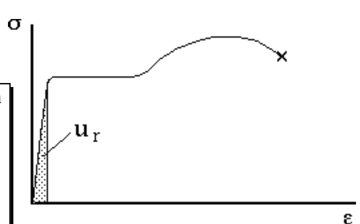
Modulus of Resilience

The modulus of resilience is the area under the elastic part of the stress strain curve. This provides a measure of a material's ability to absorb and return energy without damage.



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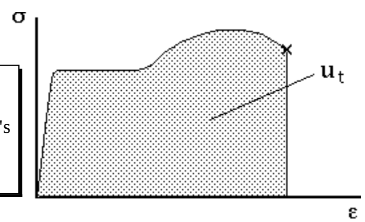
Modulus of Toughness

The modulus of toughness is the area under the entire stress-strain curve. This provides a measure of a material's ability to absorb destructive energy prior to fracturing.



Elastic Strain Energy Storage Capacities of Several Materials

Material	**Working Stress** psi	**Density** lbs/ft3	**Energy Stored** ft-lbs/ lb
Cast Iron	10,000	486	0.5
Steel Spring	100,000	486	44
Tendon	10,000	69	840
Rubber	1,000	75	2700

Here are some resilience values for common materials.

The End