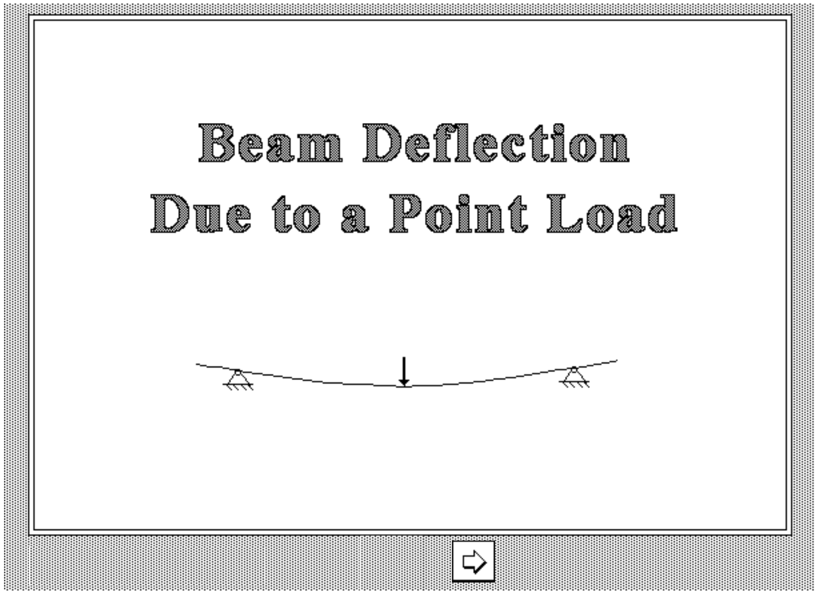
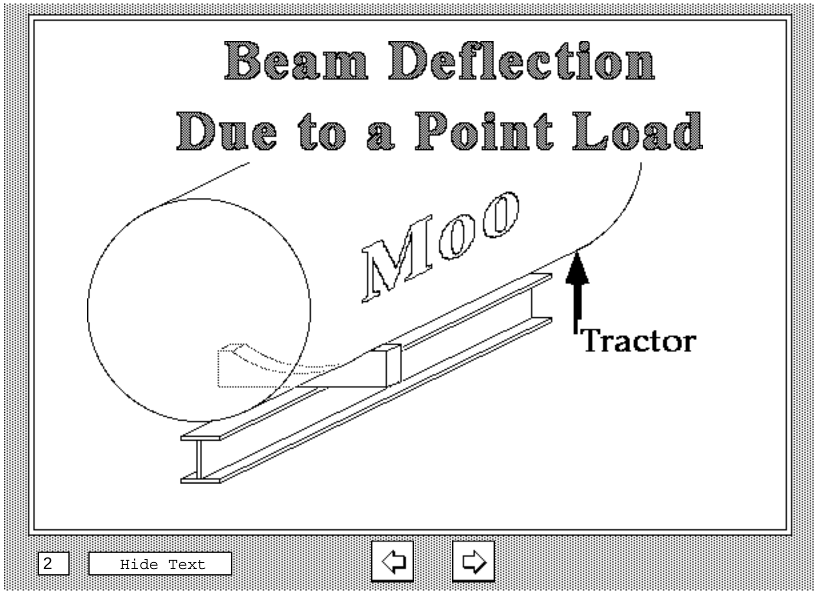
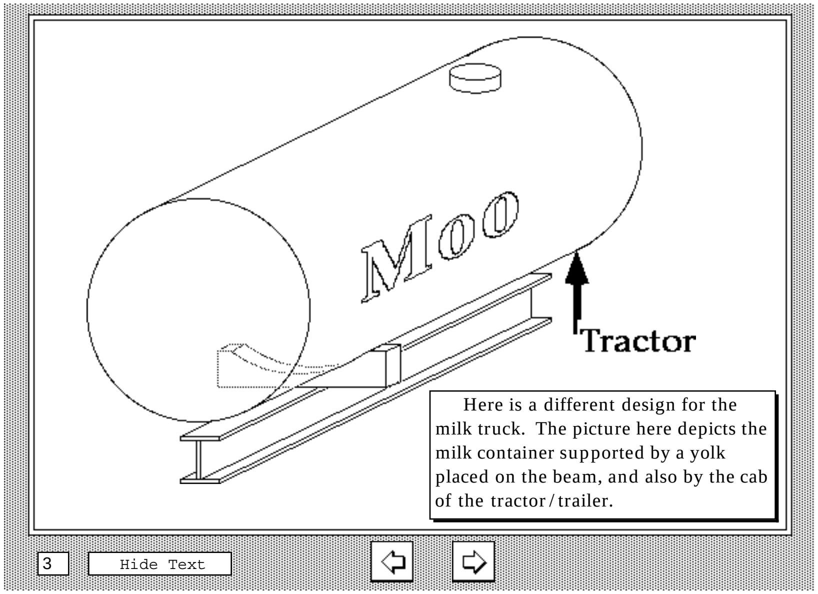


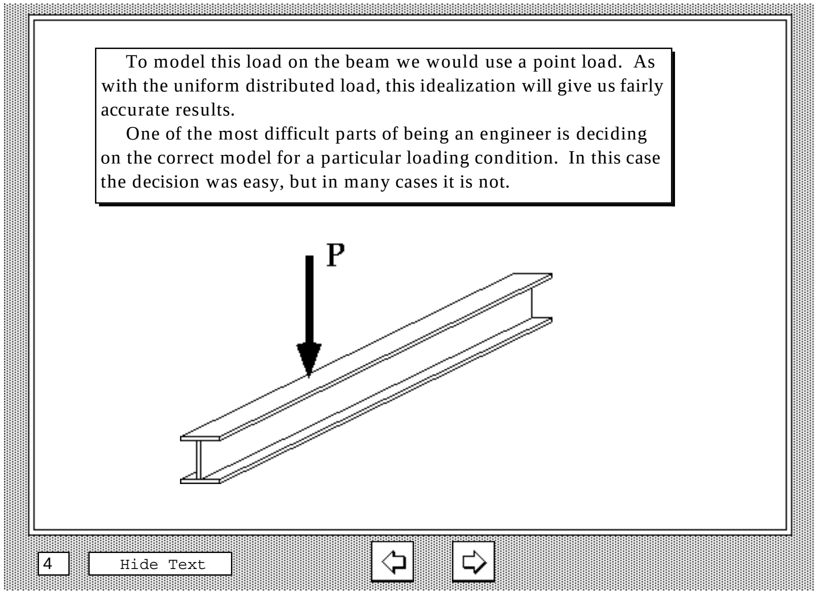
Beam Deflection
Due to a Point Load



Beam Deflection
Due to a Point Load

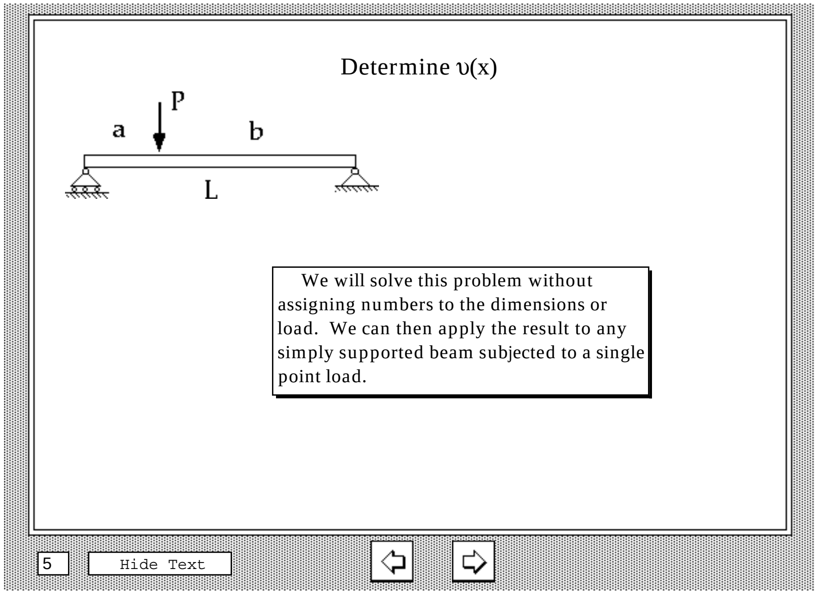


Here is a different design for the milk truck. The picture here depicts the milk container supported by a yolk placed on the beam, and also by the cab of the tractor/trailer.



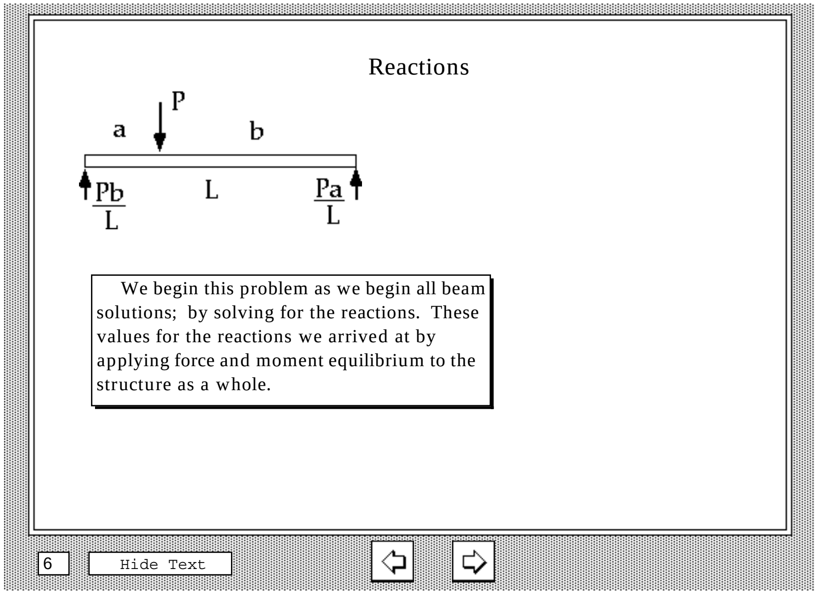
To model this load on the beam we would use a point load. As with the uniform distributed load, this idealization will give us fairly accurate results.

One of the most difficult parts of being an engineer is deciding on the correct model for a particular loading condition. In this case the decision was easy, but in many cases it is not.



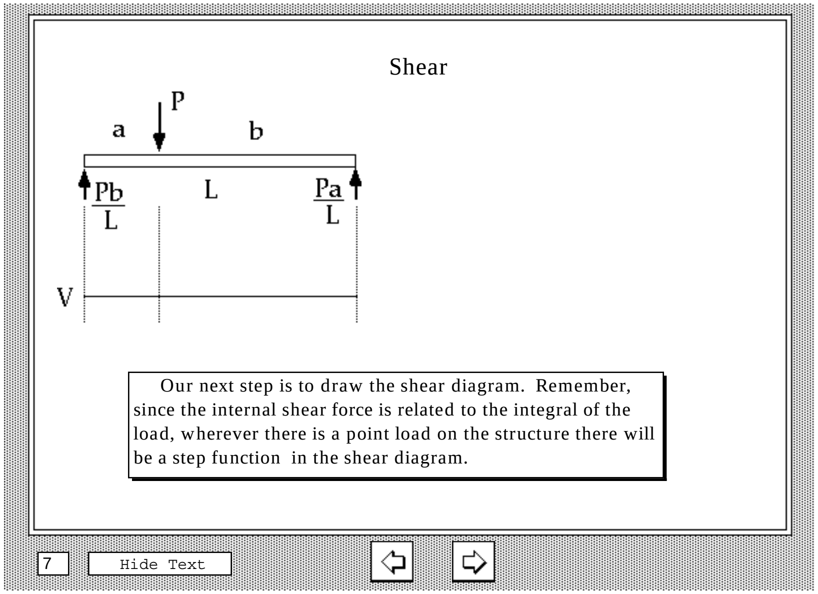
Determine υ(x)

We will solve this problem without assigning numbers to the dimensions or load. We can then apply the result to any simply supported beam subjected to a single point load.



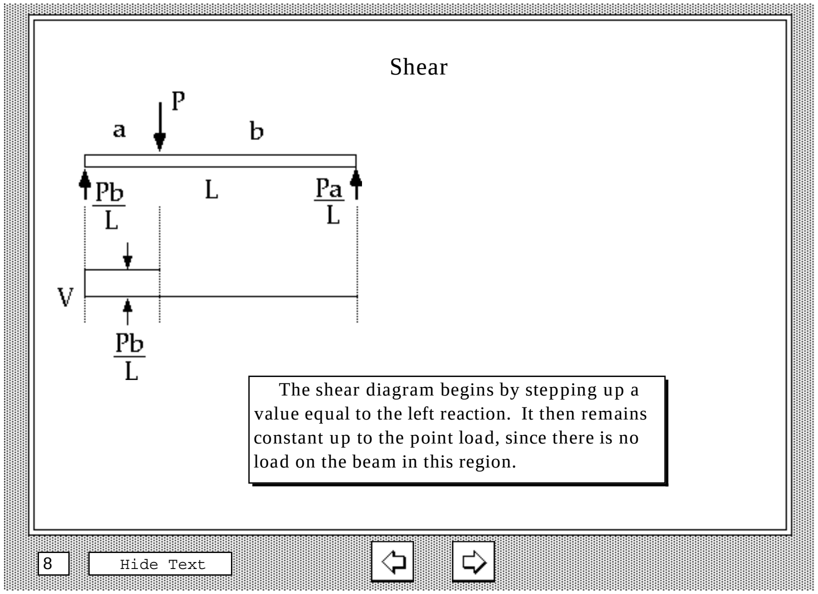
Reactions

We begin this problem as we begin all beam solutions; by solving for the reactions. These values for the reactions we arrived at by applying force and moment equilibrium to the structure as a whole.



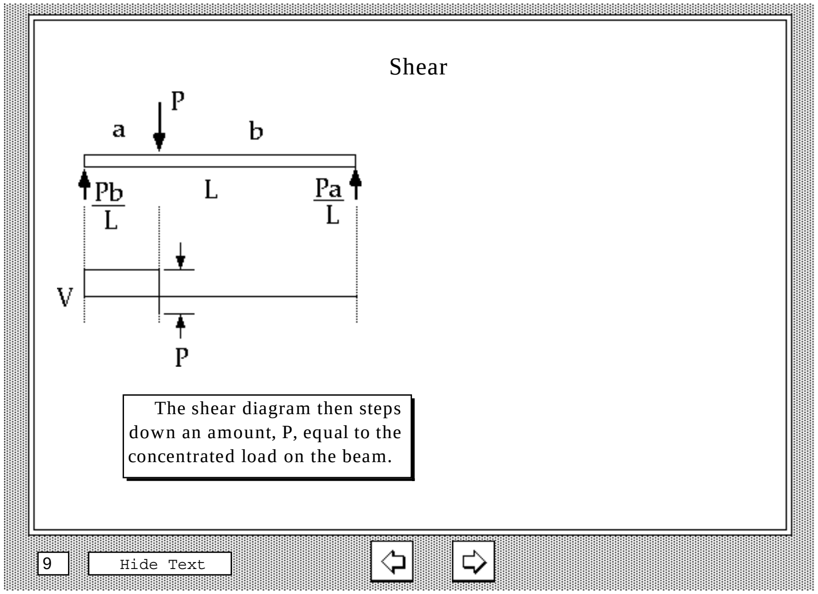
Shear

Our next step is to draw the shear diagram. Remember, since the internal shear force is related to the integral of the load, wherever there is a point load on the structure there will be a step function in the shear diagram.



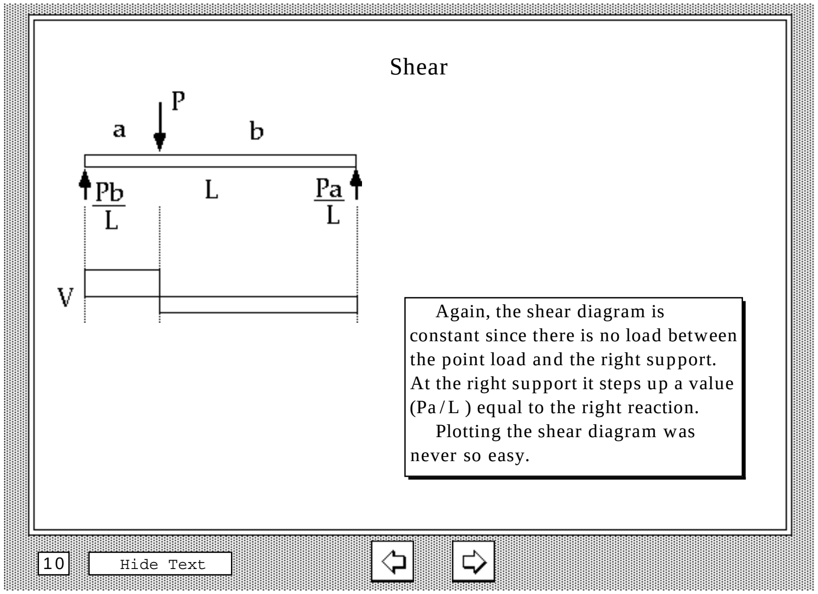
Shear

The shear diagram begins by stepping up a value equal to the left reaction. It then remains constant up to the point load, since there is no load on the beam in this region.



Shear

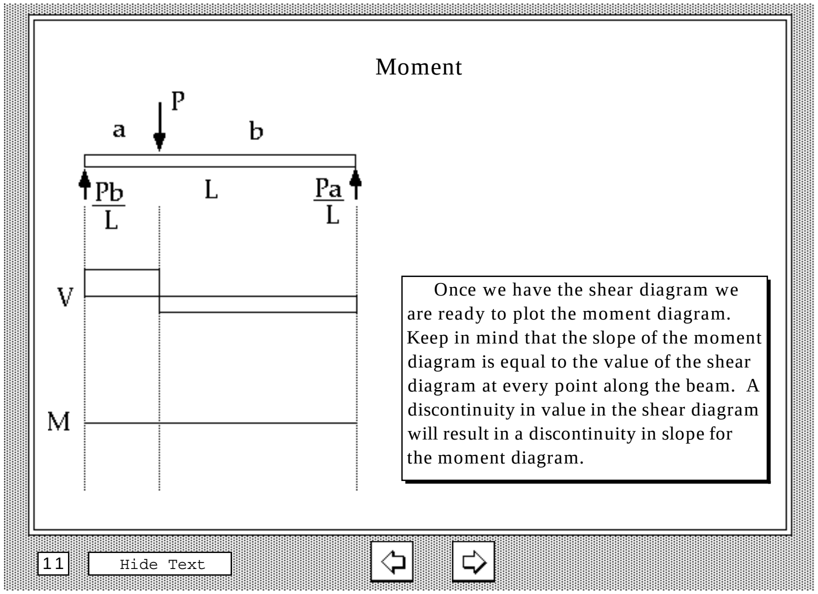
The shear diagram then steps down an amount, P, equal to the concentrated load on the beam.



Shear

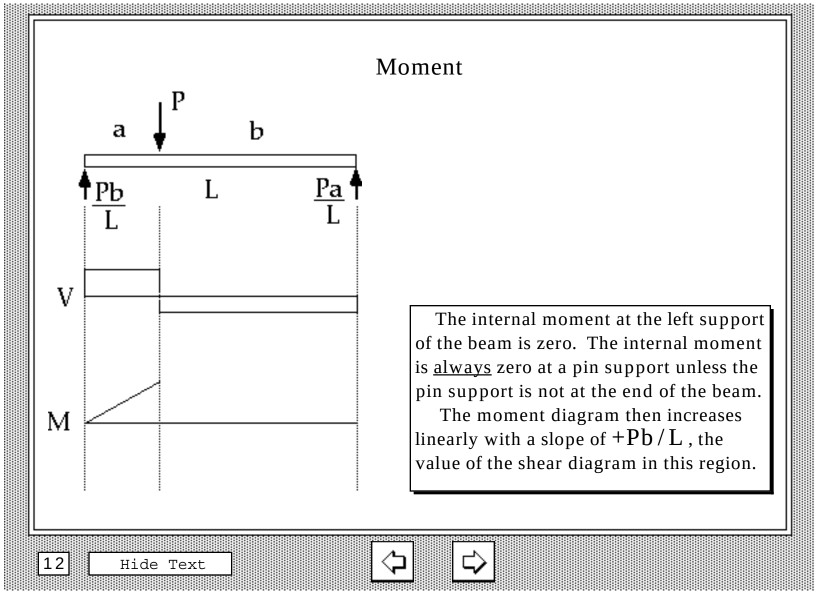
Again, the shear diagram is constant since there is no load between the point load and the right support. At the right support it steps up a value (Pa/L) equal to the right reaction.

Plotting the shear diagram was never so easy.



Moment

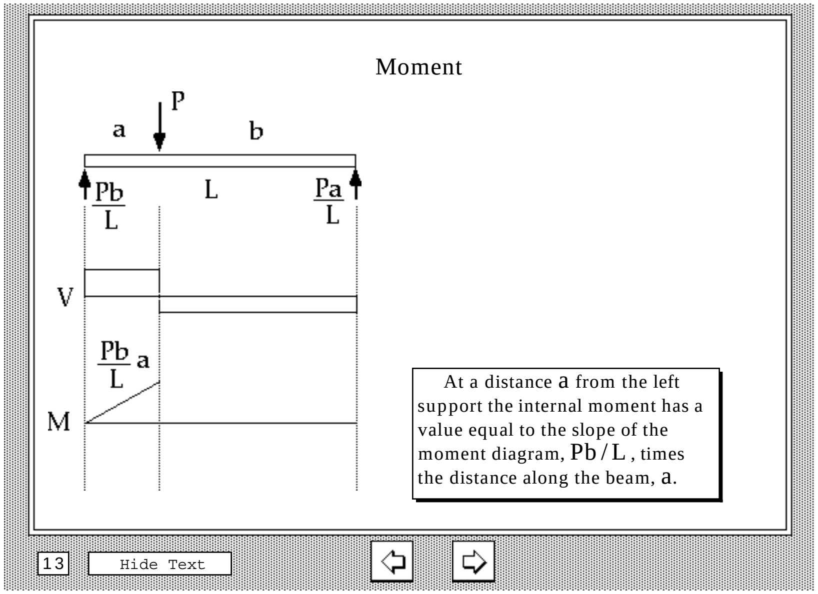
Once we have the shear diagram we are ready to plot the moment diagram. Keep in mind that the slope of the moment diagram is equal to the value of the shear diagram at every point along the beam. A discontinuity in value in the shear diagram will result in a discontinuity in slope for the moment diagram.



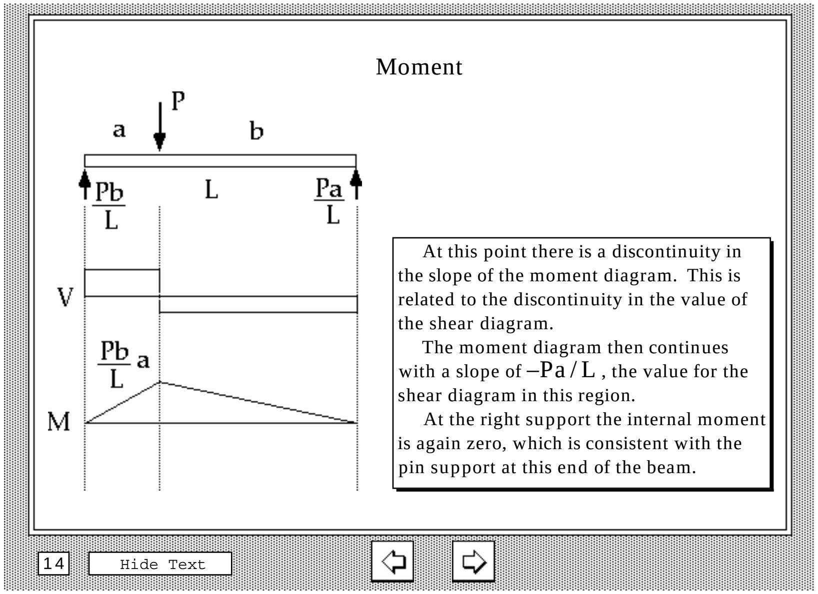
Moment

The internal moment at the left support of the beam is zero. The internal moment is always zero at a pin support unless the pin support is not at the end of the beam.

The moment diagram then increases linearly with a slope of $+Pb/L$, the value of the shear diagram in this region.



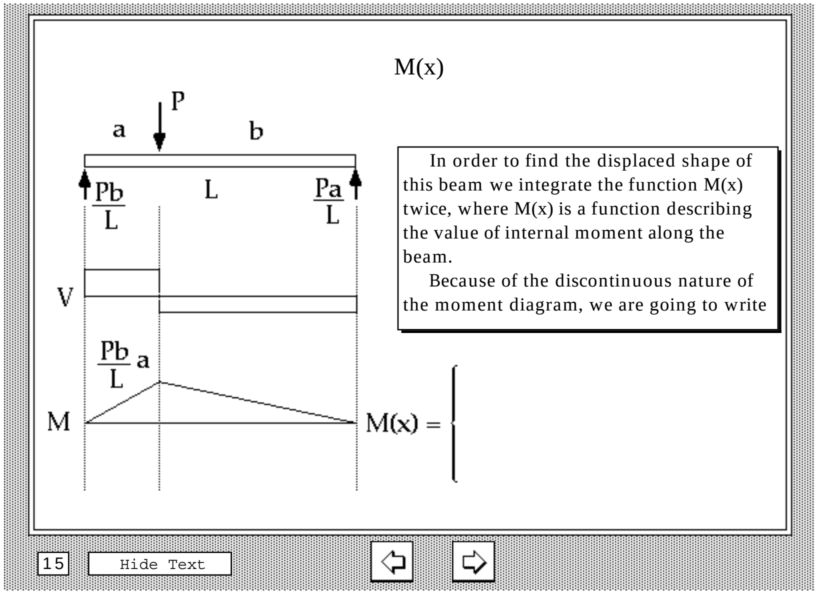
At a distance a from the left support the internal moment has a value equal to the slope of the moment diagram, Pb / L , times the distance along the beam, a.



At this point there is a discontinuity in the slope of the moment diagram. This is related to the discontinuity in the value of the shear diagram.

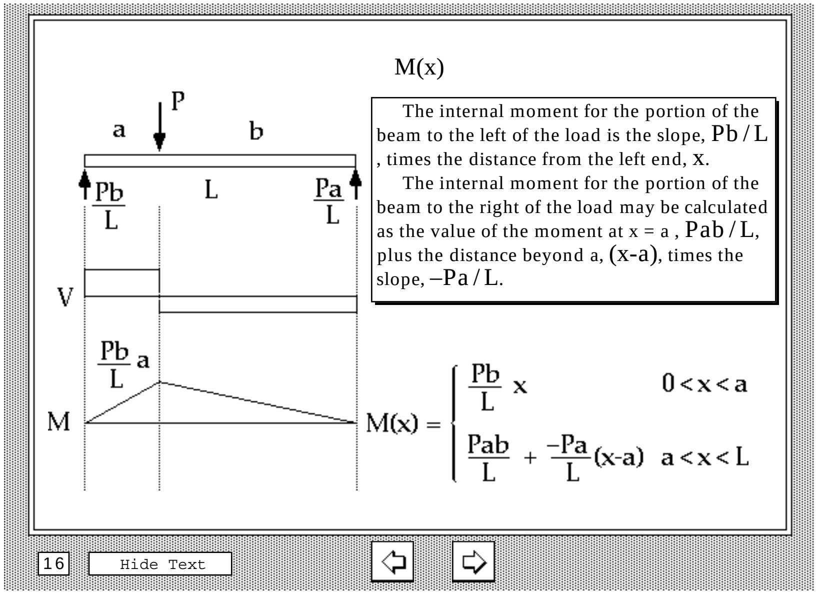
The moment diagram then continues with a slope of –Pa / L , the value for the shear diagram in this region.

At the right support the internal moment is again zero, which is consistent with the pin support at this end of the beam.



In order to find the displaced shape of this beam we integrate the function M(x) twice, where M(x) is a function describing the value of internal moment along the beam.

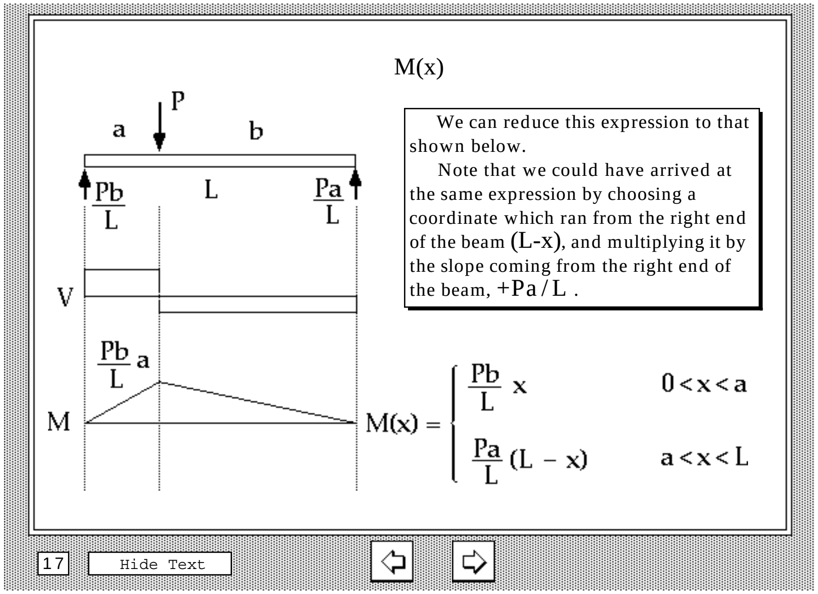
Because of the discontinuous nature of the moment diagram, we are going to write



The internal moment for the portion of the beam to the left of the load is the slope, Pb / L , times the distance from the left end, x.

The internal moment for the portion of the beam to the right of the load may be calculated as the value of the moment at x = a , Pab / L, plus the distance beyond a, (x-a), times the slope, –Pa / L.

$$M(x) = \begin{cases} \frac{Pb}{L}\,x & 0 < x < a \\ \frac{Pab}{L} + \frac{-Pa}{L}(x\text{-}a) & a < x < L \end{cases}$$

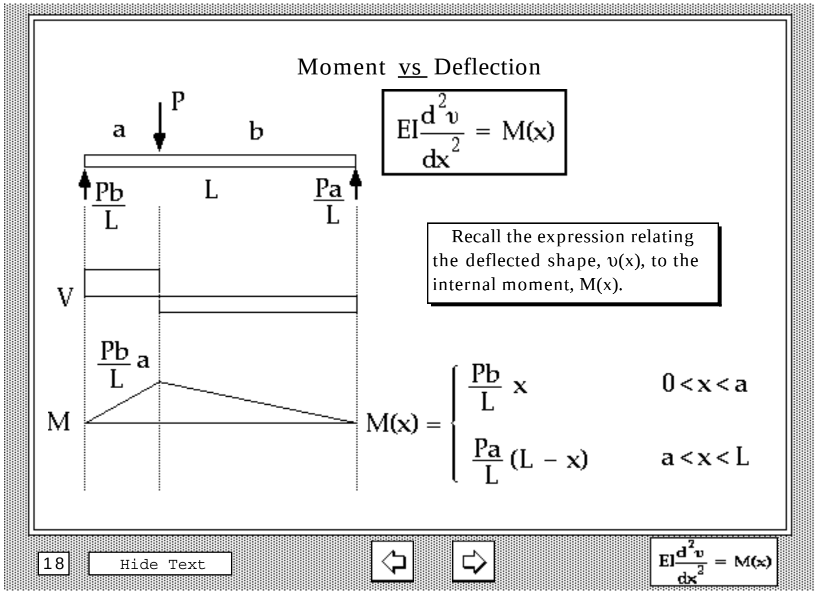


M(x)

We can reduce this expression to that shown below.

Note that we could have arrived at the same expression by choosing a coordinate which ran from the right end of the beam (L-x), and multiplying it by the slope coming from the right end of the beam, +Pa / L .

$$M(x) = \begin{cases} \frac{Pb}{L}\, x & 0 < x < a \\ \frac{Pa}{L}(L - x) & a < x < L \end{cases}$$

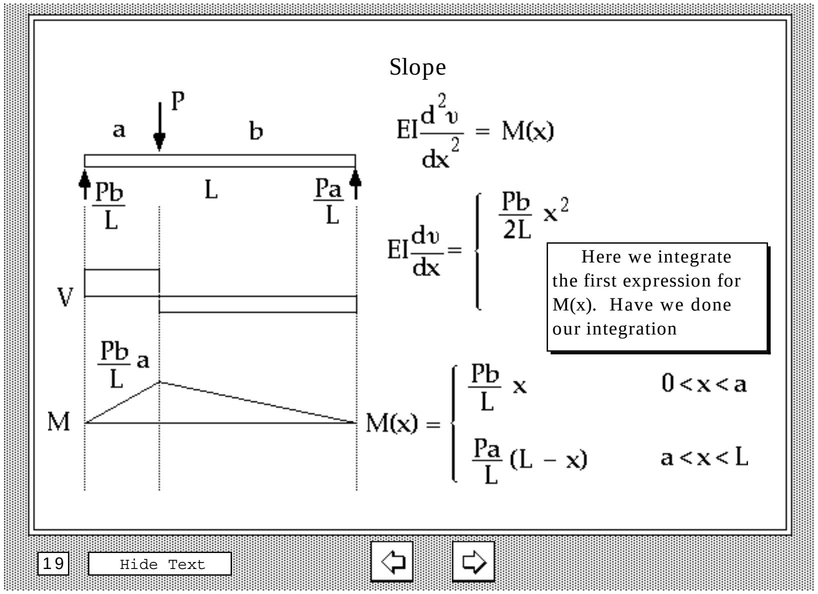


Moment vs Deflection

$$\boxed{EI\frac{d^2\upsilon}{dx^2} = M(x)}$$

Recall the expression relating the deflected shape, υ(x), to the internal moment, M(x).

$$M(x) = \begin{cases} \frac{Pb}{L}\, x & 0 < x < a \\ \frac{Pa}{L}(L - x) & a < x < L \end{cases}$$



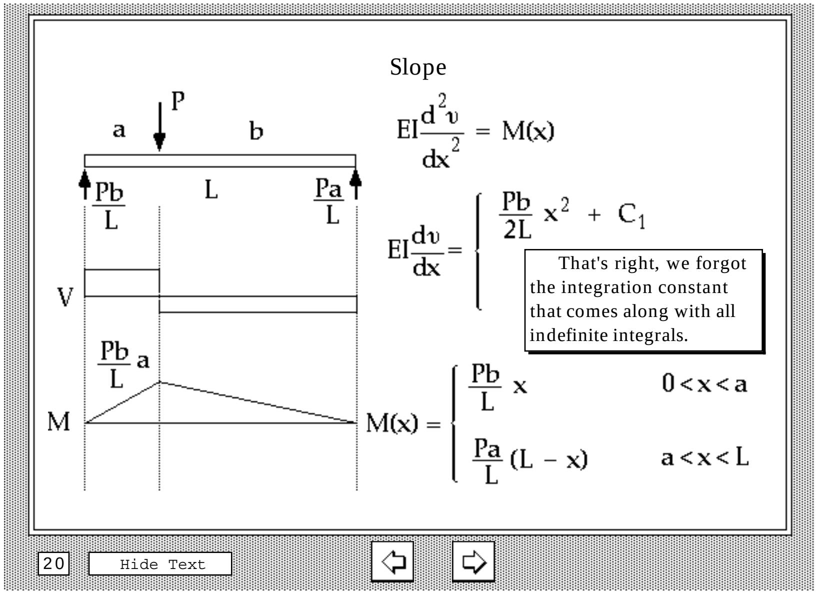
Slope

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}\, x^2 \\ \end{cases}$$

Here we integrate the first expression for M(x). Have we done our integration

$$M(x) = \begin{cases} \frac{Pb}{L}\, x & 0 < x < a \\ \frac{Pa}{L}(L - x) & a < x < L \end{cases}$$



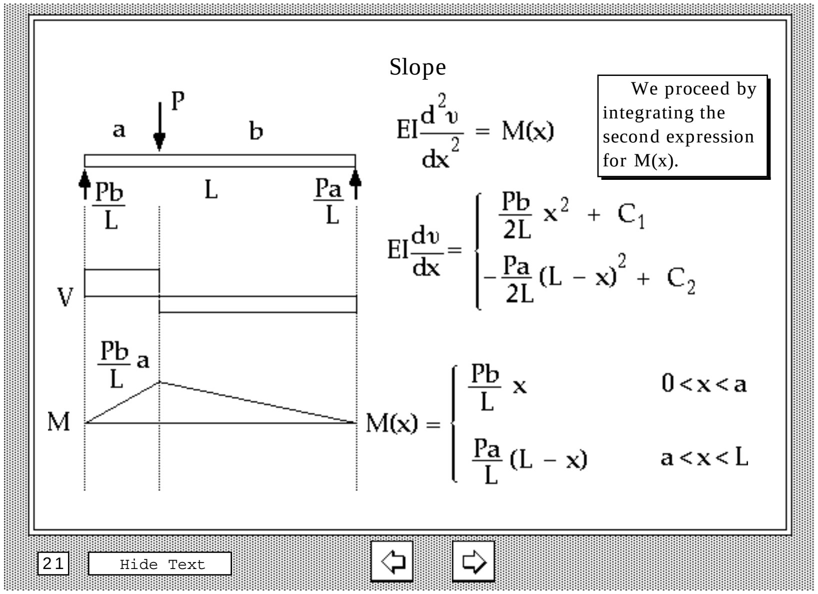
Slope

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}\, x^2 + C_1 \\ \end{cases}$$

That's right, we forgot the integration constant that comes along with all indefinite integrals.

$$M(x) = \begin{cases} \frac{Pb}{L}\, x & 0 < x < a \\ \frac{Pa}{L}(L - x) & a < x < L \end{cases}$$



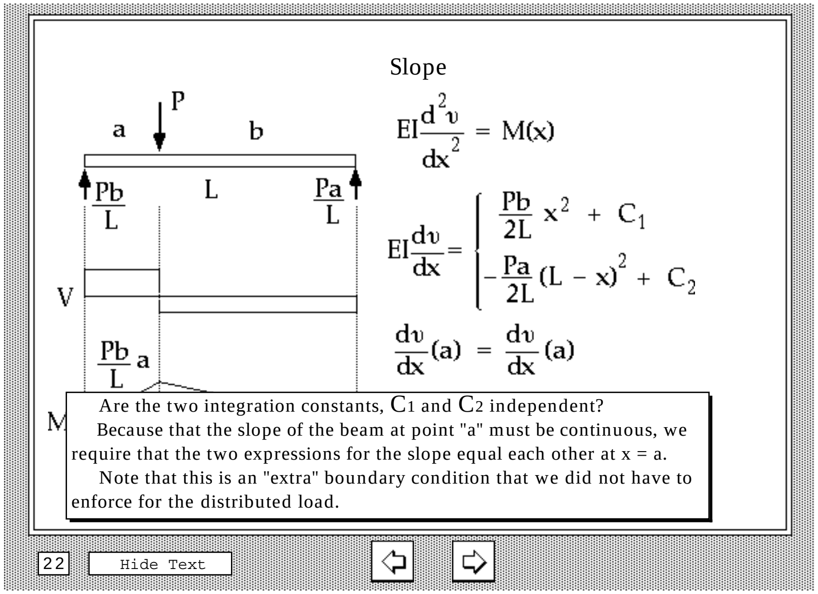
Slope

$$EI\frac{d^2v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + C_2 \end{cases}$$

$$M(x) = \begin{cases} \frac{Pb}{L}x & 0<x<a \\ \frac{Pa}{L}(L-x) & a<x<L \end{cases}$$

We proceed by integrating the second expression for M(x).



Slope

$$EI\frac{d^2v}{dx^2} = M(x)$$

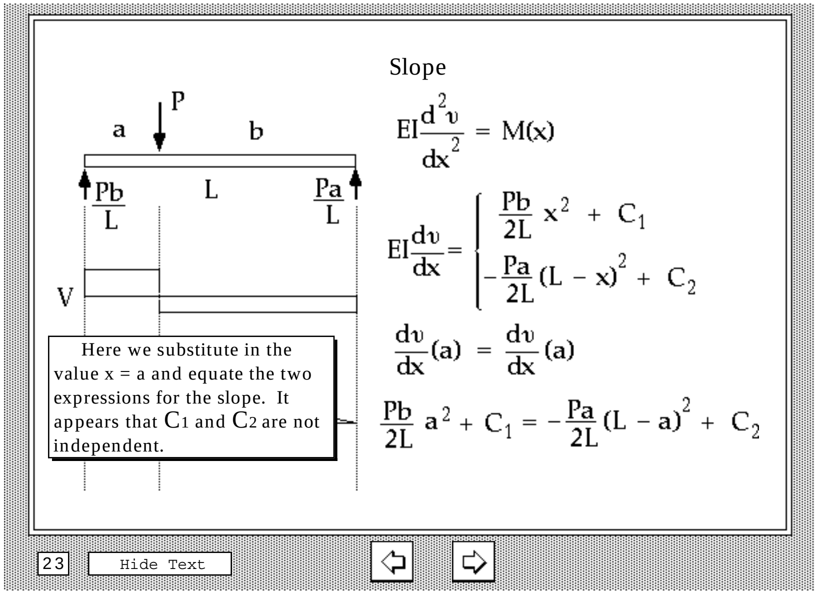
$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + C_2 \end{cases}$$

$$\frac{dv}{dx}(a) = \frac{dv}{dx}(a)$$

Are the two integration constants, C1 and C2 independent?

Because that the slope of the beam at point "a" must be continuous, we require that the two expressions for the slope equal each other at x = a.

Note that this is an "extra" boundary condition that we did not have to enforce for the distributed load.



Slope

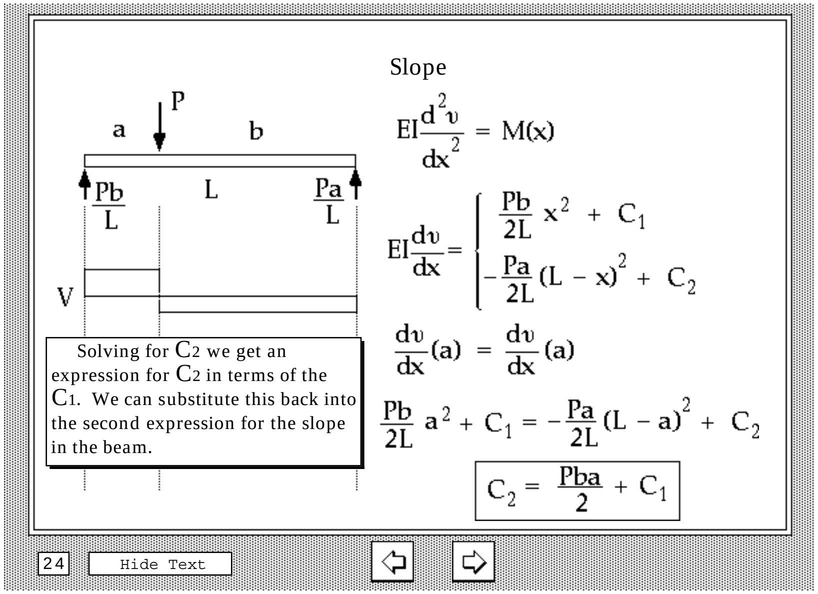
$$EI\frac{d^2v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + C_2 \end{cases}$$

$$\frac{dv}{dx}(a) = \frac{dv}{dx}(a)$$

$$\frac{Pb}{2L}a^2 + C_1 = -\frac{Pa}{2L}(L-a)^2 + C_2$$

Here we substitute in the value x = a and equate the two expressions for the slope. It appears that C1 and C2 are not independent.



Slope

$$EI\frac{d^2v}{dx^2} = M(x)$$

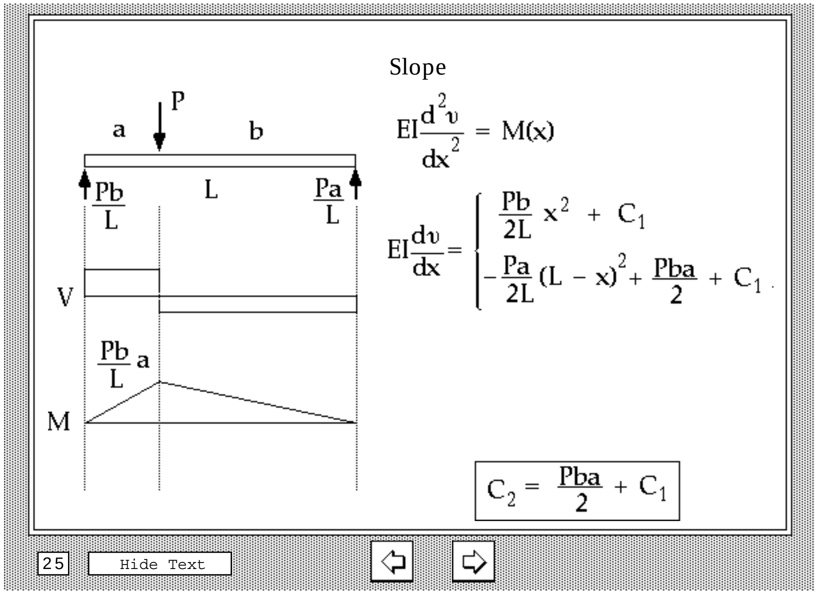
$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + C_2 \end{cases}$$

$$\frac{dv}{dx}(a) = \frac{dv}{dx}(a)$$

$$\frac{Pb}{2L}a^2 + C_1 = -\frac{Pa}{2L}(L-a)^2 + C_2$$

$$\boxed{C_2 = \frac{Pba}{2} + C_1}$$

Solving for C2 we get an expression for C2 in terms of the C1. We can substitute this back into the second expression for the slope in the beam.

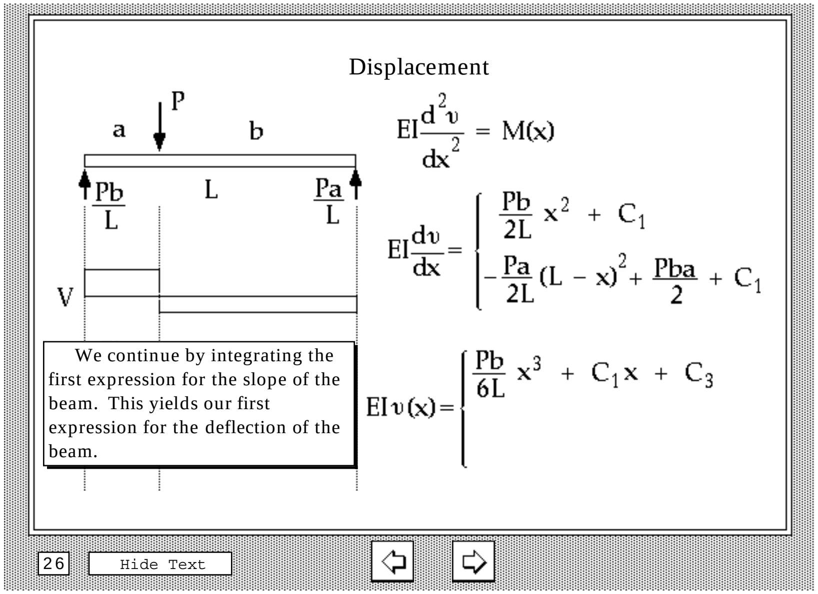


Slope

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$\boxed{C_2 = \frac{Pba}{2} + C_1}$$



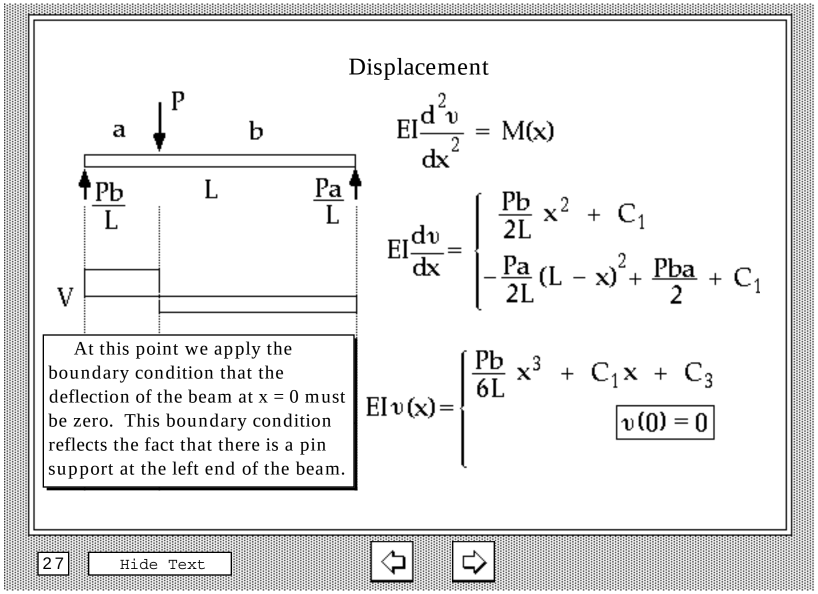
Displacement

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1x + C_3 \\ \end{cases}$$

We continue by integrating the first expression for the slope of the beam. This yields our first expression for the deflection of the beam.



Displacement

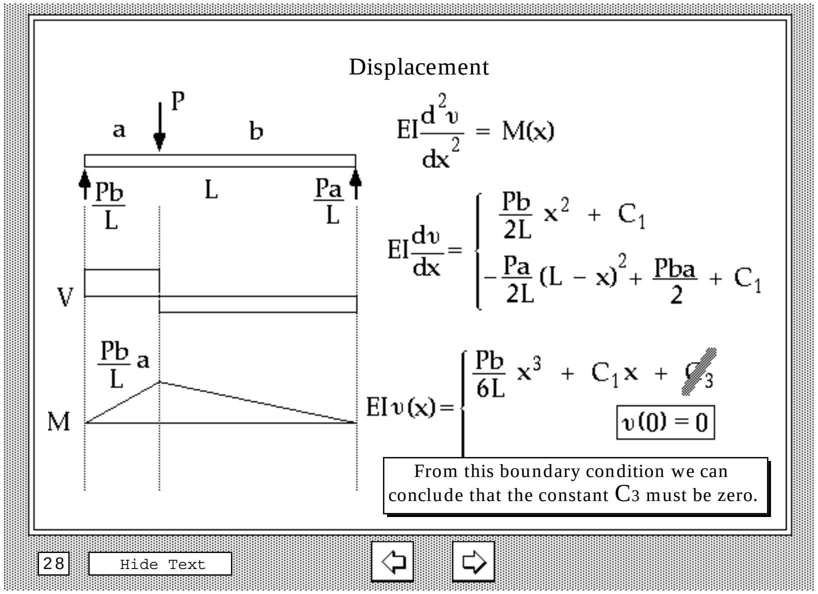
$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1x + C_3 \\ \end{cases}$$

$$\boxed{\upsilon(0) = 0}$$

At this point we apply the boundary condition that the deflection of the beam at x = 0 must be zero. This boundary condition reflects the fact that there is a pin support at the left end of the beam.



Displacement

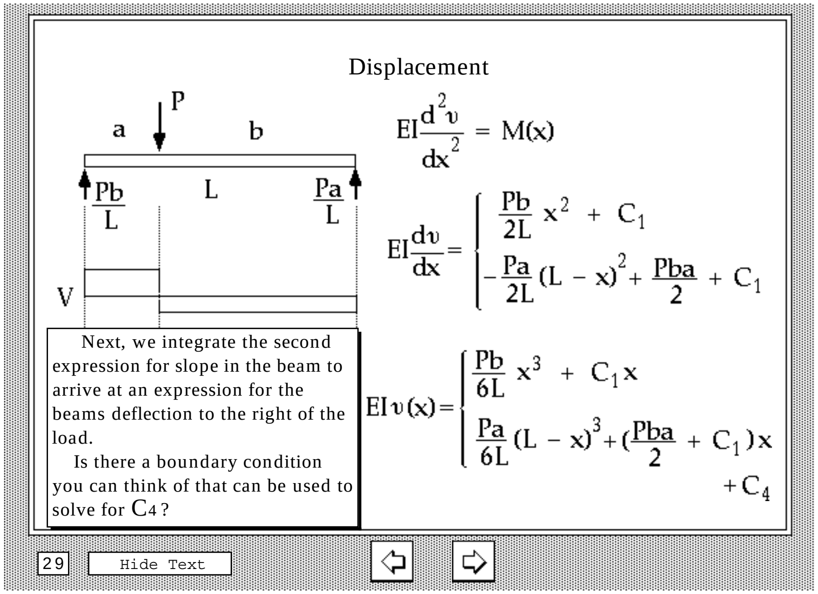
$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d\upsilon}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1x + \cancel{C_3} \\ \end{cases}$$

$$\boxed{\upsilon(0) = 0}$$

From this boundary condition we can conclude that the constant C3 must be zero.



Displacement

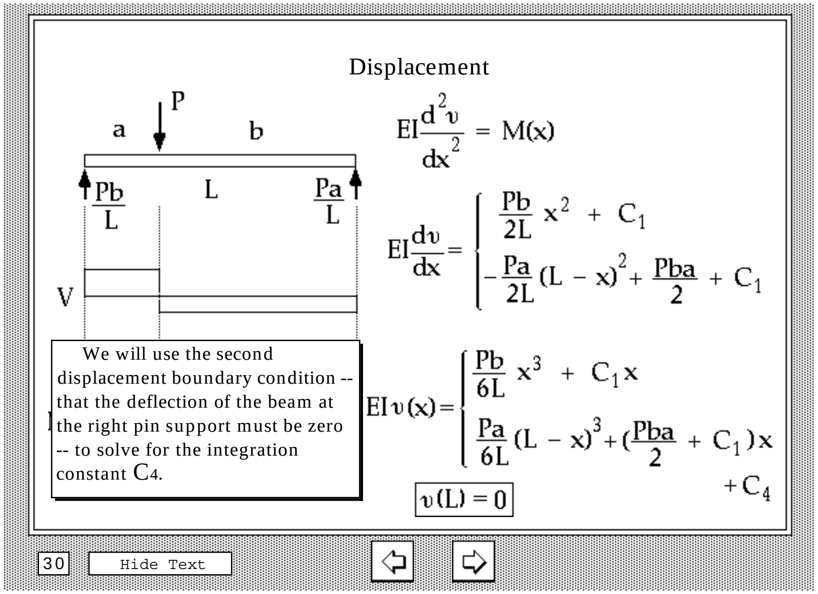
$$EI\frac{d^2 v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$EI v(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 + (\frac{Pba}{2} + C_1)x + C_4 \end{cases}$$

Next, we integrate the second expression for slope in the beam to arrive at an expression for the beams deflection to the right of the load.

Is there a boundary condition you can think of that can be used to solve for C_4?



Displacement

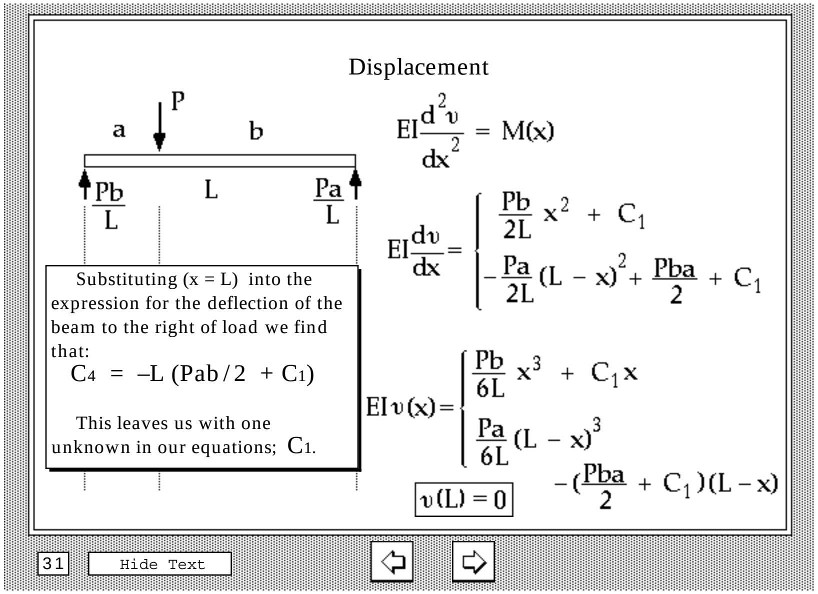
$$EI\frac{d^2 v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

$$EI v(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 + (\frac{Pba}{2} + C_1)x + C_4 \end{cases}$$

$$\boxed{v(L) = 0}$$

We will use the second displacement boundary condition -- that the deflection of the beam at the right pin support must be zero -- to solve for the integration constant C_4.



Displacement

$$EI\frac{d^2 v}{dx^2} = M(x)$$

$$EI\frac{dv}{dx} = \begin{cases} \frac{Pb}{2L}x^2 + C_1 \\ -\frac{Pa}{2L}(L-x)^2 + \frac{Pba}{2} + C_1 \end{cases}$$

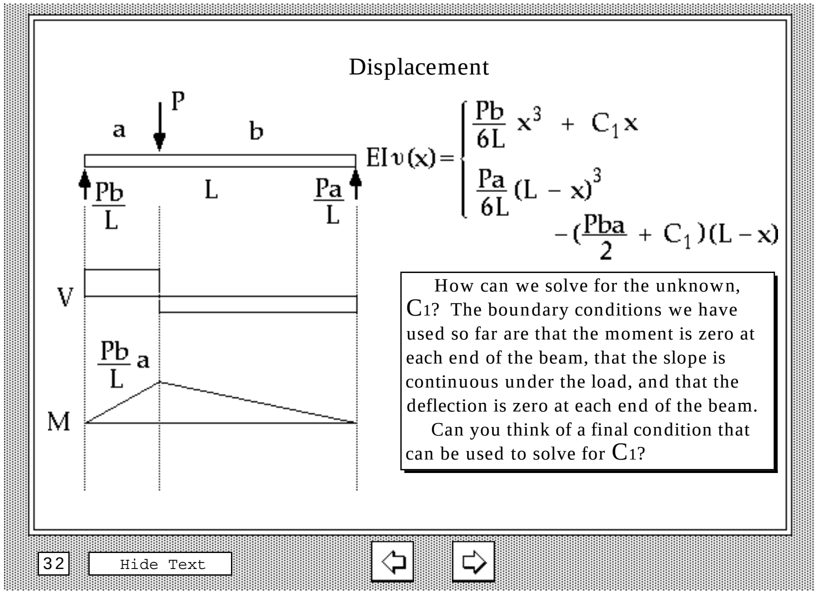
$$EI v(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

$$\boxed{v(L) = 0}$$

Substituting (x = L) into the expression for the deflection of the beam to the right of load we find that:

$$C_4 = -L\,(Pab/2 + C_1)$$

This leaves us with one unknown in our equations; C_1.

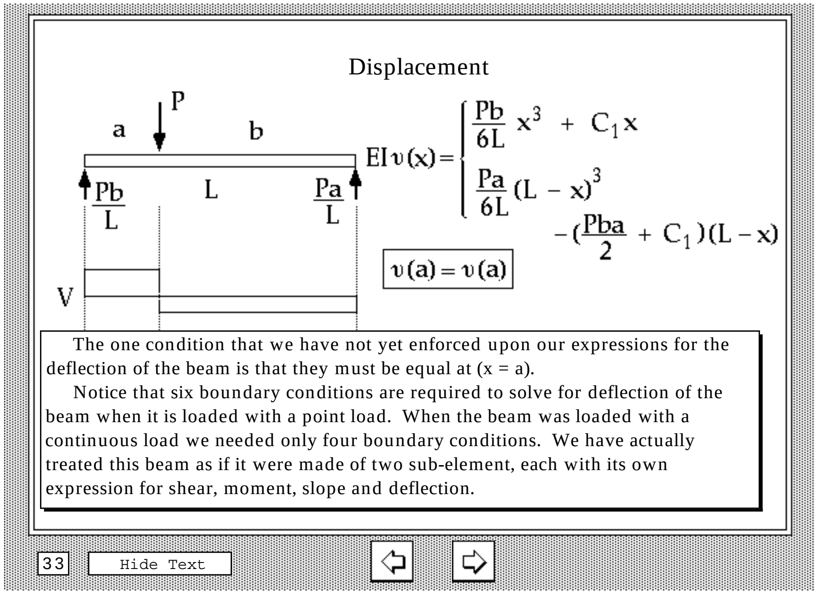


Displacement

$$EI v(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

How can we solve for the unknown, C_1? The boundary conditions we have used so far are that the moment is zero at each end of the beam, that the slope is continuous under the load, and that the deflection is zero at each end of the beam.

Can you think of a final condition that can be used to solve for C_1?



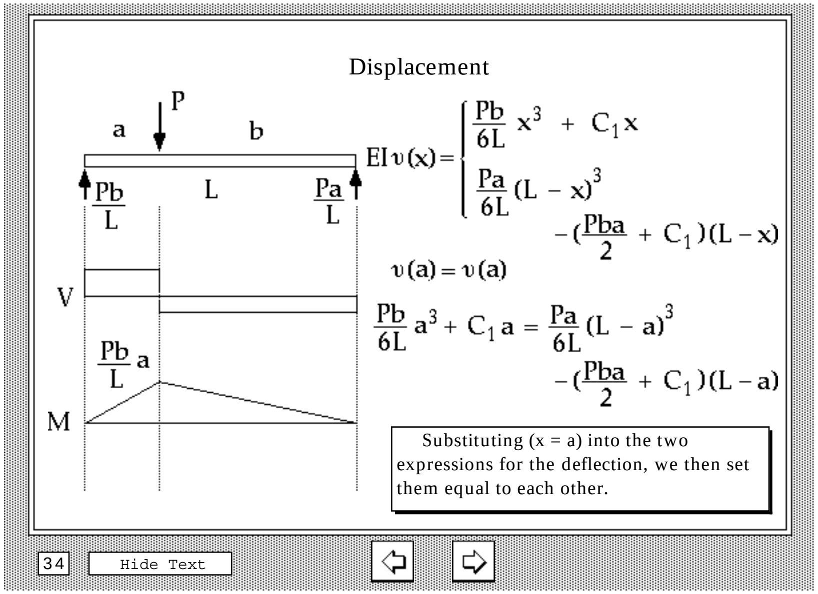
Displacement

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

$$\boxed{\upsilon(a) = \upsilon(a)}$$

The one condition that we have not yet enforced upon our expressions for the deflection of the beam is that they must be equal at (x = a).

Notice that six boundary conditions are required to solve for deflection of the beam when it is loaded with a point load. When the beam was loaded with a continuous load we needed only four boundary conditions. We have actually treated this beam as if it were made of two sub-element, each with its own expression for shear, moment, slope and deflection.



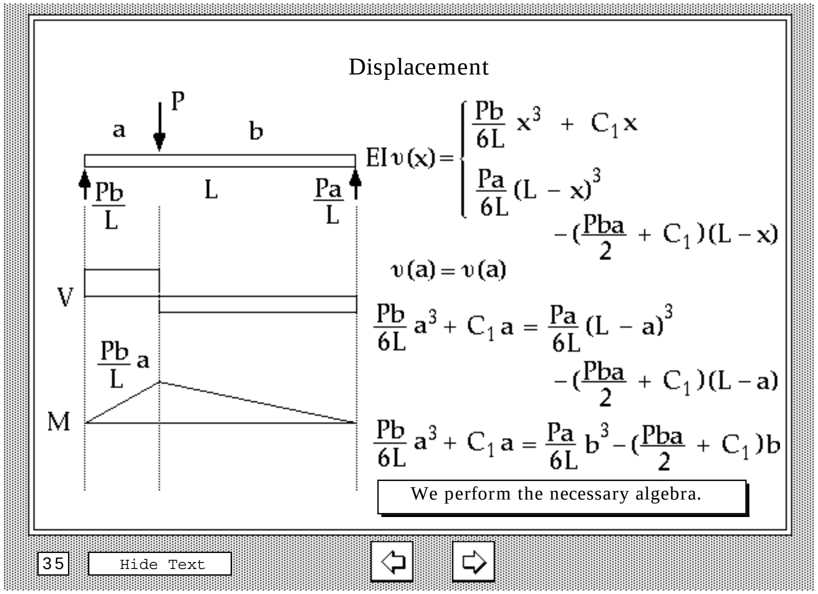
Displacement

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

$$\upsilon(a) = \upsilon(a)$$

$$\frac{Pb}{6L}a^3 + C_1 a = \frac{Pa}{6L}(L-a)^3 - (\frac{Pba}{2} + C_1)(L-a)$$

Substituting (x = a) into the two expressions for the deflection, we then set them equal to each other.



Displacement

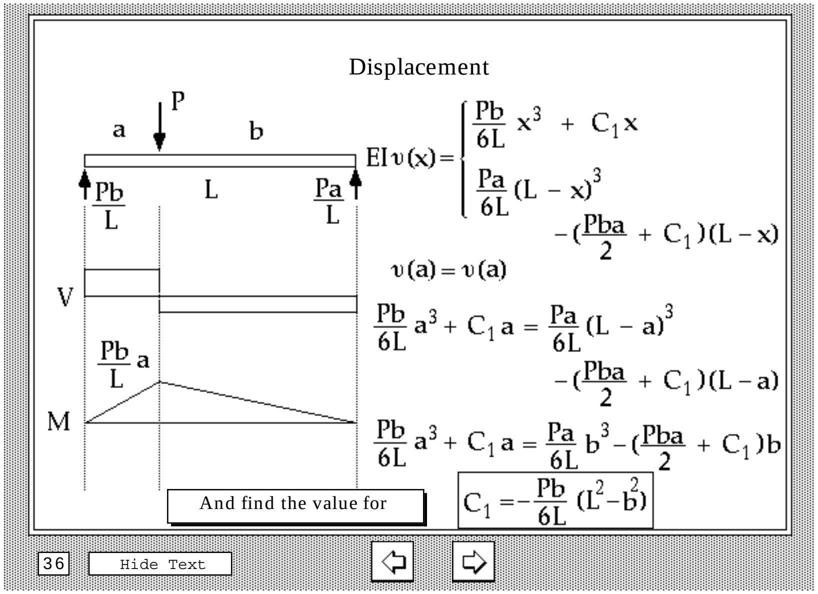
$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

$$\upsilon(a) = \upsilon(a)$$

$$\frac{Pb}{6L}a^3 + C_1 a = \frac{Pa}{6L}(L-a)^3 - (\frac{Pba}{2} + C_1)(L-a)$$

$$\frac{Pb}{6L}a^3 + C_1 a = \frac{Pa}{6L}b^3 - (\frac{Pba}{2} + C_1)b$$

We perform the necessary algebra.



Displacement

$$EI\upsilon(x) = \begin{cases} \frac{Pb}{6L}x^3 + C_1 x \\ \frac{Pa}{6L}(L-x)^3 - (\frac{Pba}{2} + C_1)(L-x) \end{cases}$$

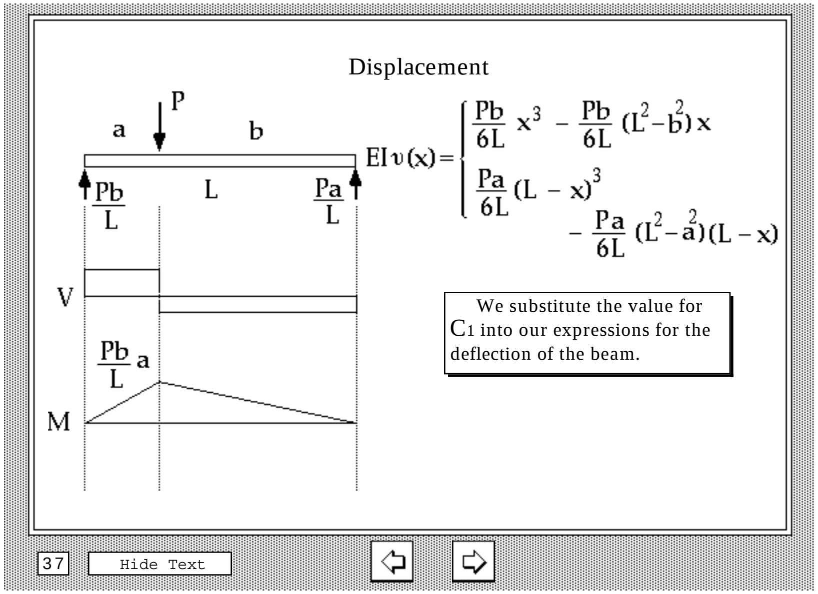
$$\upsilon(a) = \upsilon(a)$$

$$\frac{Pb}{6L}a^3 + C_1 a = \frac{Pa}{6L}(L-a)^3 - (\frac{Pba}{2} + C_1)(L-a)$$

$$\frac{Pb}{6L}a^3 + C_1 a = \frac{Pa}{6L}b^3 - (\frac{Pba}{2} + C_1)b$$

And find the value for

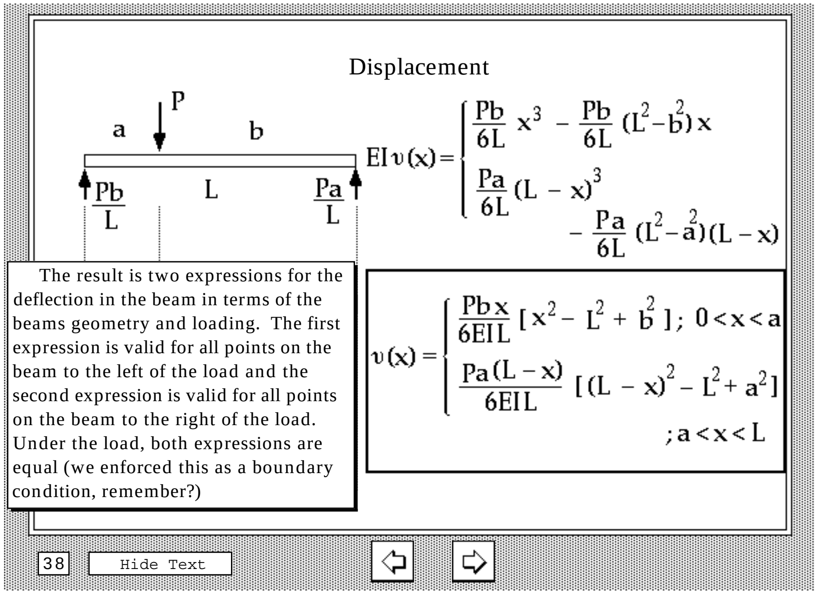
$$\boxed{C_1 = -\frac{Pb}{6L}(L^2 - b^2)}$$



Displacement

$$EI\,v(x) = \begin{cases} \frac{Pb}{6L}\,x^3 - \frac{Pb}{6L}\,(L^2 - b^2)\,x \\ \frac{Pa}{6L}\,(L - x)^3 - \frac{Pa}{6L}\,(L^2 - a^2)(L - x) \end{cases}$$

We substitute the value for C_1 into our expressions for the deflection of the beam.

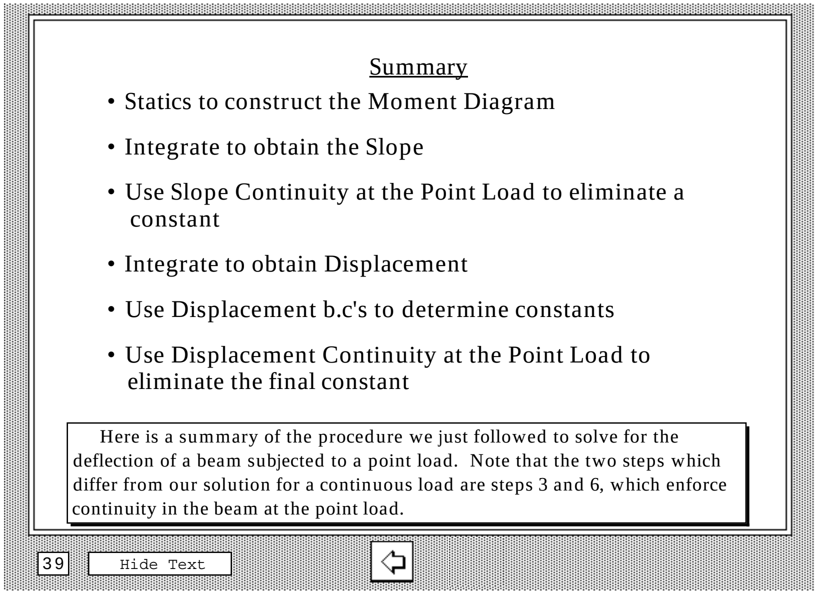


Displacement

$$EI\,v(x) = \begin{cases} \frac{Pb}{6L}\,x^3 - \frac{Pb}{6L}\,(L^2 - b^2)\,x \\ \frac{Pa}{6L}\,(L - x)^3 - \frac{Pa}{6L}\,(L^2 - a^2)(L - x) \end{cases}$$

The result is two expressions for the deflection in the beam in terms of the beams geometry and loading. The first expression is valid for all points on the beam to the left of the load and the second expression is valid for all points on the beam to the right of the load. Under the load, both expressions are equal (we enforced this as a boundary condition, remember?)

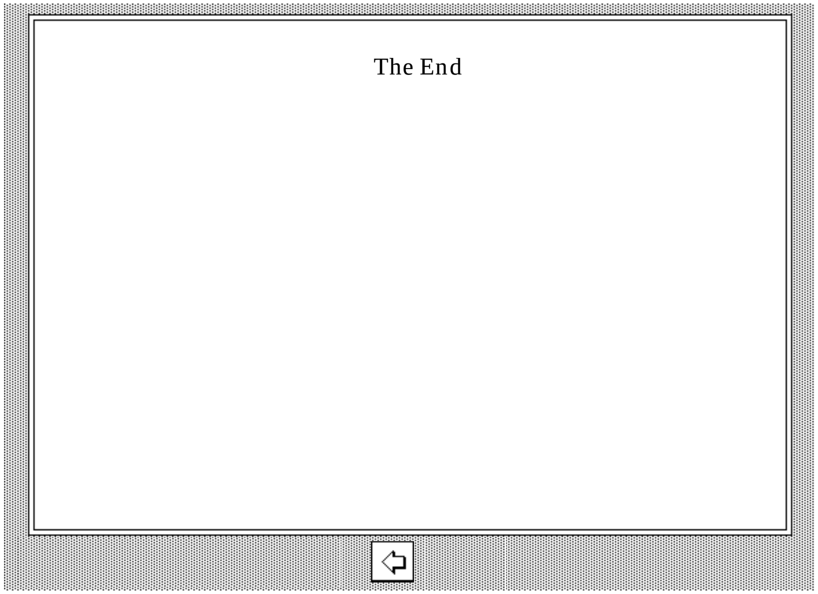
$$v(x) = \begin{cases} \frac{Pbx}{6EIL}\,[x^2 - L^2 + b^2]\,;\ 0 < x < a \\ \frac{Pa(L - x)}{6EIL}\,[(L - x)^2 - L^2 + a^2] \\ \qquad ;\,a < x < L \end{cases}$$



Summary

- Statics to construct the Moment Diagram
- Integrate to obtain the Slope
- Use Slope Continuity at the Point Load to eliminate a constant
- Integrate to obtain Displacement
- Use Displacement b.c's to determine constants
- Use Displacement Continuity at the Point Load to eliminate the final constant

Here is a summary of the procedure we just followed to solve for the deflection of a beam subjected to a point load. Note that the two steps which differ from our solution for a continuous load are steps 3 and 6, which enforce continuity in the beam at the point load.



The End