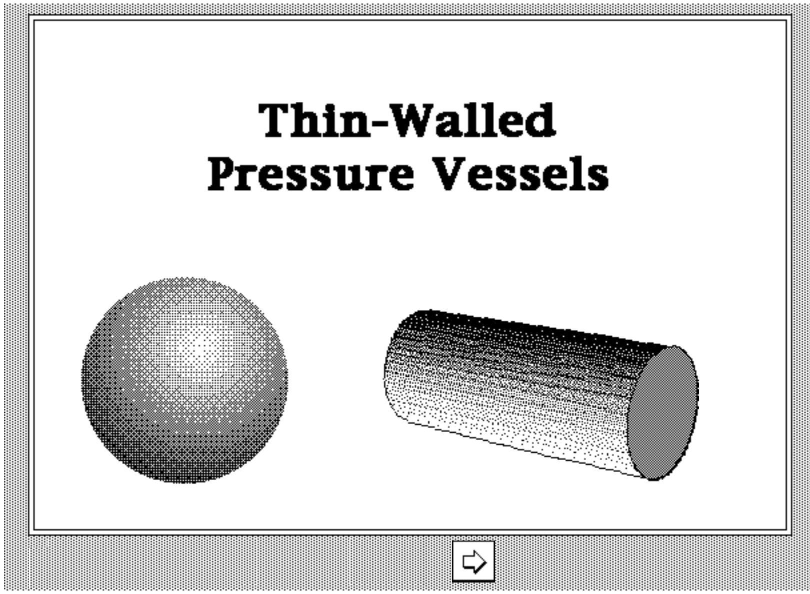
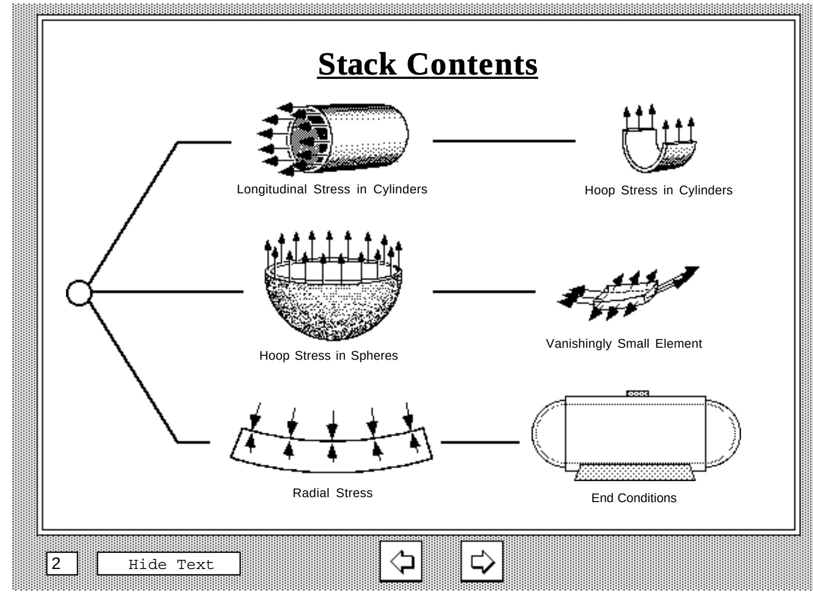
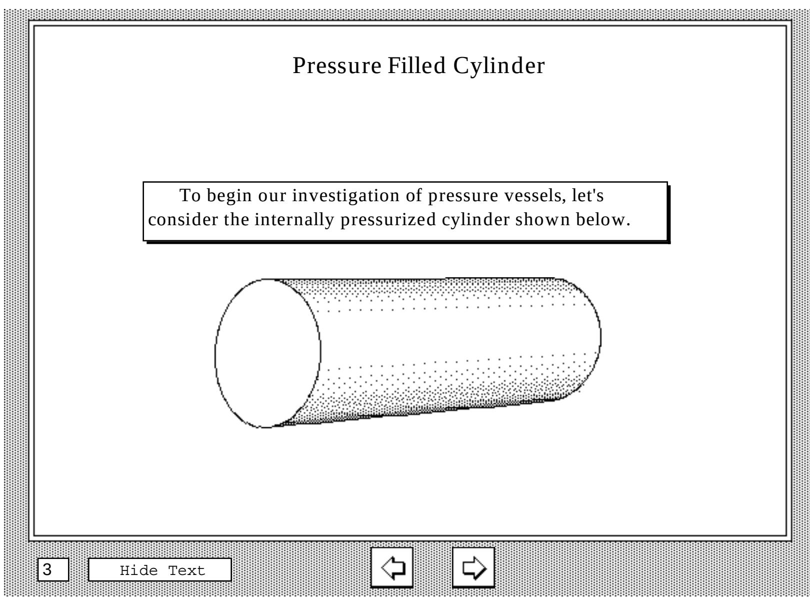
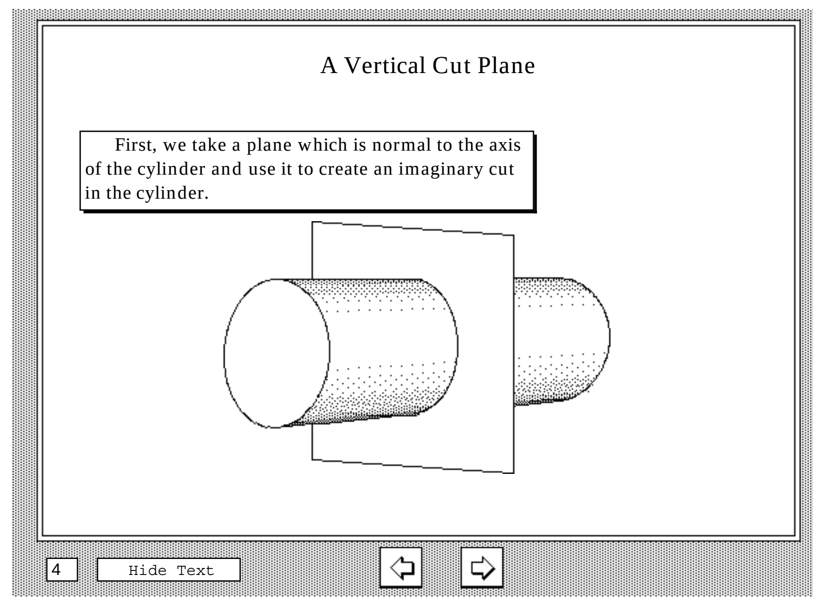

Thin-Walled
Pressure Vessels

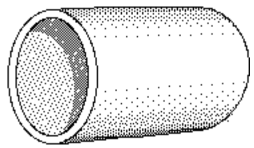

Stack Contents
Longitudinal Stress in Cylinders
Hoop Stress in Cylinders
Hoop Stress in Spheres
Vanishingly Small Element
Radial Stress
End Conditions
2
Hide Text


Pressure Filled Cylinder
To begin our investigation of pressure vessels, let's consider the internally pressurized cylinder shown below.
3
Hide Text


A Vertical Cut Plane
First, we take a plane which is normal to the axis of the cylinder and use it to create an imaginary cut in the cylinder.
4
Hide Text

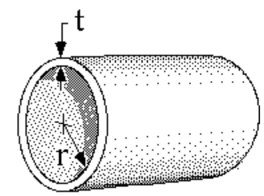
A Look at the Cut Surface

After cutting the cylinder, we retain the back half of the cylinder. Remember, since this was an imaginary cut, the gas in the remaining half of the cylinder does not escape. This portion of the cylinder is still pressurized!



Dimensions of the Cylinder

The cylinder has a thickness, t, and a radius, r. This analysis is limited to "Thin Walled Pressure Vessels". For a cylinder to qualify as "thin walled" the ratio of radius to thickness -- r / t -- must be at least 10.



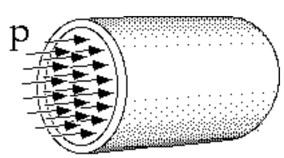
$$\frac{r}{t} \geq 10$$

radius = r
thickness = t

Internal Pressure Acting on the Cut

The internal pressure, which is still acting on this portion of the cylinder, is represented in the pressure field shown below. The pressure is constant over the entire cut of the cylinder. Note that we draw the pressure in a manner which tends to inflate the cylinder.

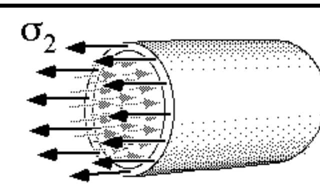
If the pressure acts in the opposite sense to that shown below, the following analysis must be used with caution. When we load vessels with "external pressure" the loading can result in buckling (crumpling) of the cylinder.



Internal Pressure = p

Stress Acting on the Cut

In order for the free-body diagram to satisfy equilibrium, there must be some force which counteracts the internal pressure. When analyzing thin walled pressure vessels it is assumed that all stresses act parallel to the surface of the vessel. This is often called "membrane action". In the cylinder below, the only stress acting on the cut which can counteract the internal pressure is the normal stress σ_2.



σ_2 = Longitudinal Stress in Cylinder

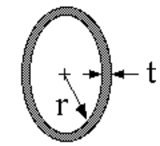
Longitudinal Force Equilibrium

At this point we have enough information to apply force equilibrium in the longitudinal direction.

The figure above represents the fact that the force caused by the internal pressure must be equilibrated by the force caused by the longitudinal normal stress.

Area Calculations

$A = \pi r^2$

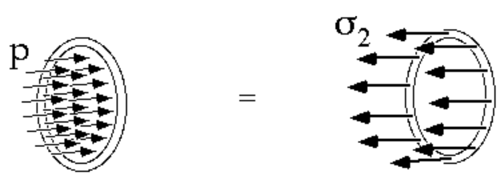


$A = 2\pi r \times t$

Remember, stress (and pressure) is expressed in units of force over area. To include stresses (and pressures) in force equilibrium equations, you must multiply the stress (or pressure) times the area on which it acts.

The areas acted on by the longitudinal stress and the pressure are calculated in the figure above.

Force Calculations



$A = \pi r^2$ $\qquad$ $A = 2\pi r \times t$

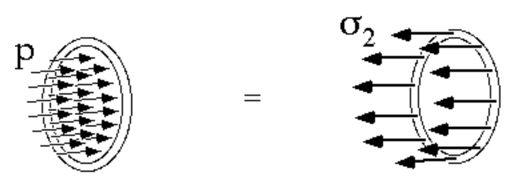
$F = p(\pi r^2)$ $\qquad$ $F = \sigma_2(2\pi r t)$

Once we have figured the areas, we can calculate the forces caused by the pressure and the normal stress as shown above.

We have made a major assumption here, do you know what it is?

Major Assumption

Equate the Forces...

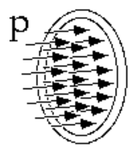


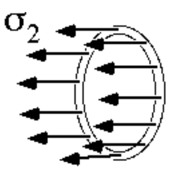
$A = \pi r^2$ $\qquad$ $A = 2\pi r \times t$

$$p(\pi r^2) = \sigma_2(2\pi r t)$$

Equilibrium requires that the two forces be equal if the cylinder is to remain stationary.

The Longitudinal Stress





$A = \pi r^2$ $\qquad$ $A = 2\pi r \times t$

$$p(\pi r^2) = \sigma_2 (2\pi r t)$$

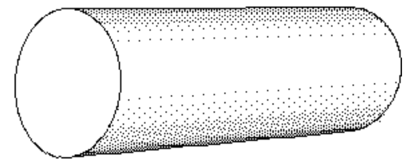
And the Result is.....

$$\sigma_2 = \frac{pr}{2t}$$

We call this stress in a cylinder the "longitudinal stress" because it acts parallel to the long axis of the cylinder.

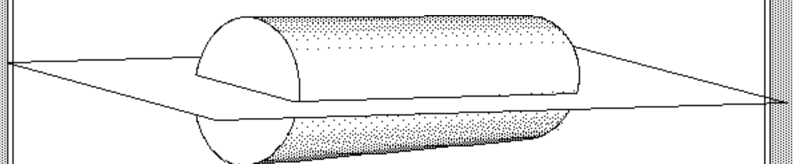
Return to the Original Cylinder

We just calculated the longitudinal stress. Is this the only stress acting in a pressure filled cylinder? Let's cut the cylinder in a different manner and see.



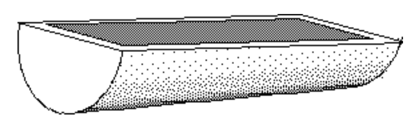
Horizontal Cut Plane

This time our cut is parallel to the axis of the cylinder.



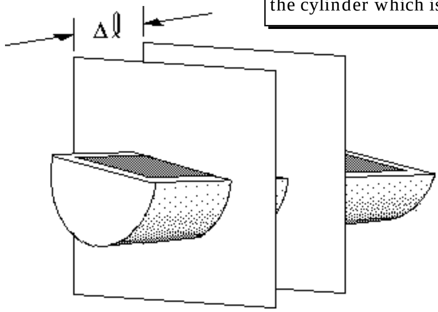
Retain the Lower Half of the Cylinder

After the cut we retain the bottom half of the cylinder. Again, the gas has not left the portion of the cylinder shown below: it is still pressurized.



Take a Slice of the Remaining Material...

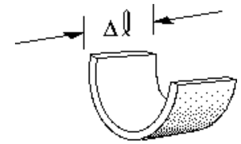
But our cutting does not stop there. Next, we remove a slice of the cylinder which is of length Δl.



17 Hide Text

The Slice

What remains after our slicing and dicing is a "half pipe" of length Δl. Our previous analysis addressed the pressure which acts on this element in the longitudinal direction, what other pressure would act on the



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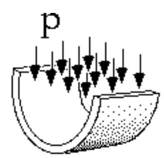
Internal Pressure Acting on the Slice

This time we will look at the pressure acting on the element in the vertical direction. Again, as we have drawn it the pressure tends to inflate the element.

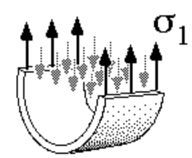


Internal Pressure = p

19 Hide Text

Stress Acting on the Slice

The only stress acting in the vertical direction which can counteract the pressure shown on the previous page is the normal stress σ_1. This stress is called the "Hoop stress" because it acts like a steel hoop around a wooden barrel.



σ_1 = Hoop Stress in Cylinder

20 Hide Text

Vertical Force Equilibrium

It is time again to enforce equilibrium. Here we are focusing on forces which act in the vertical direction.

21 Hide Text

Area Calculations

$A = 2r \times \Delta l$ $\quad$ $A = 2\,(t \times \Delta l)$

The necessary area calculations are shown above.

22 Hide Text

Force Calculations

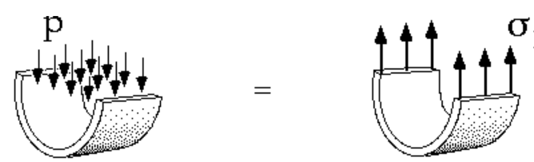
$A = 2r \times \Delta l$ $\quad$ $A = 2\,(t \times \Delta l)$

$F = p\,(2r \times \Delta l)$ $\quad$ $F = \sigma_1\,(2\,t \times \Delta l\,)$

And here are the equivalent forces.

23 Hide Text

The Hoop Stress!



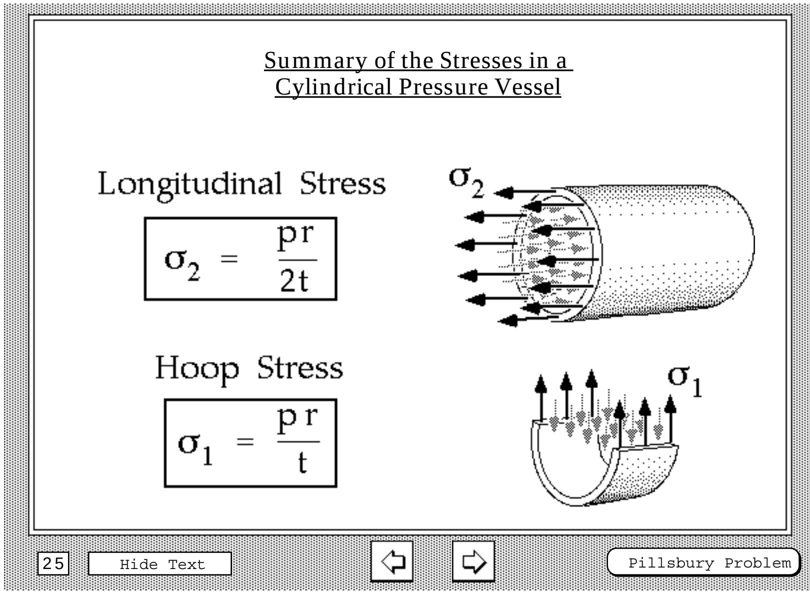
$A = 2r \times \Delta l$ $\quad$ $A = 2\,(t \times \Delta l)$

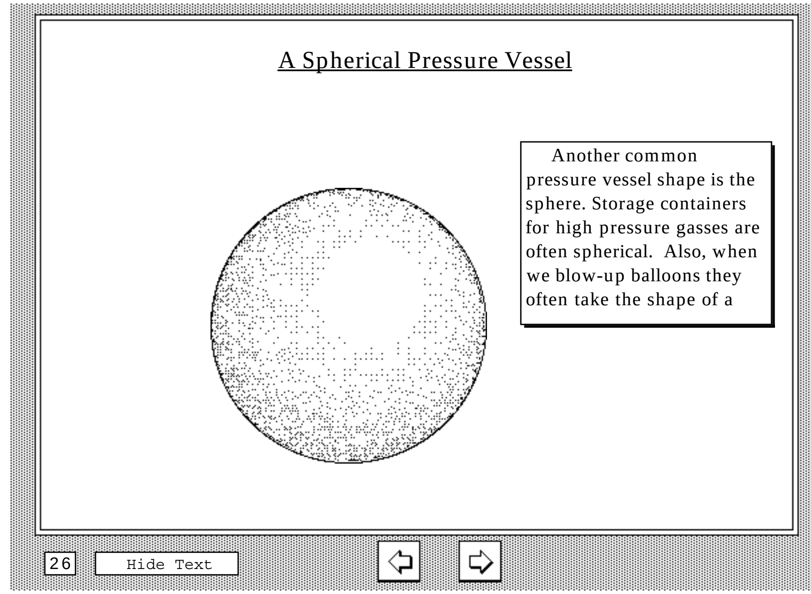
$F = p\,(2r \times \Delta l) = F = \sigma_1\,(2\,t \times \Delta l\,)$

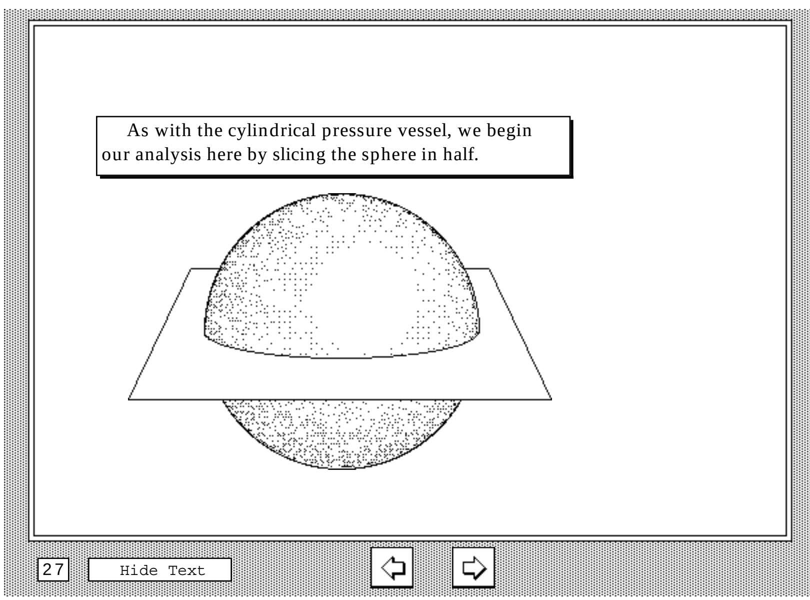
Equating the two forces we arrive at the expression for the hoop stress shown at the right.

$$\sigma_1 = \frac{p\,r}{t}$$

24 Hide Text


Summary of the Stresses in a Cylindrical Pressure Vessel
Longitudinal Stress
$\sigma_2 = \frac{pr}{2t}$
σ_2
Hoop Stress
$\sigma_1 = \frac{pr}{t}$
σ_1
25
Hide Text
Pillsbury Problem


A Spherical Pressure Vessel
Another common pressure vessel shape is the sphere. Storage containers for high pressure gasses are often spherical. Also, when we blow-up balloons they often take the shape of a
26
Hide Text


As with the cylindrical pressure vessel, we begin our analysis here by slicing the sphere in half.
27
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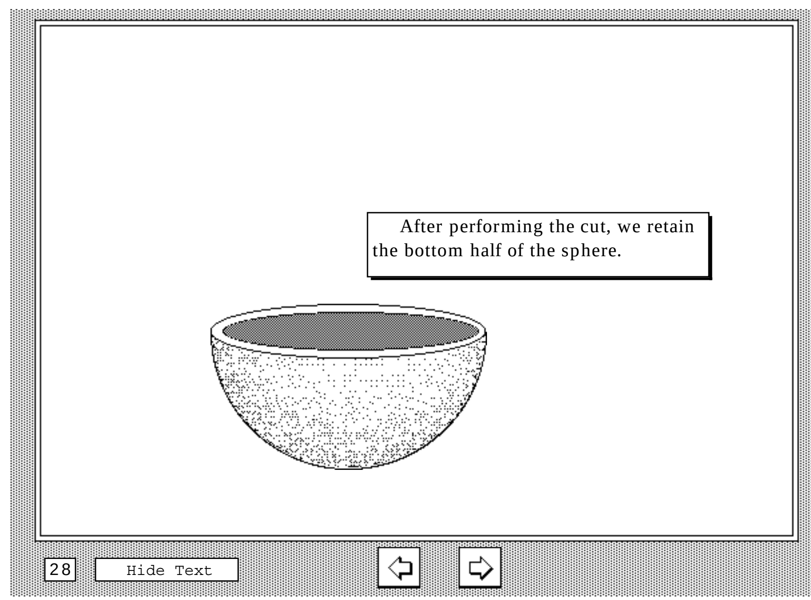
After performing the cut, we retain the bottom half of the sphere.
28
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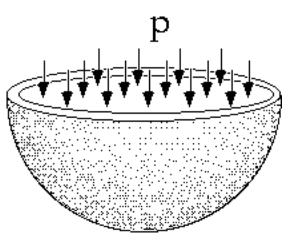
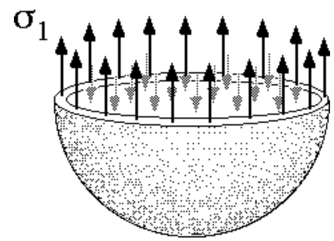
The internal pressure acting on the cut surface is shown below.



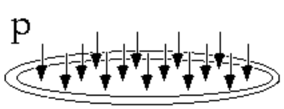
29 Hide Text

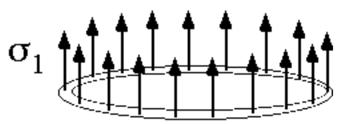
And here are the stresses which counteract the internal pressure.



30 Hide Text

Vertical Force Equilibrium





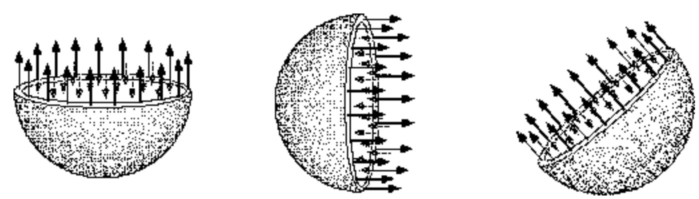
$A = \pi r^2$ $\qquad$ $A = 2\pi r \times t$

$$p(\pi r^2) = \sigma_1 (2\pi r t)$$

$$\boxed{\sigma_1 = \frac{pr}{2t}}$$

Calculations identical to those for the longitudinal stress in a cylinder can be used to calculate the hoop stress in a sphere. And that's all there is! We are done.

31 Hide Text



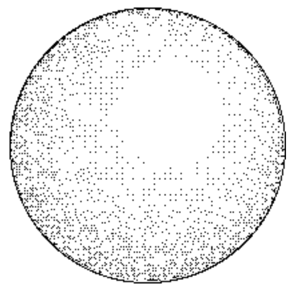
How did I know that we had completely solved the pressurized sphere problem? You can imagine that it is possible to cut the sphere in half using many different planes. Yet after each cut the analysis would be identical to that on the previous page. We can therefore conclude that a spherical pressure vessel is under "uniform stress".

Previously, we have created Mohr's Circle for the case of uniform stress. Click on the button below to view the circle.

Mohr's Circle

32 Hide Text

An Alternative Derivation: Consider a Small Element...

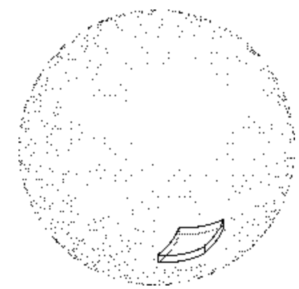


It is a very useful exercise to derive the stress equation for a pressurized sphere using an alternative method.

In this method we look at all of the forces acting on a vanishingly small element of the sphere. By enforcing equilibrium at this level we can arrive at the result shown previously.

We begin by focusing on a small element as a free body.

33 Hide Text



34 Hide Text

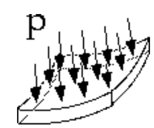
A Spherical Element

This element is acted upon by two sets of forces.

35 Hide Text

Internal Pressure Acting on the Element

The first force is caused by the internal pressure.

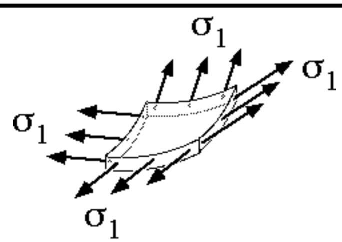


36 Hide Text

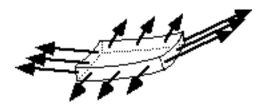
Stresses Acting on the Element

The other set of forces is due to the normal stresses acting parallel to the surface of the sphere.

We can simplify our analysis by looking at looking at a side view of this element.



37 Hide Text



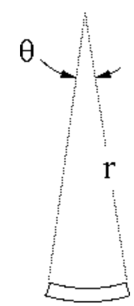
38 Hide Text

Side View

From this view we see three of the four normal stresses acting on the element.

39 Hide Text

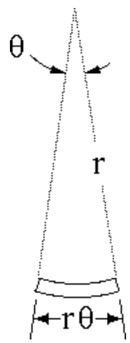
Geometery



Before we proceed any further we must determine the geometric properties of the element. We can imagine that the element was formed by removing a small slice of the sphere defined by the angle θ.

40 Hide Text

Length of each Edge

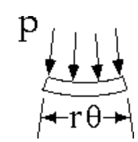


The length of the each edge of the element (assuming that the element is squarish) is calculated as $r\theta$.

41 Hide Text

Internal Pressure

We now have enough information to calculate total force on the element caused by the internal pressure.



42 Hide Text

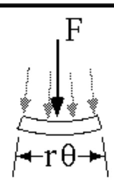
Force caused by Internal Pressure

Force due to Internal Pressure

$F = p \times A$

$$F = p(r\theta)^2$$

The force is calculated as pressure times the area of the element. Note that the resultant force has no horizontal component. The portion of the pressure which acts to the left is equated by an equal amount of pressure that acts to the right.



43 Hide Text

What is the Vertical Force due to Stress

Force due to Internal Pressure

$F = p(r\theta)^2$

What will counteract the downward force exerted by the internal pressure? The vertical component of the normal stress, σ_1.

Note that the horizontal components of the normal stresses result in no net force.

44 Hide Text

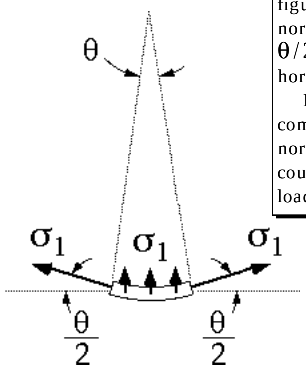
Geometery

Recall that the arc of the element was defined by the angle θ. From the figure we see that each normal stress is oriented $\theta/2$ above the horizontal plane.

It is the vertical component of these normal stresses which will counteract the pressure loading.

Force due to Internal Pressure

$F = p(r\theta)^2$



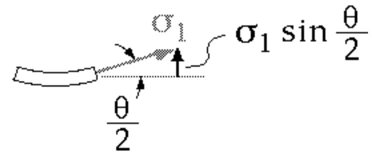
45 Hide Text

Focus on One Edge of the Element

Looking at one edge of the element, it is seen that the component of the stress acting in the vertical direction may be calculated in terms of the sine of the angle $\theta/2$.

Force due to Internal Pressure

$F = p(r\theta)^2$



46 Hide Text

A Bit of Trigonometery

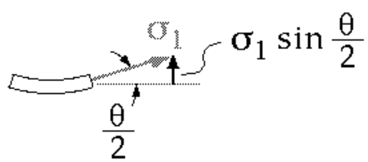
Recall that for small values of θ the sine of θ is approximately equal to θ. Even at $\theta = \pi/6$ radians the error is less than 5%.

Force due to Internal Pressure

$F = p(r\theta)^2$

For small values of θ:

$$\sin\theta \approx \theta$$



47 Hide Text

Substituting $\theta/2$ for $\sin(\theta/2)$

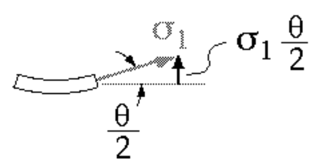
Therefore, we substitute $\theta/2$ in place of $\sin(\theta/2)$.

Force due to Internal Pressure

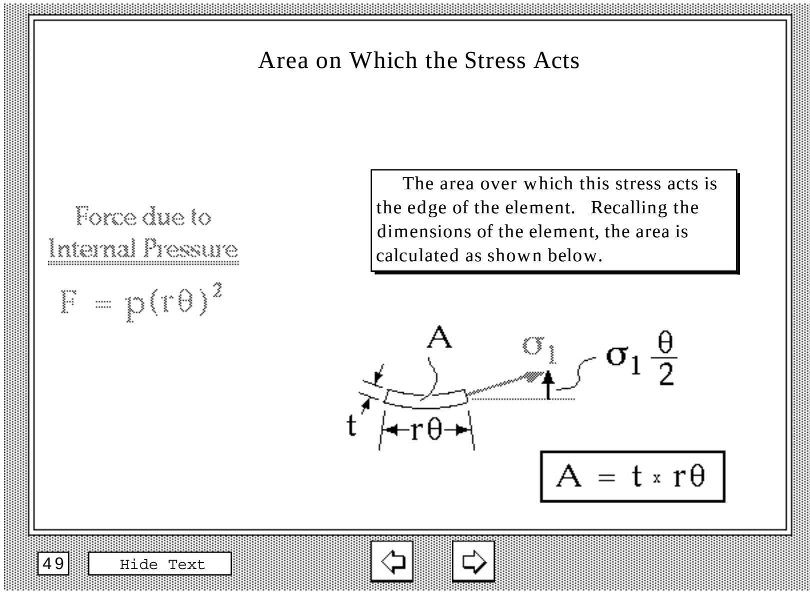
$F = p(r\theta)^2$

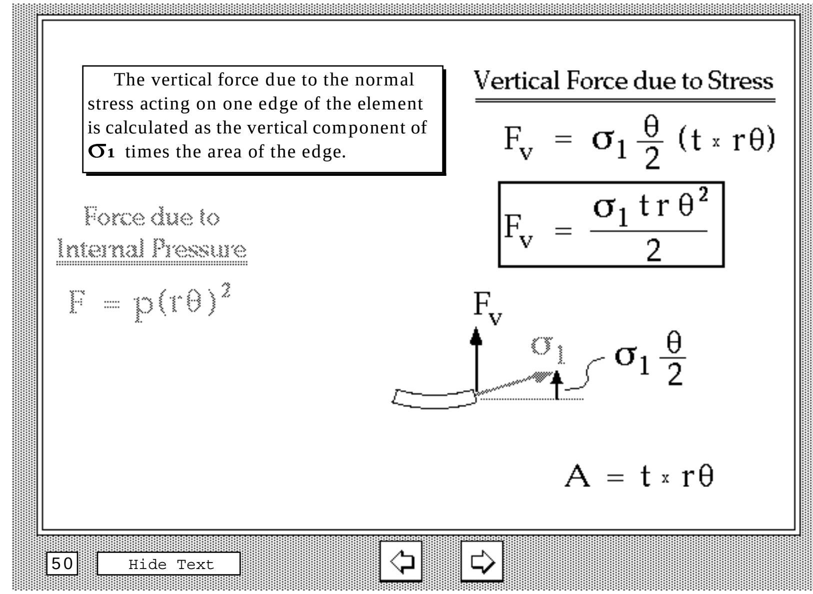
For small values of θ:

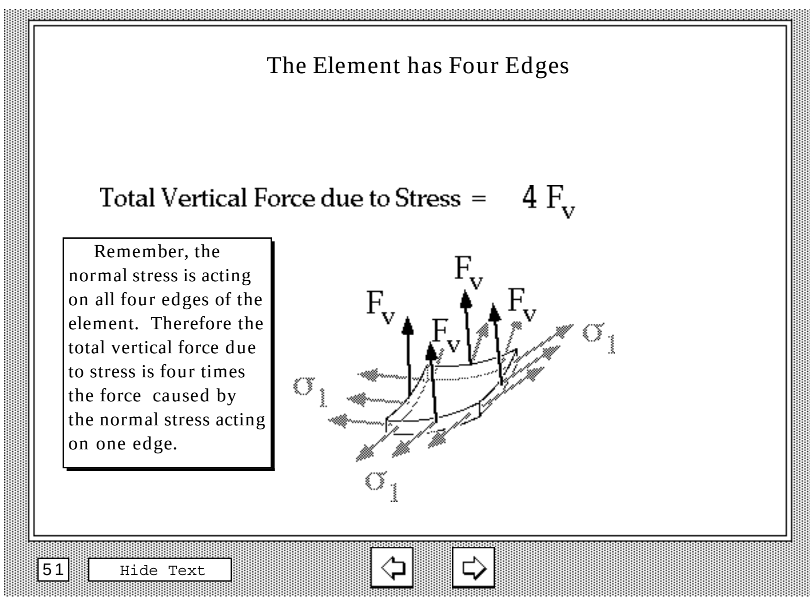
$$\sin\theta \approx \theta$$

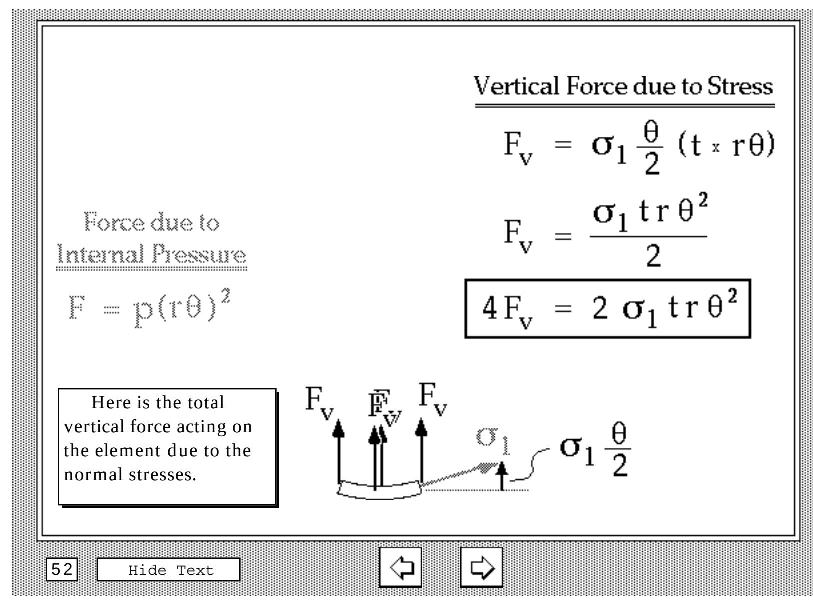


48 Hide Text


Area on Which the Stress Acts
Force due to Internal Pressure
F = p(rθ)²
The area over which this stress acts is the edge of the element. Recalling the dimensions of the element, the area is calculated as shown below.
A
t
rθ
σ1
σ1 θ/2
A = t x rθ
49
Hide Text


The vertical force due to the normal stress acting on one edge of the element is calculated as the vertical component of σ1 times the area of the edge.
Vertical Force due to Stress
Fv = σ1 θ/2 (t x rθ)
Fv = σ1 t r θ² / 2
Force due to Internal Pressure
F = p(rθ)²
Fv
σ1
σ1 θ/2
A = t x rθ
50
Hide Text


The Element has Four Edges
Total Vertical Force due to Stress = 4 Fv
Remember, the normal stress is acting on all four edges of the element. Therefore the total vertical force due to stress is four times the force caused by the normal stress acting on one edge.
Fv
Fv
Fv
Fv
σ1
σ1
σ1
51
Hide Text


Vertical Force due to Stress
Fv = σ1 θ/2 (t x rθ)
Fv = σ1 t r θ² / 2
4 Fv = 2 σ1 t r θ²
Force due to Internal Pressure
F = p(rθ)²
Here is the total vertical force acting on the element due to the normal stresses.
Fv
Fv
Fv
σ1
σ1 θ/2
52
Hide Text

Apply Equilibrium...

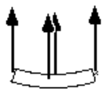
We use equilibrium to equate the two vertical forces acting on the element.

Force due to Internal Pressure

$$F = p(r\theta)^2$$

Vertical Force due to Stress

$$4F_v = 2\,\sigma_1\,t\,r\,\theta^2$$



Apply Equilibrium...

Force due to Internal Pressure

$$F = p(r\theta)^2$$

Vertical Force due to Stress

$$4F_v = 2\,\sigma_1\,t\,r\,\theta^2$$

$$p(r\theta)^2 = 2\,\sigma_1\,t\,r\,\theta^2$$

The algebra is simple enough.

The Hoop Stress for a Spherical Pressure Vessel

Force due to Internal Pressure

$$F = p(r\theta)^2$$

Vertical Force due to Stress

$$4F_v = 2\,\sigma_1\,t\,r\,\theta^2$$

$$p(r\theta)^2 = 2\,\sigma_1\,t\,r\,\theta^2$$

$$\boxed{\sigma_1 = \frac{pr}{2t}}$$

And the result is identical to that achieved by applying equilibrium to half of the sphere.

Summary

Cylinder

Longitudinal Stress

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$

Hoop Stress

$$\boxed{\sigma_1 = \frac{pr}{t}}$$

Sphere

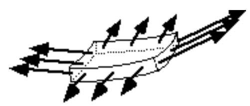
Hoop Stress

$$\boxed{\sigma_1 = \frac{pr}{2t}}$$

These are the stresses present in thin-walled pressure vessels.

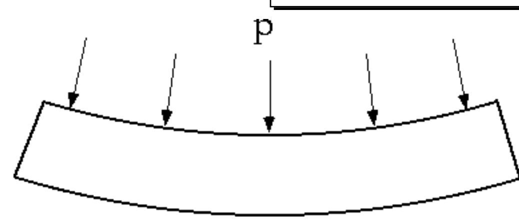
We Assume Biaxial Stress

In our derivation of the stresses in pressure vessels we assumed a "biaxial" stress state. Another way of stating this is that we assumed that all stresses act parallel to the surface of the vessel.



Is the Stress Really Biaxial?

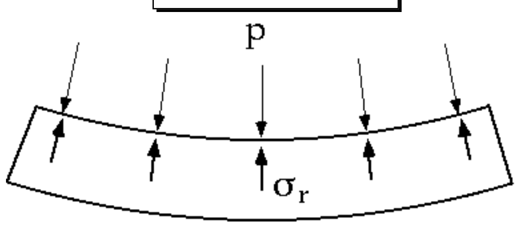
As we can see from the figure below, there clearly must be some radial stresses produced by the pressure itself, at least on the inside surface of the vessel.



Radial Stress!

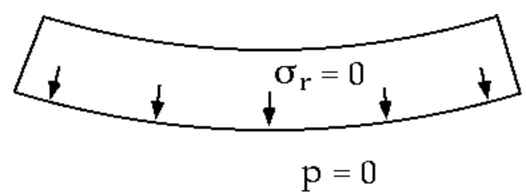
This radial stress on the inner surface is, in fact, equal to the pressure.

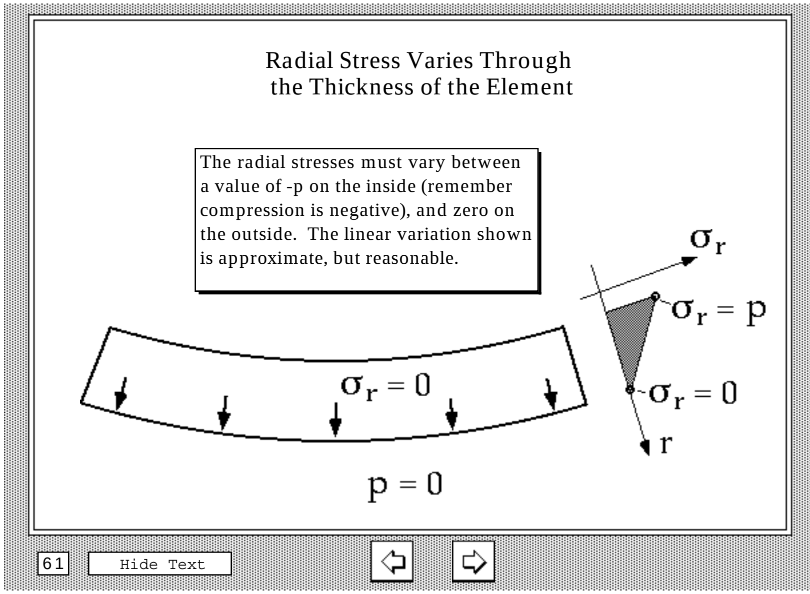
$$\sigma_r = p$$



Radial Stress at the Outer Surface

However, on the outer surface there is no radial stress, since there is no applied pressure on the outside of the vessel.




Radial Stress Varies Through
the Thickness of the Element
The radial stresses must vary between a value of -p on the inside (remember compression is negative), and zero on the outside. The linear variation shown is approximate, but reasonable.
σr
σr = p
σr = 0
σr = 0
r
p = 0
61
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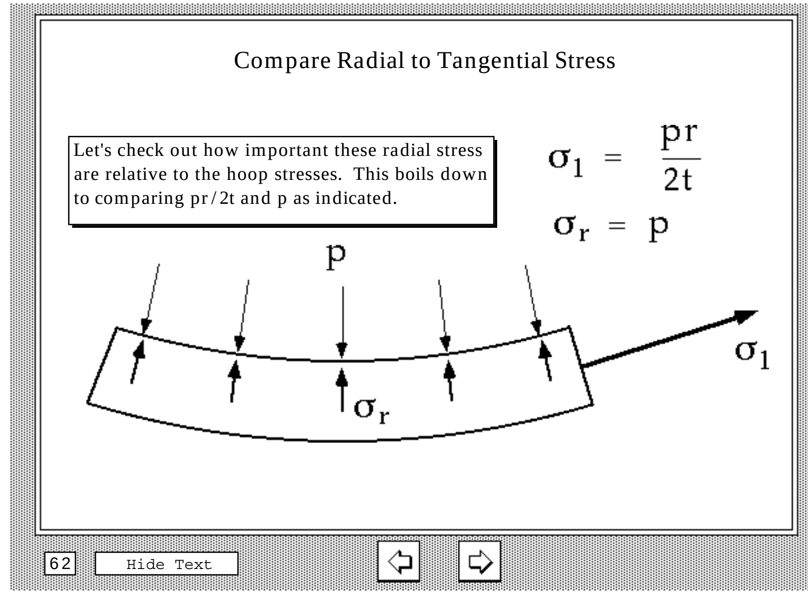
Compare Radial to Tangential Stress
Let's check out how important these radial stress are relative to the hoop stresses. This boils down to comparing pr/2t and p as indicated.
σ1 = pr/2t
σr = p
p
σr
σ1
62
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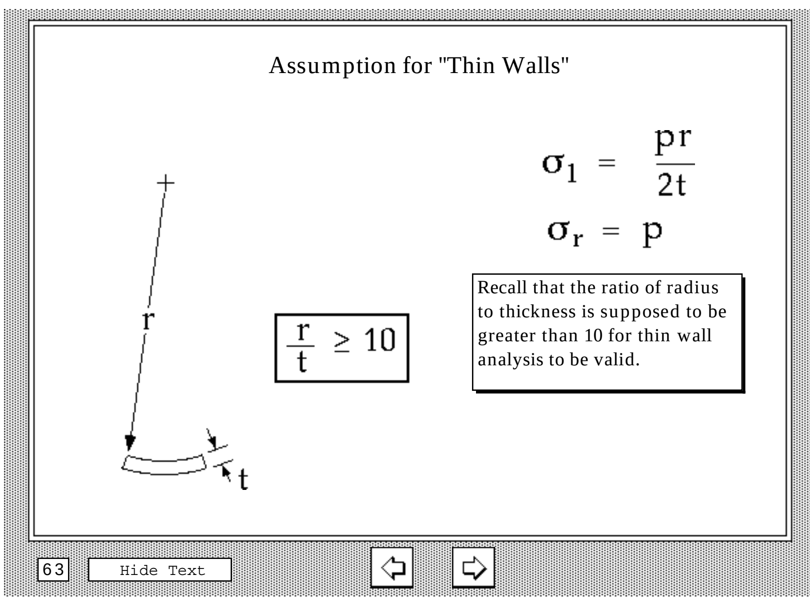
Assumption for "Thin Walls"
σ1 = pr/2t
σr = p
r
t
r/t ≥ 10
Recall that the ratio of radius to thickness is supposed to be greater than 10 for thin wall analysis to be valid.
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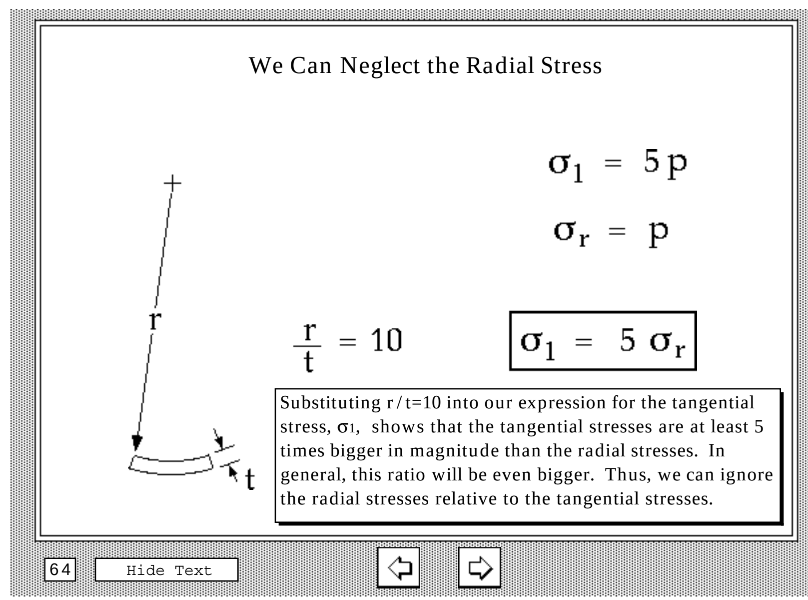
We Can Neglect the Radial Stress
σ1 = 5 p
σr = p
r
t
r/t = 10
σ1 = 5 σr
Substituting r/t=10 into our expression for the tangential stress, σ1, shows that the tangential stresses are at least 5 times bigger in magnitude than the radial stresses. In general, this ratio will be even bigger. Thus, we can ignore the radial stresses relative to the tangential stresses.
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Compare the Cylinder to the Sphere

Cylinder

Longitudinal Stress

$$\sigma_2 = \frac{pr}{2t}$$

Hoop Stress

$$\sigma_1 = \frac{pr}{t}$$

Sphere

Hoop Stress

$$\sigma_1 = \frac{pr}{2t}$$

?

Note that the hoop stresses in the sphere are 1/2 those in the cylinder. Any idea why?

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Cylinder

Sphere

One way of thinking about it is to note that a sphere is curved in two directions, while a cylinder is only curved in one. This double curvature makes the sphere more efficient.

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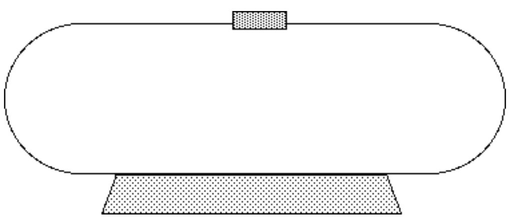
End Conditions on a Cylinder



For purposes of design, it turns out that the critical locations are often near the ends of a pressure vessel. For example, the tank shown above has spherical ends.

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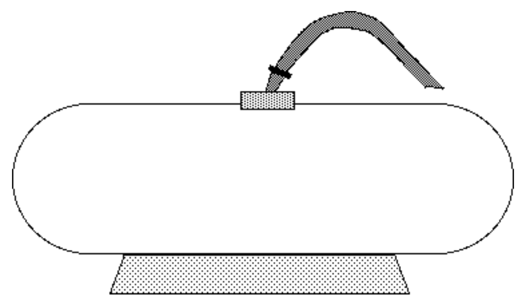
Increase the Internal Pressure



As the tank is pressurized, it tends to expand. However, the spherical ends and the cylindrical inner portion will tend to expand different amounts.

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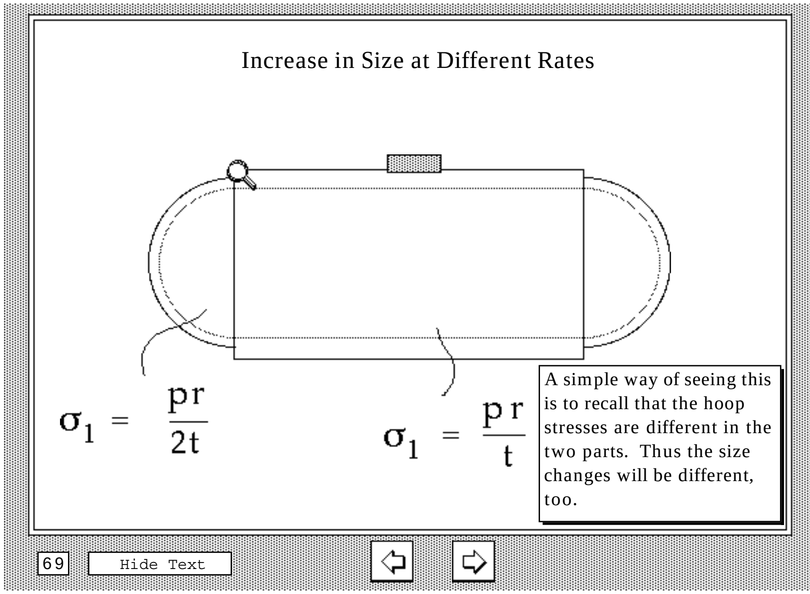
Increase in Size at Different Rates
σ1 = pr / 2t
σ1 = p r / t
A simple way of seeing this is to recall that the hoop stresses are different in the two parts. Thus the size changes will be different, too.
69
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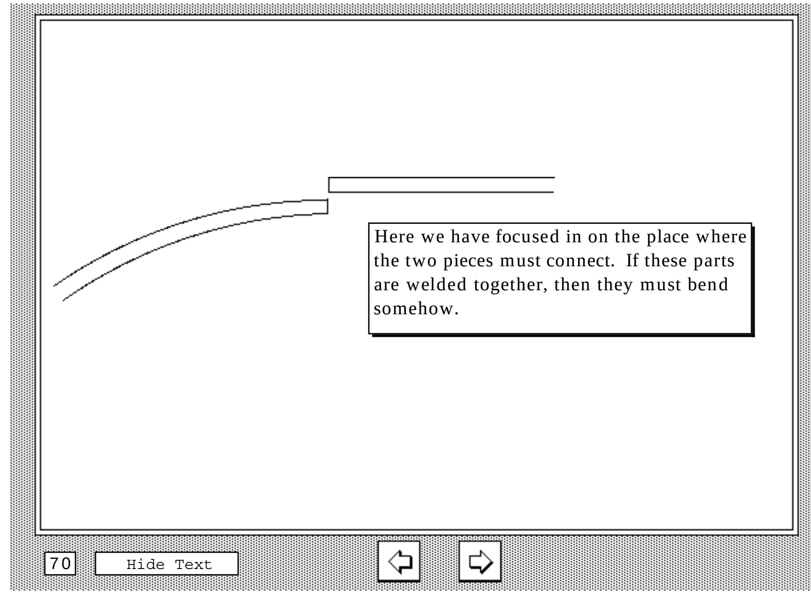
Here we have focused in on the place where the two pieces must connect. If these parts are welded together, then they must bend somehow.
70
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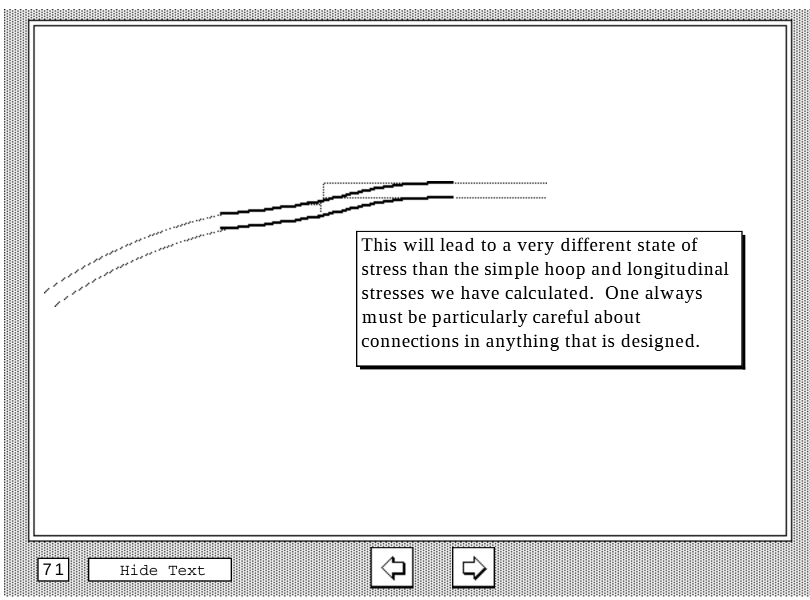
This will lead to a very different state of stress than the simple hoop and longitudinal stresses we have calculated. One always must be particularly careful about connections in anything that is designed.
71
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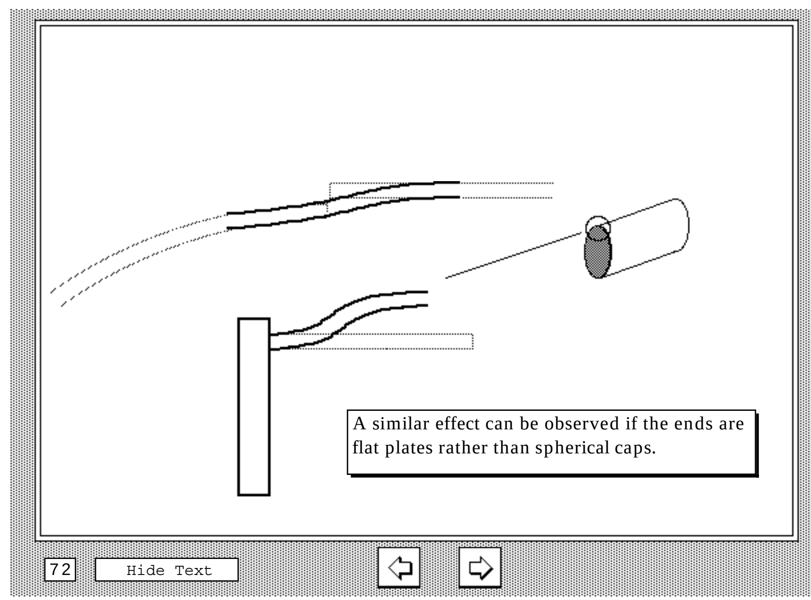
A similar effect can be observed if the ends are flat plates rather than spherical caps.
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Conclusion

- **Ignore Radial Stresses**
- **Spheres are twice as efficient as Cylinders**
- **Ends of Pressure Vessels Require Additional Design**
- **The Preceding Analysis only**