Hooke's Law

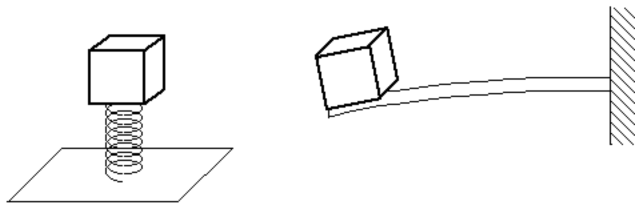
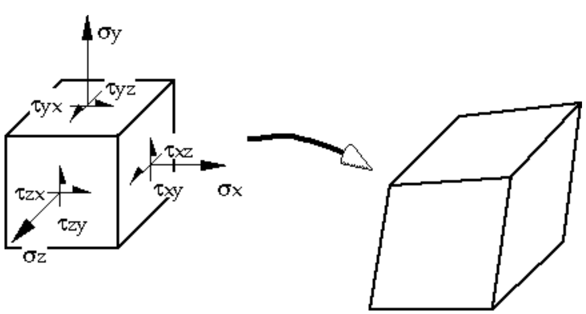
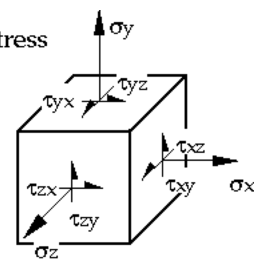


Table of Contents

Basic Stress-Strain Relations

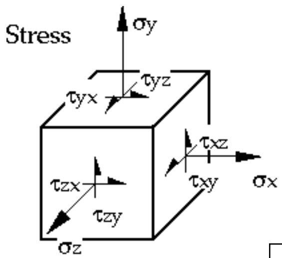


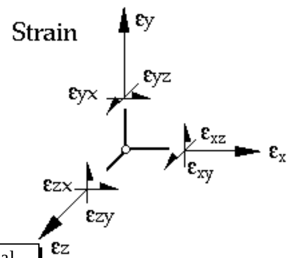
Stress



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

So far we have developed a general description of stress...



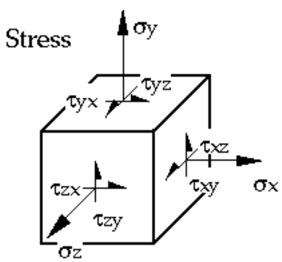


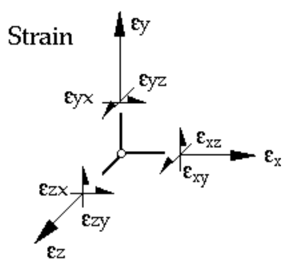
$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

...and a general description of strain. We now need to tie these two together. That is, we must figure out how much strain is caused by a given stress.

In a mathematical sense, we can assume that stress and strain are related in some functional fashion as shown. Determining the appropriate function to use, however, is a formidable task, since every material will have its own behavior, and this behavior can be very complicated. As long as we restrict ourselves to modest loads, however, most materials used by engineers exhibit quite simple behavior: elastic and linear. We have already discussed the concept of elastic versus plastic behavior; we now consider linearity. In simple terms this means that stress = strain * factor. For our general stress and strain descriptions, the picture is more complicated.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = F\left(\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} \right)$$

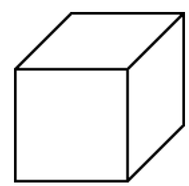




$$\sigma_x = a_{xx}\varepsilon_x + a_{xxy}\varepsilon_{xy} + a_{xxz}\varepsilon_{xz} + a_{xy}\varepsilon_y + a_{xyz}\varepsilon_{yz} + a_{xz}\varepsilon_z$$

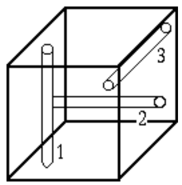
Since there are six stress components and six strain components, it would take 36 coefficients to relate an arbitrary stress state to the corresponding strain. Six of these constants are shown above. Fortunately, most materials do not require such a complex set of factors. In fact, we will see that more often than not we only need 2 coefficients.

Isotropic Material



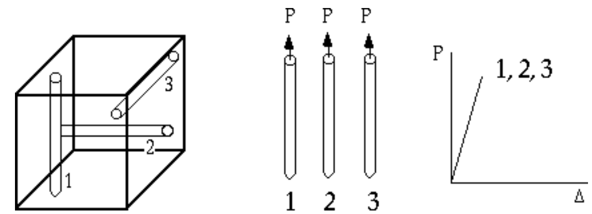
The important feature a material must have in order to allow a 2-parameter characterization is ***isotropy***. This means it exhibits the same behavior in all directions. For example if we were to take specimens from the block

Isotropic Material



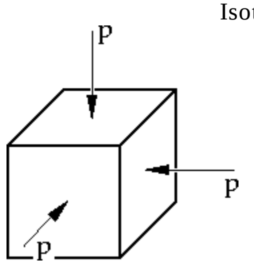
…in several different orientations, and then test each sample…

Isotropic Material



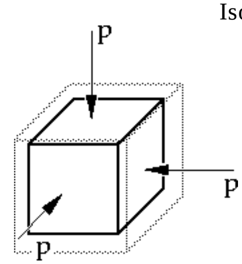
…we would observe the same behavior for each specimen. (For a material like wood, this would not be the case; such materials are called ***anisotropic***.). As we are about to see, an important consequence of isotropy is that we can decouple shape changes from size changes.

Isotropic Pressure



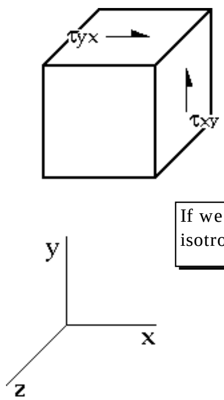
If we apply uniform pressure to a cube of isotropic material…

Isotropic Pressure



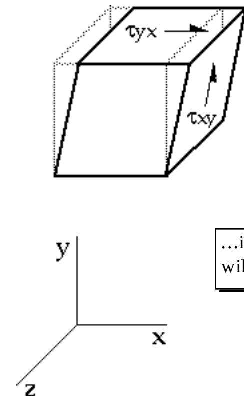
…its volume will decrease, but it will remain a cube.

Pure Shear

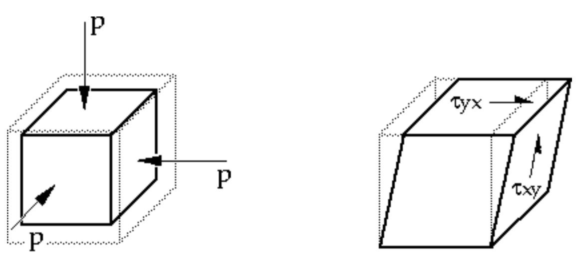


If we apply pure shear to a cube of isotropic material…

Pure Shear



…its shape will change, but its volume will remain constant.

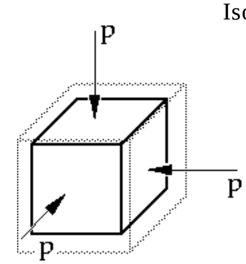


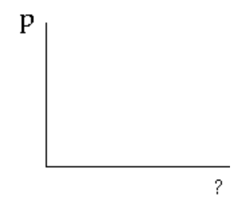
Pure Pressure

Pure Shear

The linear stress-strain behavior of an isotropic body can be characterized by considering these two types of loading separately. Any general loading can be decomposed into a pressure part and a shearing or shape changing part. There will be one linear coefficient associated with pressure / volume change, and another associated with shear / shape change. These two constants will be all we need to characterize the stress-strain behavior of an isotropic material.

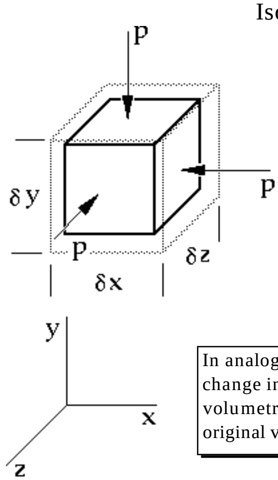
Isotropic Pressure

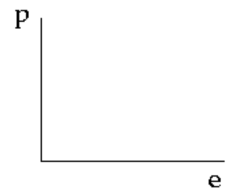




Consider the case of pure pressure. We wish to obtain a relation between pressure and volume change, but we do not yet have a general method for measuring volume change appropriately.

Isotropic Pressure



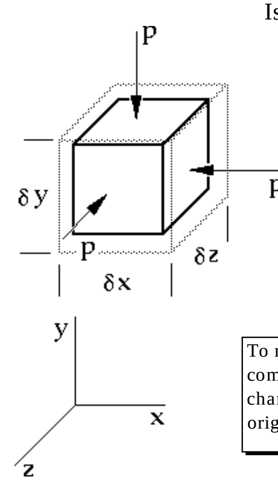


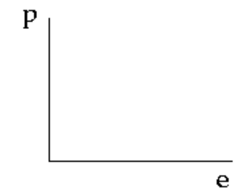
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$

In analogy with our definition for basic strain, change in length over original length, we define the volumetric strain, e, as change in volume over original volume.

17 | Hide Text | ⇦ | ⇨ | Skip the Details

Isotropic Pressure



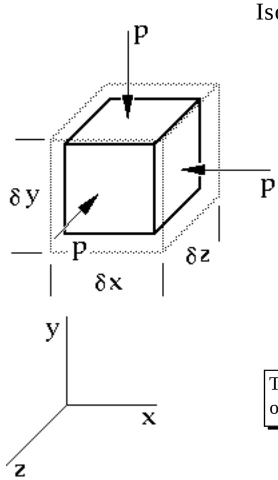


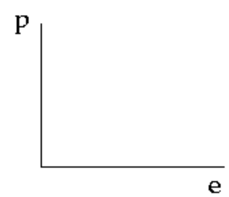
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$

To relate the volumetric strain, e, to the basic strain components, ε_x, ε_y, etc., we will need to examine the change in the block dimensions. Shown here are the original block side lengths.

18 | Hide Text | ⇦ | ⇨

Isotropic Pressure





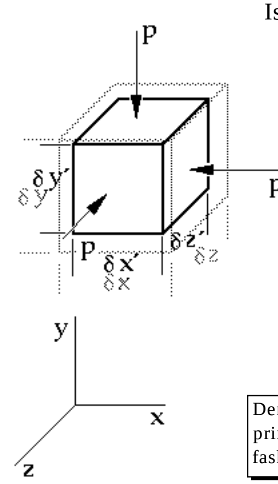
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$

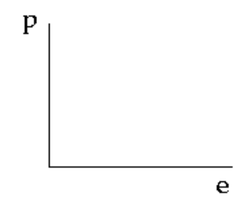
$\delta V = \delta x \ \delta y \ \delta z$

The original volume is simple to calculate in terms of the original block dimensions.

19 | Hide Text | ⇦ | ⇨

Isotropic Pressure





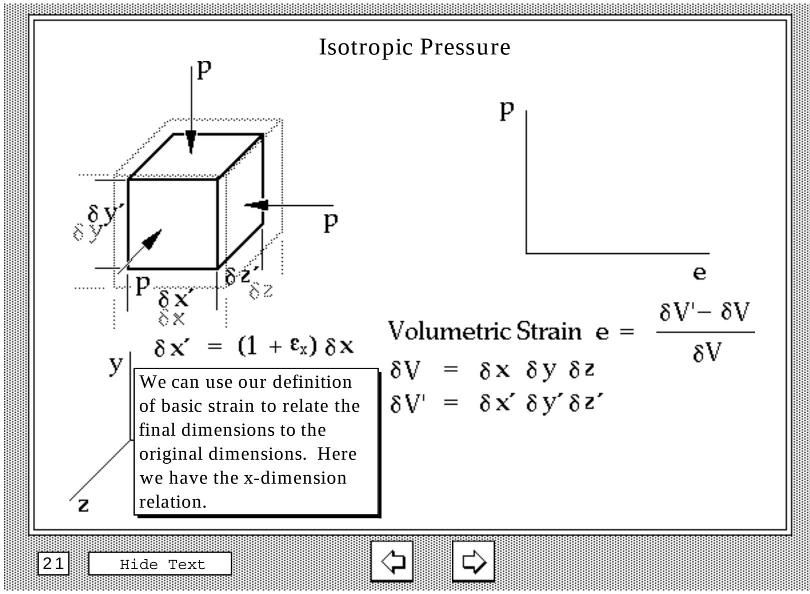
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$

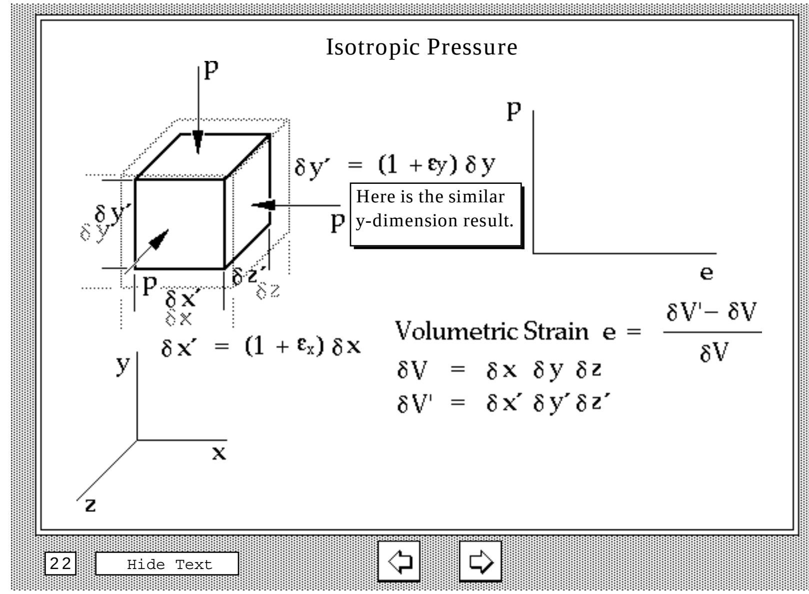
$\delta V = \delta x \ \delta y \ \delta z$

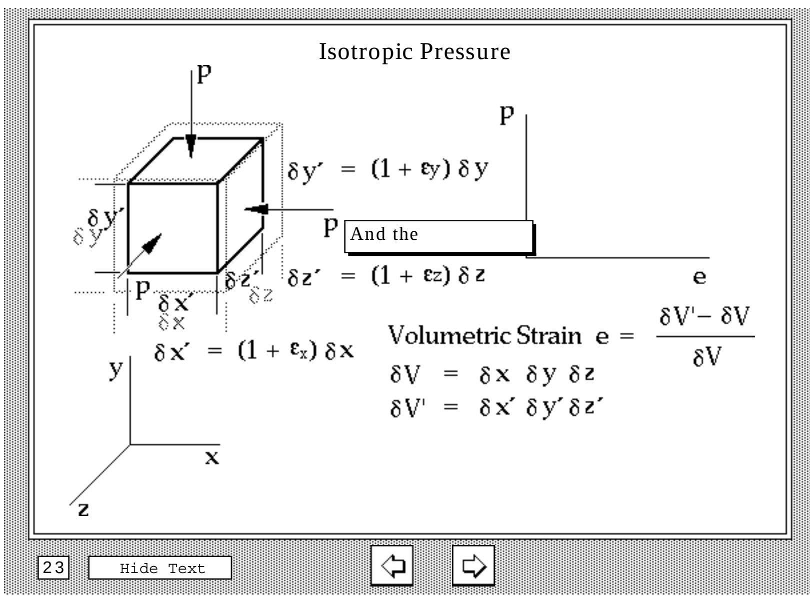
$\delta V' = \delta x' \ \delta y' \ \delta z'$

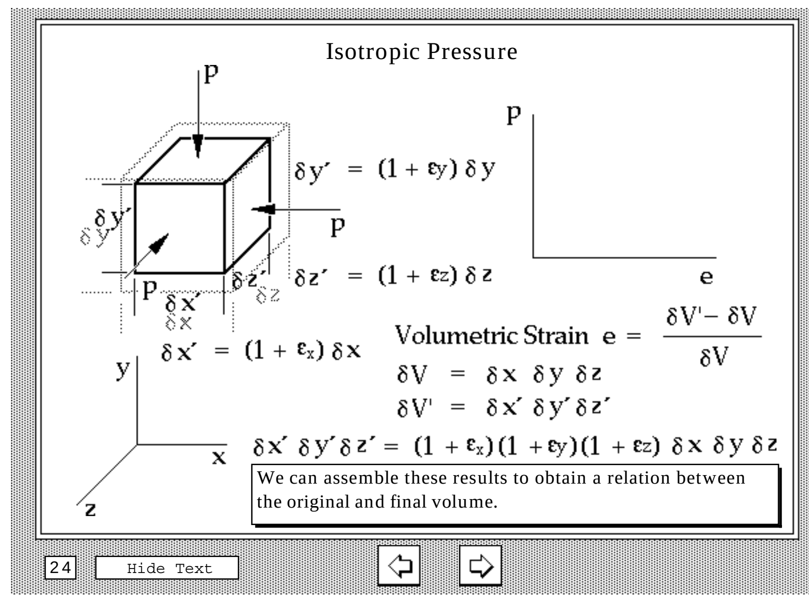
Denoting the block dimensions after deformation with primes, we can calculate the final volume in similar fashion.

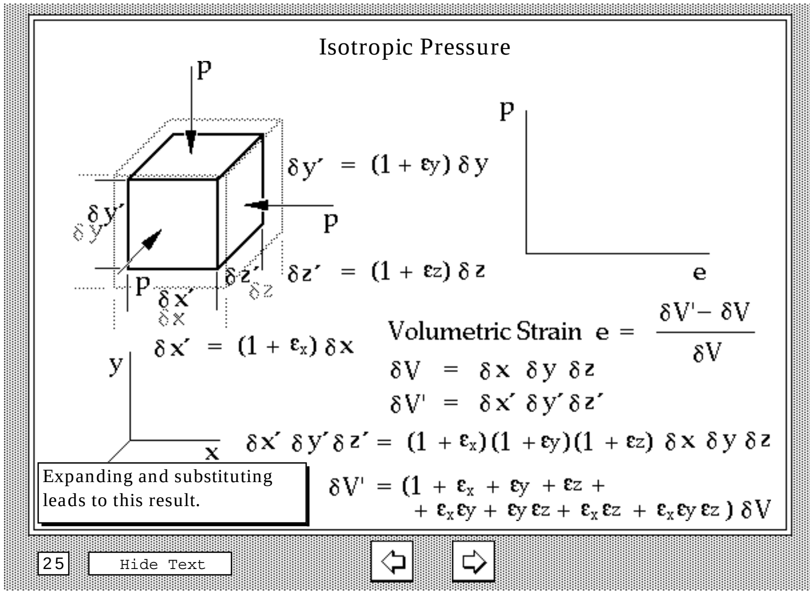
20 | Hide Text | ⇦ | ⇨

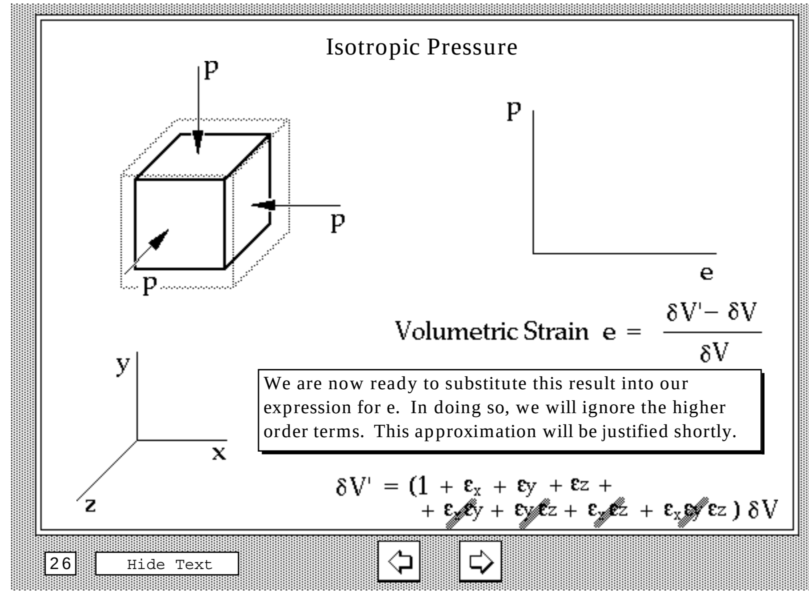

Isotropic Pressure
p
p
p
δy′
δx′
δz′
y
z
p
e
Volumetric Strain e = (δV′ − δV) / δV
δx′ = (1 + εx) δx
δV = δx δy δz
δV′ = δx′ δy′ δz′
We can use our definition of basic strain to relate the final dimensions to the original dimensions. Here we have the x-dimension relation.
21
Hide Text

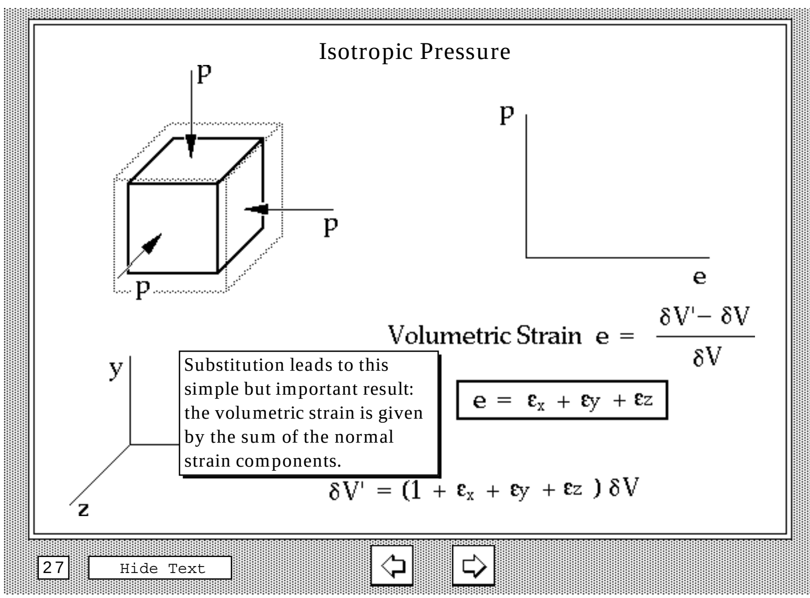

Isotropic Pressure
p
p
p
δy′
δx′
δz′
δy′ = (1 + εy) δy
Here is the similar y-dimension result.
p
e
Volumetric Strain e = (δV′ − δV) / δV
δx′ = (1 + εx) δx
δV = δx δy δz
δV′ = δx′ δy′ δz′
y
x
z
22
Hide Text

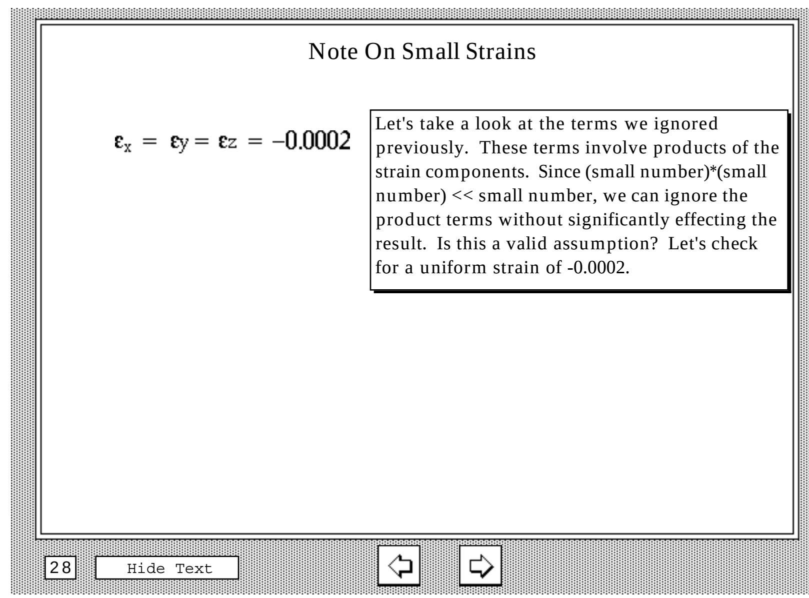

Isotropic Pressure
p
p
p
δy′
δx′
δz′
δy′ = (1 + εy) δy
And the
δz′ = (1 + εz) δz
p
e
Volumetric Strain e = (δV′ − δV) / δV
δx′ = (1 + εx) δx
δV = δx δy δz
δV′ = δx′ δy′ δz′
y
x
z
23
Hide Text


Isotropic Pressure
p
p
p
δy′
δx′
δz′
δy′ = (1 + εy) δy
δz′ = (1 + εz) δz
p
e
Volumetric Strain e = (δV′ − δV) / δV
δx′ = (1 + εx) δx
δV = δx δy δz
δV′ = δx′ δy′ δz′
δx′ δy′ δz′ = (1 + εx)(1 + εy)(1 + εz) δx δy δz
We can assemble these results to obtain a relation between the original and final volume.
y
x
z
24
Hide Text


Isotropic Pressure
p
p
p
p
e
$\delta y' = (1 + \varepsilon_y)\,\delta y$
$\delta z' = (1 + \varepsilon_z)\,\delta z$
$\delta x' = (1 + \varepsilon_x)\,\delta x$
$\delta y'$
δy
$\delta z'$
δz
$\delta x'$
δx
y
x
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$
$\delta V = \delta x\ \delta y\ \delta z$
$\delta V' = \delta x'\ \delta y'\ \delta z'$
$\delta x'\ \delta y'\ \delta z' = (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)\ \delta x\ \delta y\ \delta z$
Expanding and substituting leads to this result.
$\delta V' = (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)\ \delta V$
25
Hide Text


Isotropic Pressure
p
p
p
p
e
y
x
z
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$
We are now ready to substitute this result into our expression for e. In doing so, we will ignore the higher order terms. This approximation will be justified shortly.
$\delta V' = (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_y\varepsilon_z)\ \delta V$
26
Hide Text


Isotropic Pressure
p
p
p
p
e
y
z
Volumetric Strain $e = \dfrac{\delta V' - \delta V}{\delta V}$
Substitution leads to this simple but important result: the volumetric strain is given by the sum of the normal strain components.
$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$
$\delta V' = (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)\ \delta V$
27
Hide Text


Note On Small Strains
$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$
Let's take a look at the terms we ignored previously. These terms involve products of the strain components. Since (small number)*(small number) << small number, we can ignore the product terms without significantly effecting the result. Is this a valid assumption? Let's check for a uniform strain of -0.0002.
28
Hide Text

Note On Small Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$$

$$\delta V' = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,\delta V$$

Recall the complete expression for the volume change -- including the higher order terms.

29 Hide Text

Note On Small Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$$

$$\delta V' = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,\delta V$$

$$e = \frac{\delta V' - \delta V}{\delta V} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$

This is the corresponding complete expression for the volumetric strain, e.

30 Hide Text

Note On Small Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$$

$$\delta V' = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,\delta V$$

$$e = \frac{\delta V' - \delta V}{\delta V} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$
$$= (1 - 0.0002)^3 - 1$$

If we substitute our numerical values from above...

31 Hide Text

Note On Small Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$$

$$\delta V' = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,\delta V$$

$$e = \frac{\delta V' - \delta V}{\delta V} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$
$$= (1 - 0.0002)^3 - 1$$
$$= -0.00059988$$

This is the result.

32 Hide Text

Note On Small Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -0.0002$$

$$\delta V' = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\,\delta V$$

$$e = \frac{\delta V' - \delta V}{\delta V} = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$

$$= (1 - 0.0002)^3 - 1$$

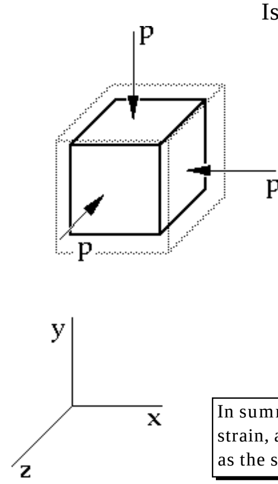
$$= -0.00059988$$

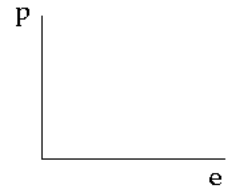
$$\varepsilon_x + \varepsilon_y + \varepsilon_z = -0.000600$$

Here is our linear approximation to e, which is very easy to compute. Our approximation is accurate to within 0.02%. This degree of accuracy is consistent with our original assumption of small strains.

33 Hide Text

Isotropic Pressure





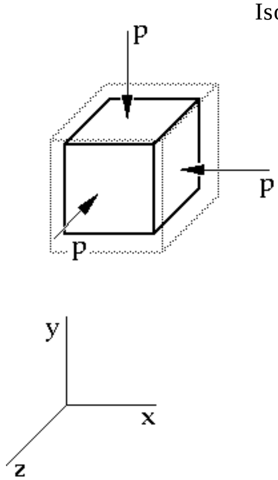
$$\text{Volumetric Strain } e = \frac{\delta V' - \delta V}{\delta V}$$

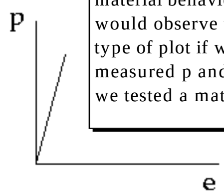
$$\boxed{e = \varepsilon_x + \varepsilon_y + \varepsilon_z}$$

In summary, isotropic pressure causes volumetric strain, and this volumetric strain can be computed as the sum of the normal strain components.

34 Hide Text Detailed Derivation

Isotropic Pressure





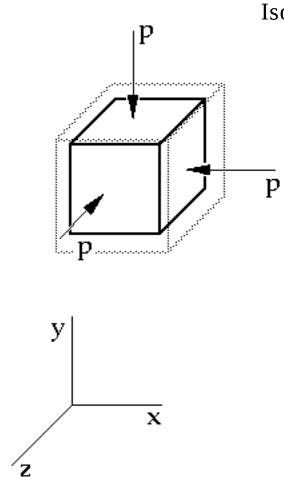
Assuming linear material behavior, we would observe this type of plot if we measured p and e as we tested a material.

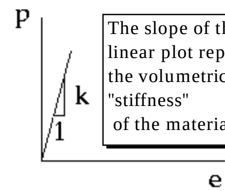
$$\text{Volumetric Strain } e = \frac{\delta V' - \delta V}{\delta V}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

35 Hide Text

Isotropic Pressure



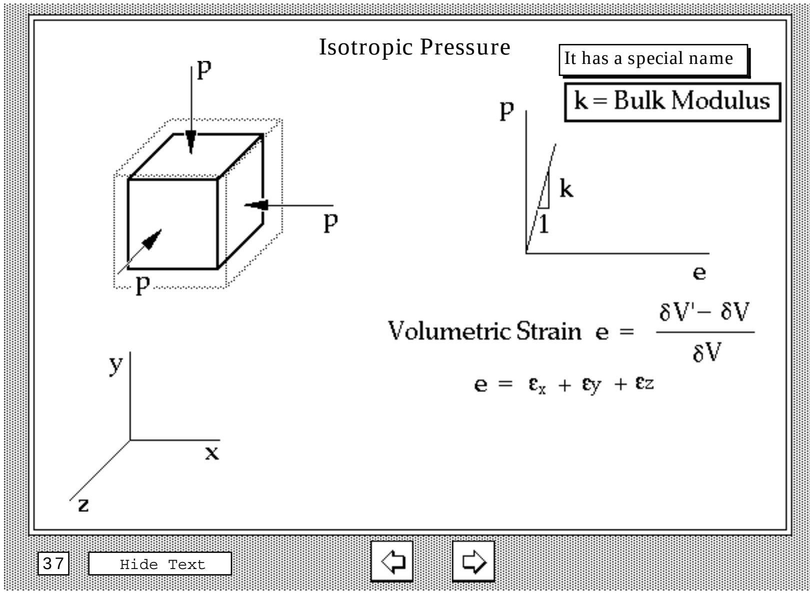


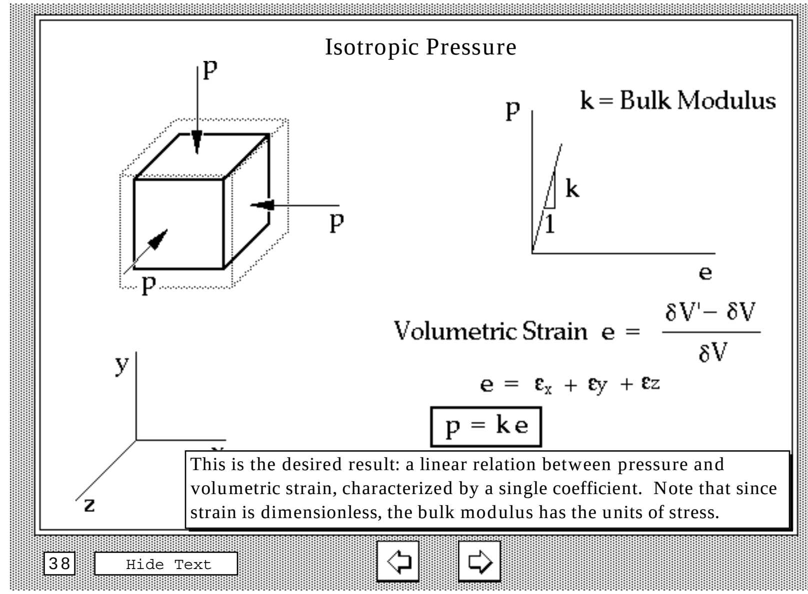
The slope of this linear plot represents the volumetric "stiffness" of the material.

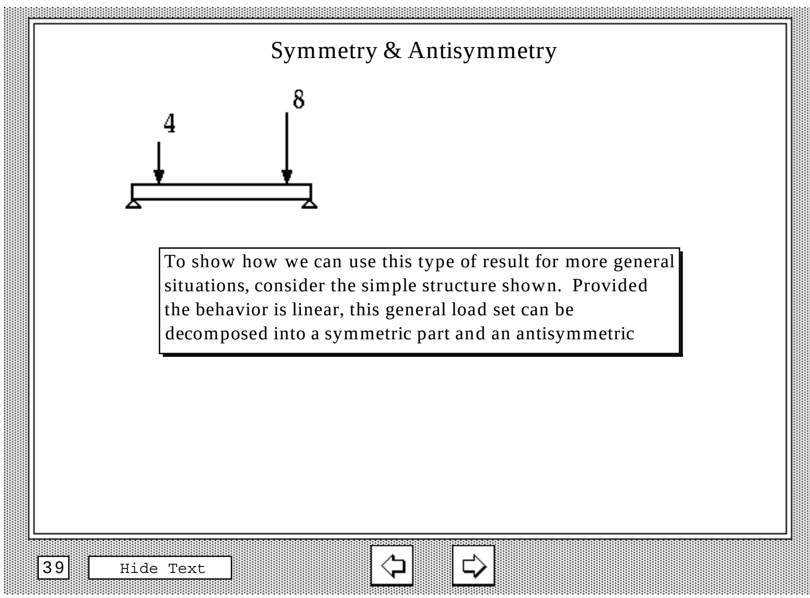
$$\text{Volumetric Strain } e = \frac{\delta V' - \delta V}{\delta V}$$

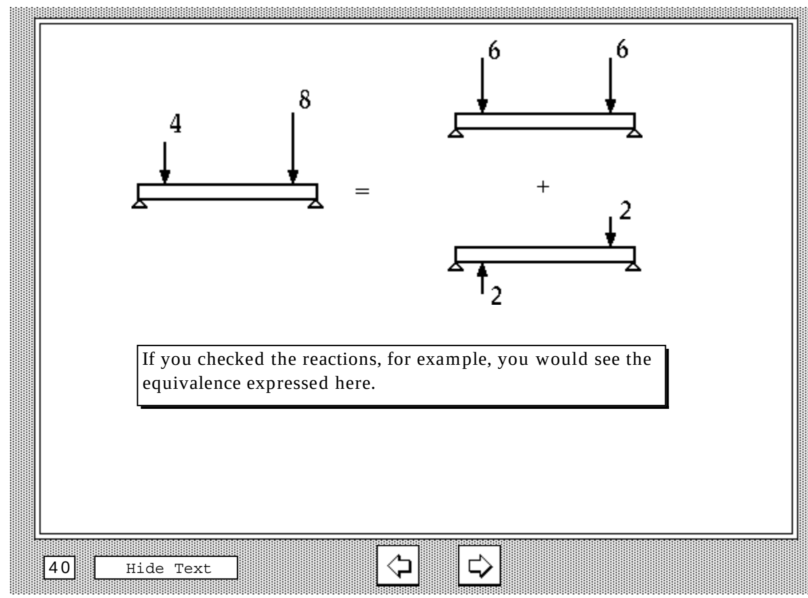
$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

36 Hide Text

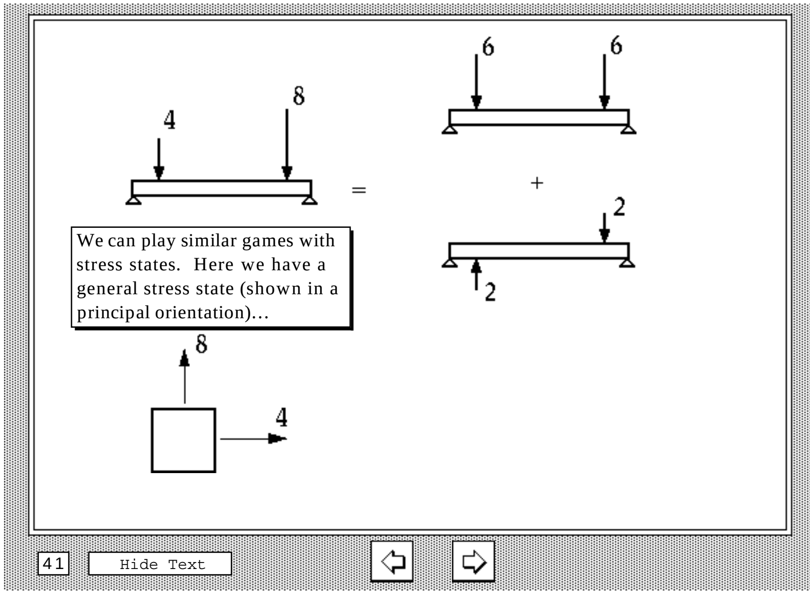

Isotropic Pressure
It has a special name
k = Bulk Modulus
p
p
p
p
k
1
e
y
x
z
Volumetric Strain e = δV'− δV / δV
e = εx + εy + εz
37
Hide Text

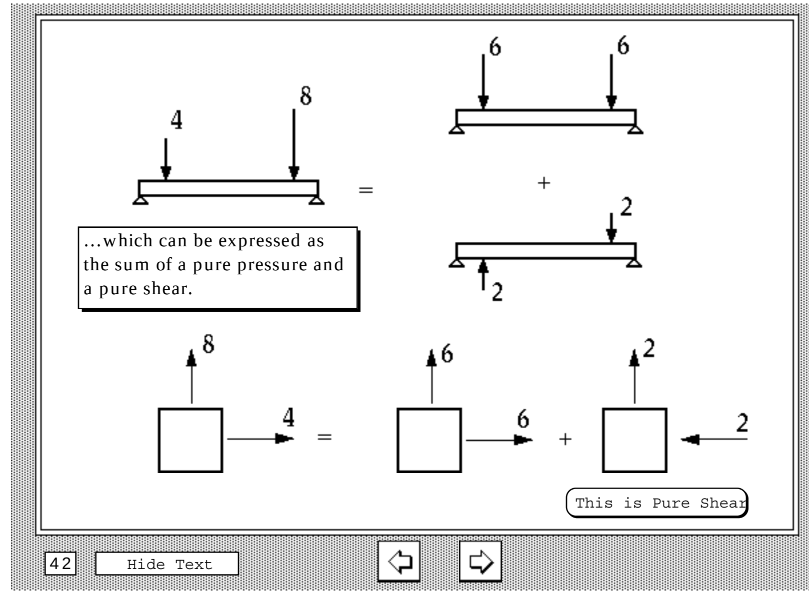

Isotropic Pressure
k = Bulk Modulus
p
p
p
p
k
1
e
y
z
Volumetric Strain e = δV'− δV / δV
e = εx + εy + εz
p = k e
This is the desired result: a linear relation between pressure and volumetric strain, characterized by a single coefficient. Note that since strain is dimensionless, the bulk modulus has the units of stress.
38
Hide Text

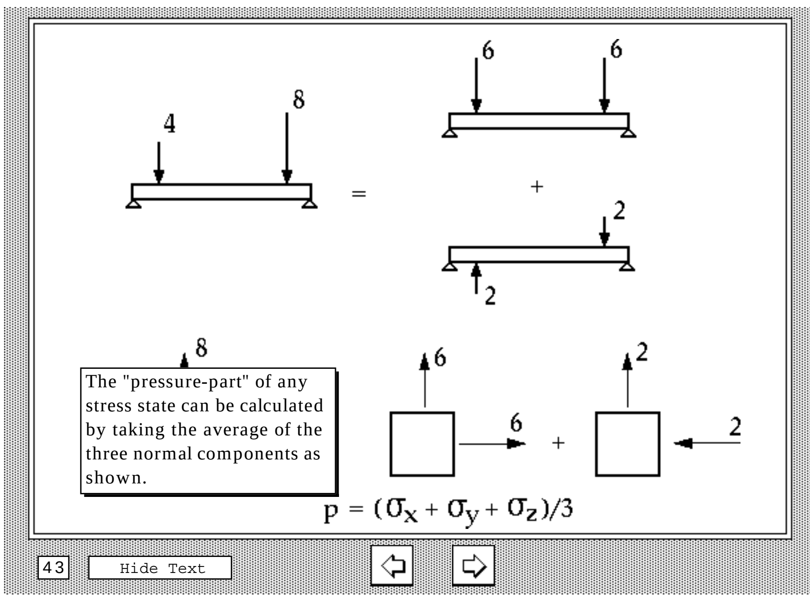

Symmetry & Antisymmetry
4
8
To show how we can use this type of result for more general situations, consider the simple structure shown. Provided the behavior is linear, this general load set can be decomposed into a symmetric part and an antisymmetric
39
Hide Text


4
8
=
6
6
+
2
2
If you checked the reactions, for example, you would see the equivalence expressed here.
40
Hide Text

Hooke's Law: 3/30/00







General Stress State

$$p = (\sigma_x + \sigma_y + \sigma_z)/3$$

$$p = k e$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

We can combine this fact with our previous results.

44 Hide Text

General Stress State

$$p = (\sigma_x + \sigma_y + \sigma_z)/3$$

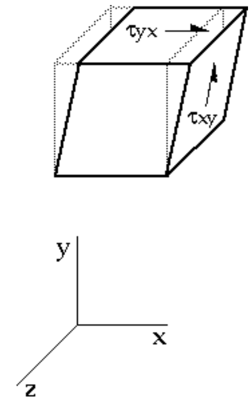
$$p = k e$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$(\sigma_x + \sigma_y + \sigma_z) = 3k(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

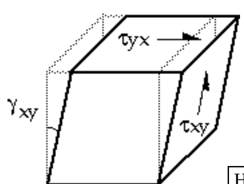
Here we have the pressure / volume change relation expressed entirely in terms of stress and strain components. This equation is valid for general stress-strain states.

Pure Shear

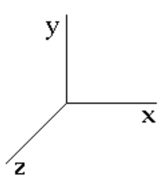


The other primary load / deformation response involves pure shear.

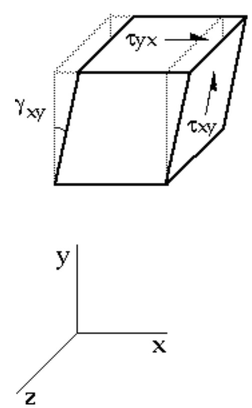
Pure Shear

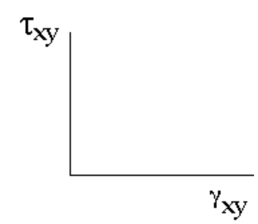


Here we can use our stress and strain components directly. Note that this is an example in which it is convenient to use the engineering strain component, $\gamma = 2\varepsilon$.

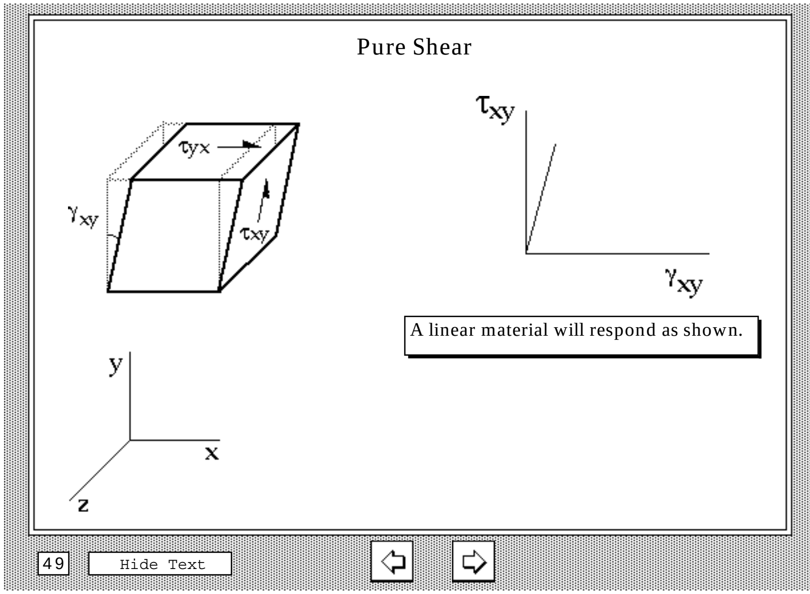


Pure Shear

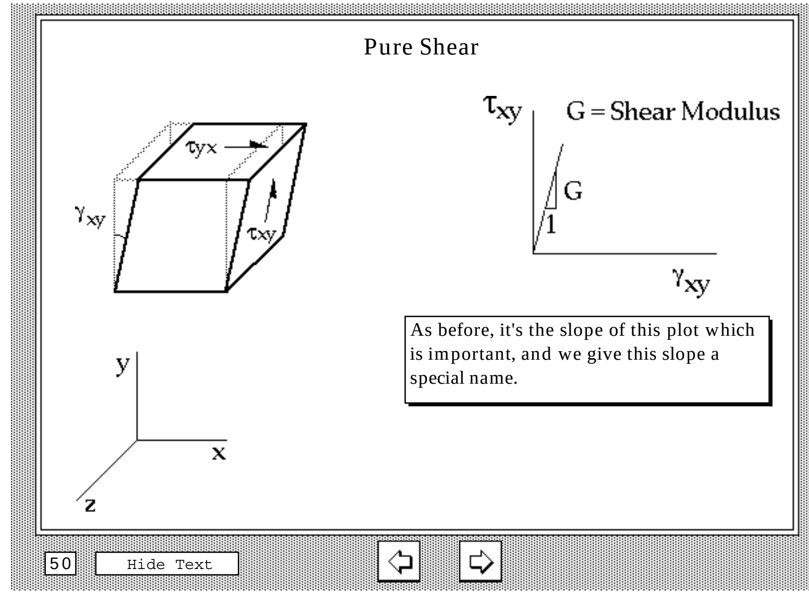




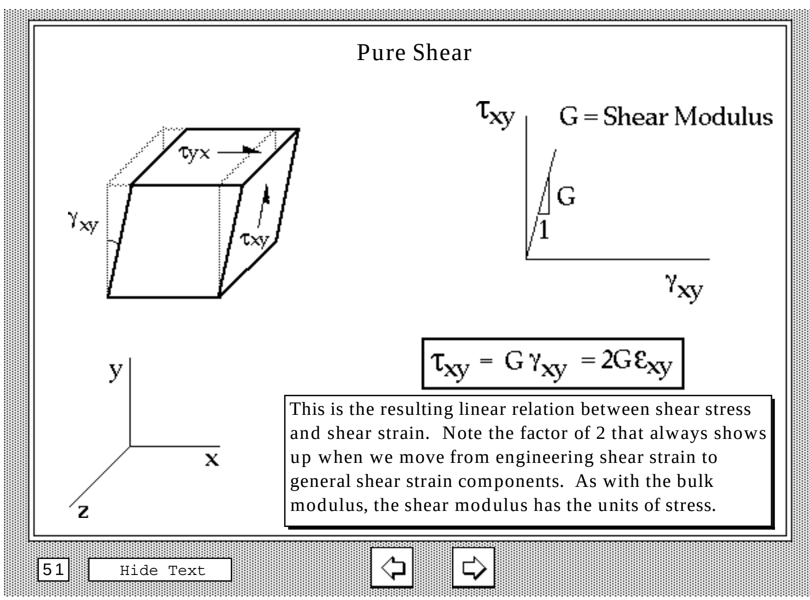
As before, we imagine we run a test and measure the load and deformation.



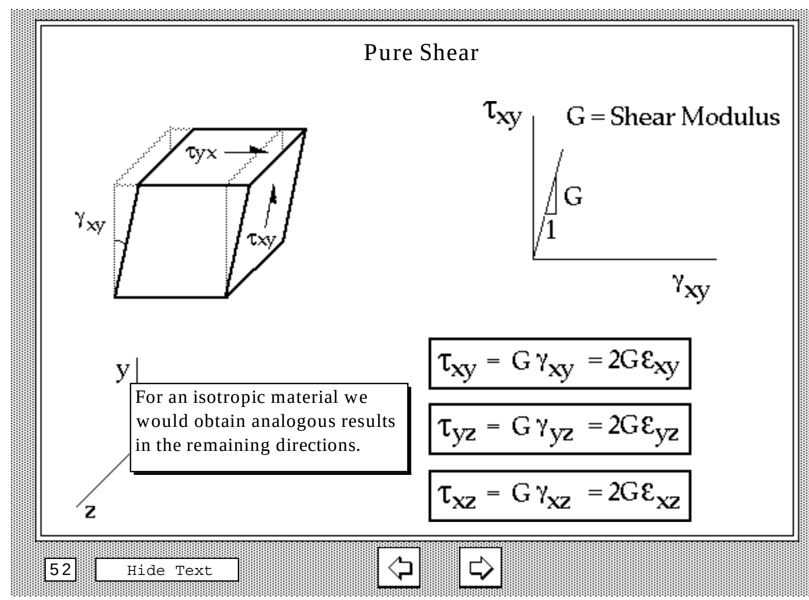
Pure Shear
τ_{yx}
γ_{xy}
τ_{xy}
τ_{xy}
γ_{xy}
A linear material will respond as shown.
y
x
z
49
Hide Text



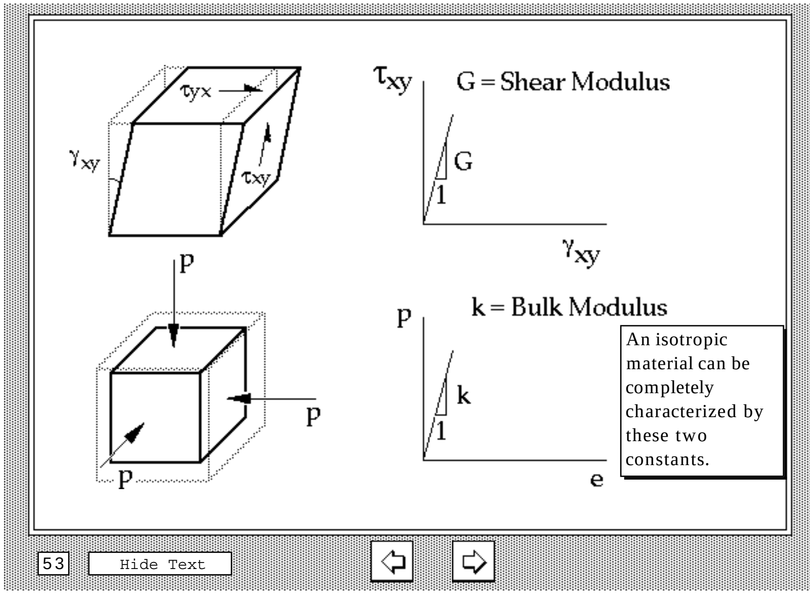
Pure Shear
τ_{yx}
γ_{xy}
τ_{xy}
τ_{xy}
G = Shear Modulus
G
1
γ_{xy}
As before, it's the slope of this plot which is important, and we give this slope a special name.
y
x
z
50
Hide Text

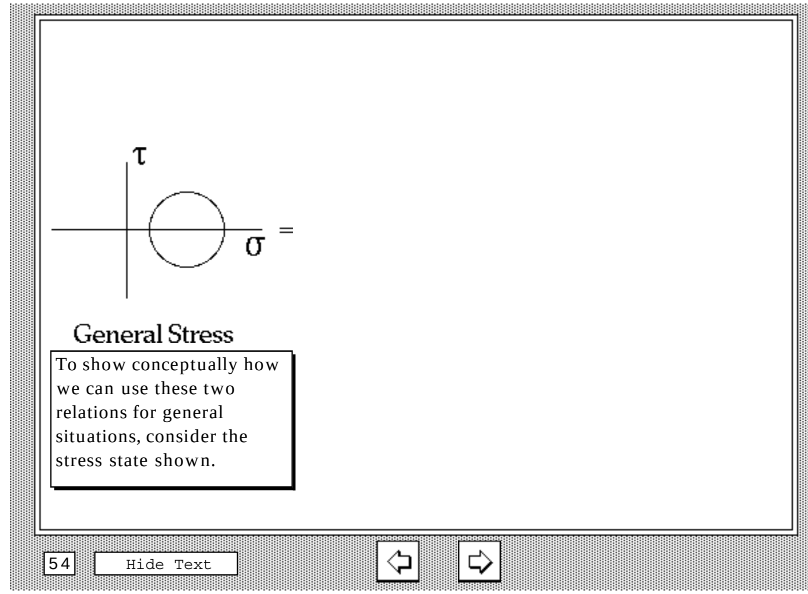


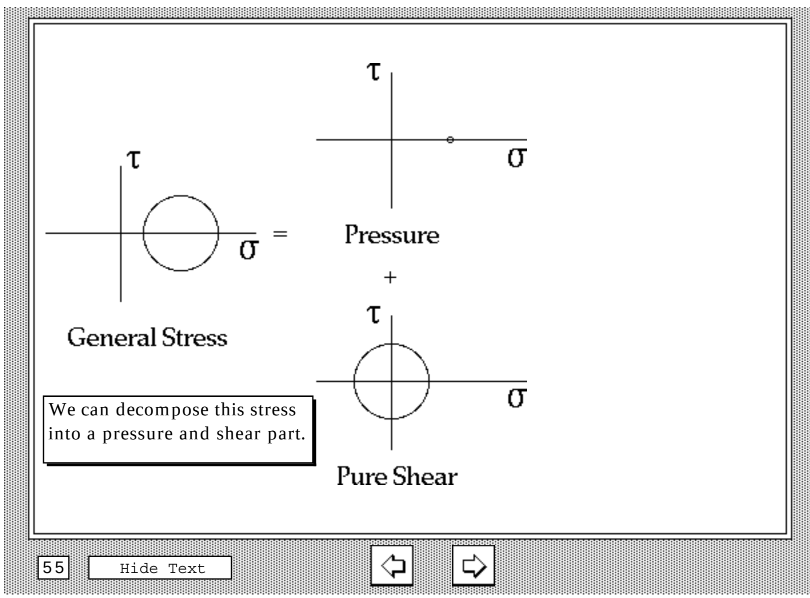
Pure Shear
τ_{yx}
γ_{xy}
τ_{xy}
τ_{xy}
G = Shear Modulus
G
1
γ_{xy}
$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$
This is the resulting linear relation between shear stress and shear strain. Note the factor of 2 that always shows up when we move from engineering shear strain to general shear strain components. As with the bulk modulus, the shear modulus has the units of stress.
y
x
z
51
Hide Text

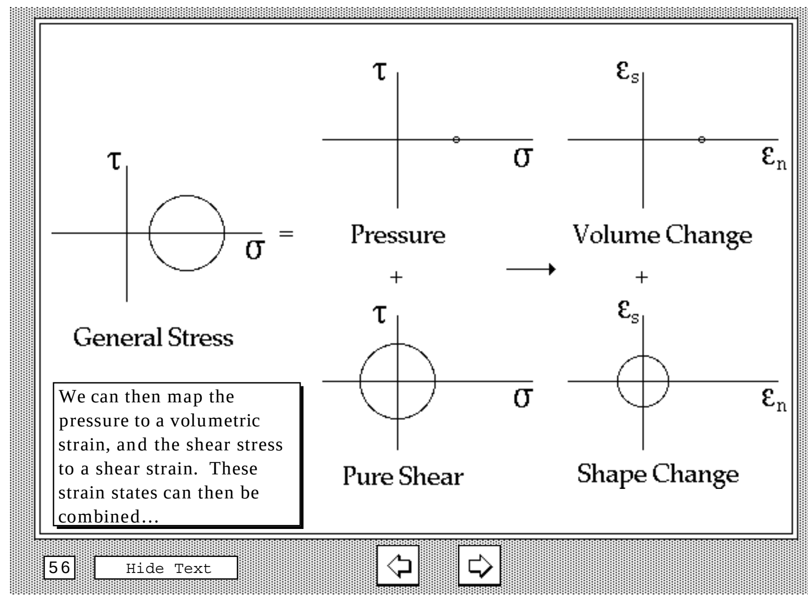


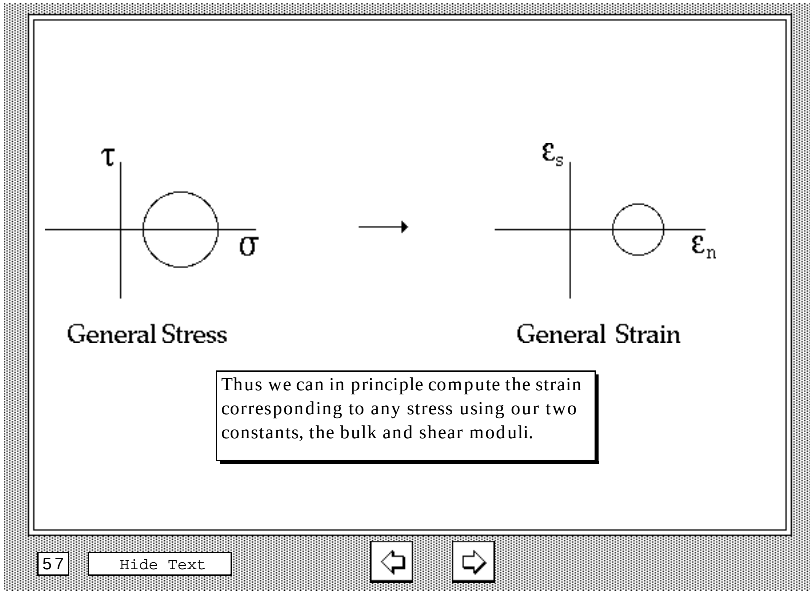
Pure Shear
τ_{yx}
γ_{xy}
τ_{xy}
τ_{xy}
G = Shear Modulus
G
1
γ_{xy}
$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$
$\tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$
$\tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz}$
For an isotropic material we would obtain analogous results in the remaining directions.
y
z
52
Hide Text

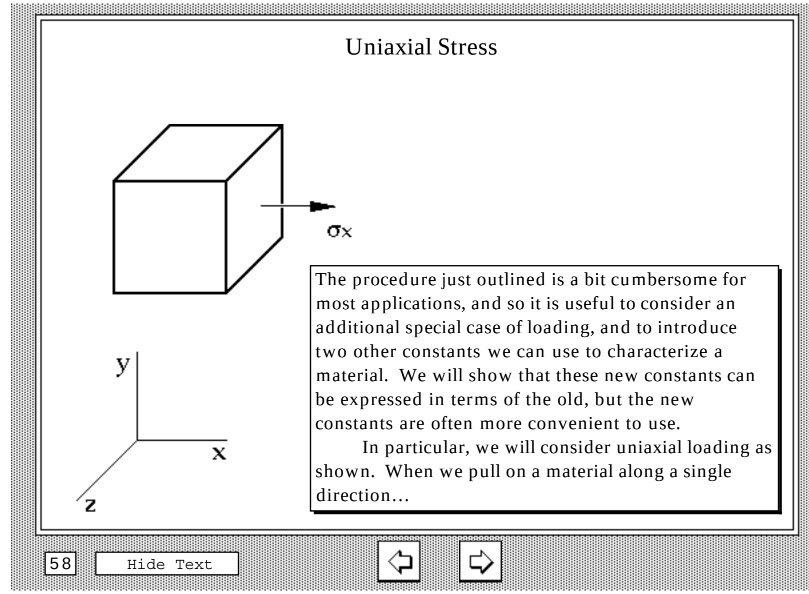

τyx
γxy
τxy
p
p
p
τxy
G = Shear Modulus
G
1
γxy
p
k = Bulk Modulus
k
1
e
An isotropic material can be completely characterized by these two constants.
53
Hide Text

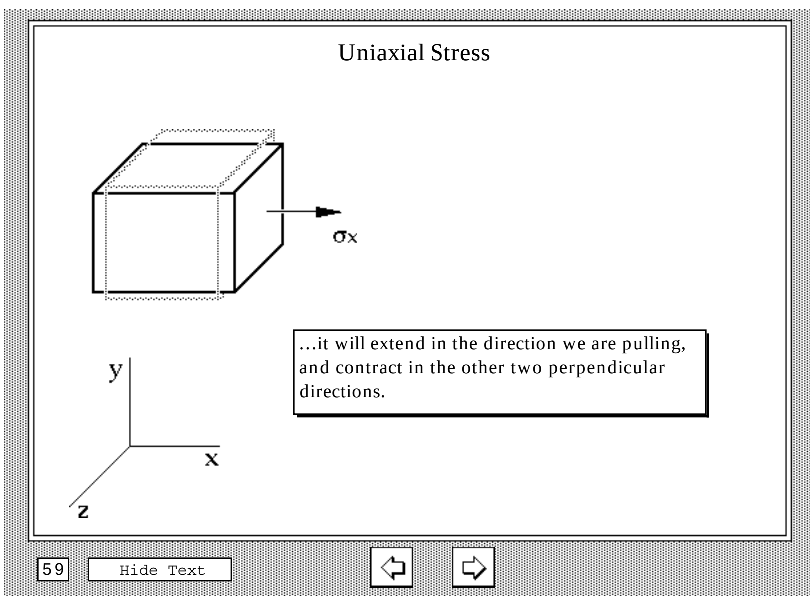

τ
σ
=
General Stress
To show conceptually how we can use these two relations for general situations, consider the stress state shown.
54
Hide Text

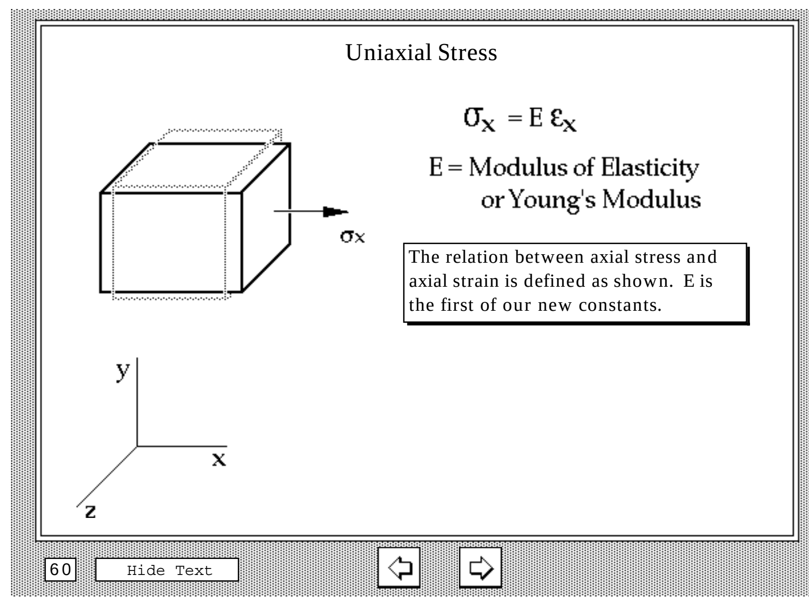

τ
σ
=
General Stress
τ
σ
Pressure
+
τ
σ
Pure Shear
We can decompose this stress into a pressure and shear part.
55
Hide Text


τ
σ
=
General Stress
τ
σ
Pressure
+
τ
σ
Pure Shear
εs
εn
Volume Change
+
εs
εn
Shape Change
We can then map the pressure to a volumetric strain, and the shear stress to a shear strain. These strain states can then be combined...
56
Hide Text

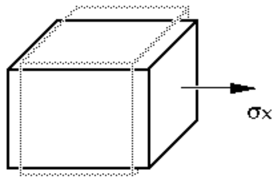

τ
σ
εs
εn
General Stress
General Strain
Thus we can in principle compute the strain corresponding to any stress using our two constants, the bulk and shear moduli.
57
Hide Text


Uniaxial Stress
σx
y
x
z
The procedure just outlined is a bit cumbersome for most applications, and so it is useful to consider an additional special case of loading, and to introduce two other constants we can use to characterize a material. We will show that these new constants can be expressed in terms of the old, but the new constants are often more convenient to use.
In particular, we will consider uniaxial loading as shown. When we pull on a material along a single direction…
58
Hide Text


Uniaxial Stress
σx
y
x
z
…it will extend in the direction we are pulling, and contract in the other two perpendicular directions.
59
Hide Text


Uniaxial Stress
σx = E εx
E = Modulus of Elasticity
or Young's Modulus
σx
y
x
z
The relation between axial stress and axial strain is defined as shown. E is the first of our new constants.
60
Hide Text

Uniaxial Stress



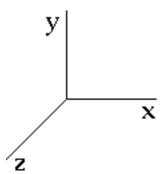
$$\sigma_x = E\,\varepsilon_x$$

E = Modulus of Elasticity
or Young's Modulus

$$\varepsilon_y = -\nu\,\varepsilon_x$$

$$\varepsilon_z = -\nu\,\varepsilon_x$$

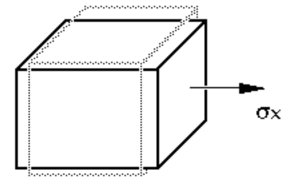
ν = Poisson's Ratio



The relation between the extension in the direction of pulling and the contraction in the perpendicular directions is expressed in terms of the dimensionless parameter, ν -- pronounced 'gnu'. (Recall that the modulus constants, k, E, and G, all have units of stress). This is the second of our new

61 | Hide Text

Uniaxial Stress



$$\sigma_x = E\,\varepsilon_x$$

E = Modulus of Elasticity
or Young's Modulus

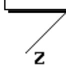
$$\varepsilon_y = -\nu\,\varepsilon_x$$

$$\varepsilon_z = -\nu\,\varepsilon_x$$

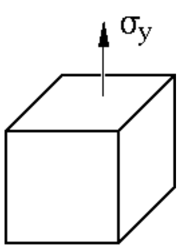
ν = Poisson's Ratio

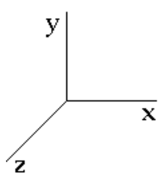
It is important to note that even though we have applied stress in 1-dimension, the resulting strain is 3-dimensional as indicated to the right. This characteristic makes general stress analysis very difficult.

$$\sigma_x \longrightarrow \begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} \\ \varepsilon_y = -\nu\dfrac{\sigma_x}{E} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} \end{cases}$$



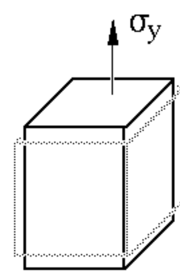
62 | Hide Text





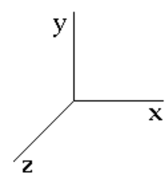
For an isotropic material, we can repeat our experiment in the y-direction…

63 | Hide Text

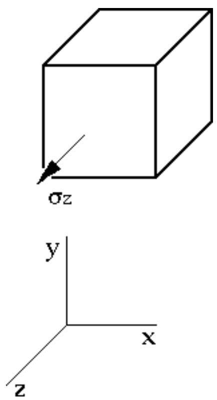


…and obtain a similar result (note which strains have ν, and which do not).

$$\sigma_y \longrightarrow \begin{cases} \varepsilon_x = -\nu\dfrac{\sigma_y}{E} \\ \varepsilon_y = \dfrac{\sigma_y}{E} \\ \varepsilon_z = -\nu\dfrac{\sigma_y}{E} \end{cases}$$

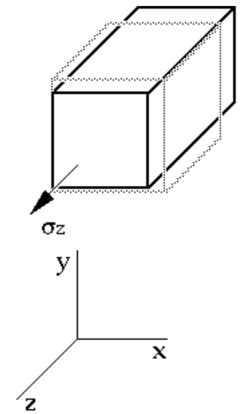


64 | Hide Text



We can pull in the z-direction…

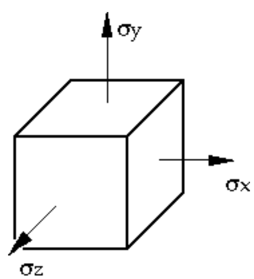
65 Hide Text



…and again end up with similar expressions.

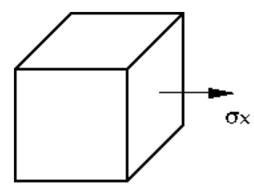
$$\sigma_z \longrightarrow \begin{cases} \varepsilon_x = -\nu \dfrac{\sigma_z}{E} \\ \varepsilon_y = -\nu \dfrac{\sigma_z}{E} \\ \varepsilon_z = \dfrac{\sigma_z}{E} \end{cases}$$

66 Hide Text



If we have axial stresses in all three directions, then we can calculate the total strain by adding up the terms for each direction individually.

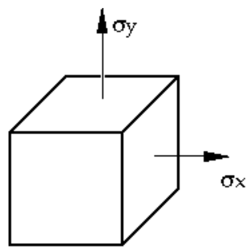
67 Hide Text



For x-direction stress we have these terms.

$$\sigma_x \longrightarrow \begin{cases} \varepsilon_x = \dfrac{1}{E}[\sigma_x \quad] \\ \varepsilon_y = \dfrac{1}{E}[\quad -\nu\,(\sigma_x \quad)] \\ \varepsilon_z = \dfrac{1}{E}[\quad -\nu\,(\sigma_x \quad)] \end{cases}$$

68 Hide Text

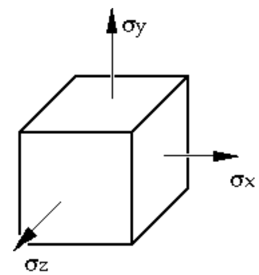


If we add y-direction stress, we add in the terms shown.

σ_x & σ_y →

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y \quad)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x \quad)]$$
$$\varepsilon_z = \frac{1}{E}[\quad - \nu(\sigma_x + \sigma_y)]$$

69 Hide Text



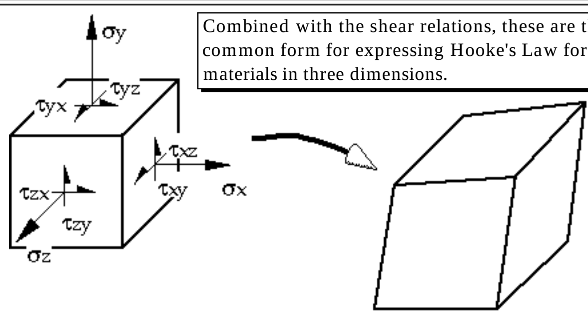
And finally if we add z-direction stress, we end up with a complete relation for each individual normal strain component in term of the three normal stress components. This form is more convenient to use than our earlier bulk modulus equation, since the strain components here are expressed individually.

σ_x & σ_y & σ_z →

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

70 Hide Text

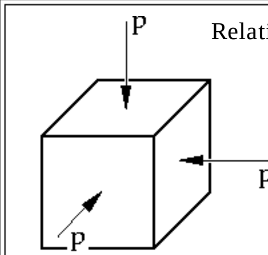
Combined with the shear relations, these are the most common form for expressing Hooke's Law for isotropic materials in three dimensions.



$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \qquad \tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \qquad \tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz}$$

71 Hide Text

Relations Between the Constants

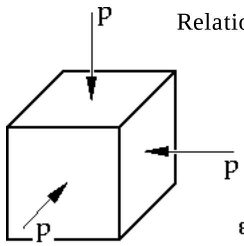


$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Our final task is to relate the "new" constants, E and ν, to our original constants, G and k. We will consider k first, and apply a uniform pressure to a block as shown.

72 Hide Text Skip the Details

Relations Between the Constants



$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = -p(1-2\nu)/E$$

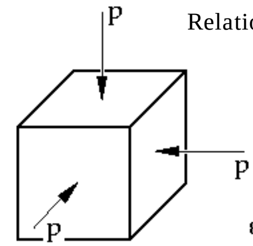
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = -p(1-2\nu)/E$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = -p(1-2\nu)/E$$

Substituting $\sigma_x = \sigma_y = \sigma_z =$ **p** into the stress-strain relations gives the result shown.

73 Hide Text

Relations Between the Constants



$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = -p(1-2\nu)/E$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = -p(1-2\nu)/E$$

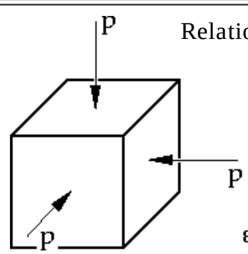
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = -p(1-2\nu)/E$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -3p(1-2\nu)/E$$

We can sum these three equations to obtain the volumetric strain.

74 Hide Text

Relations Between the Constants



$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = -p(1-2\nu)/E$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = -p(1-2\nu)/E$$

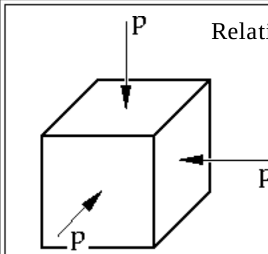
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = -p(1-2\nu)/E$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -3p(1-2\nu)/E = -p/k$$

By definition of the bulk modulus, k, we can relate the volumetric strain and pressure as shown.

75 Hide Text

Relations Between the Constants



$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = -p(1-2\nu)/E$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = -p(1-2\nu)/E$$

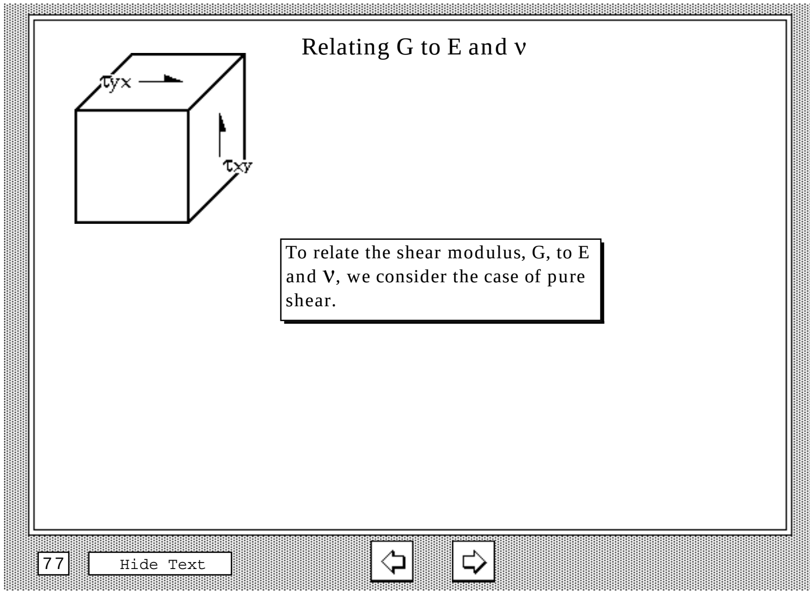
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] = -p(1-2\nu)/E$$

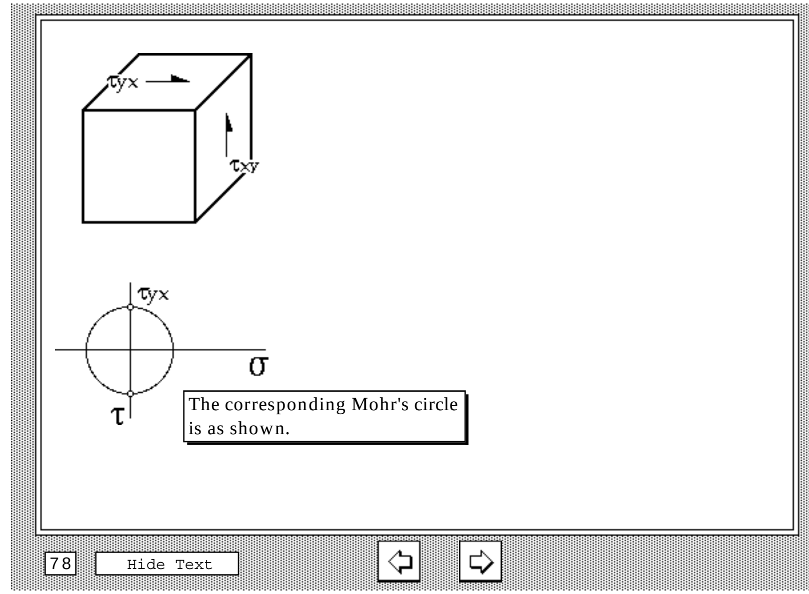
$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -3p(1-2\nu)/E = -p/k$$

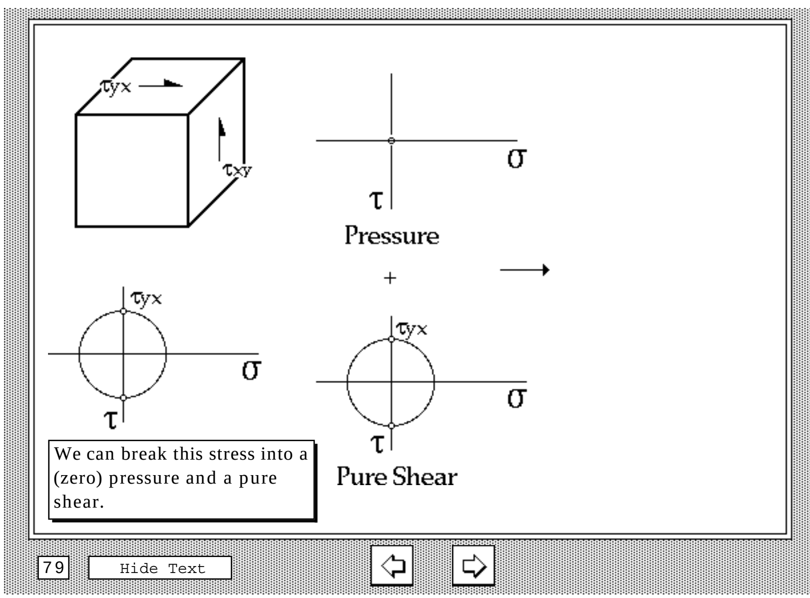
$$\longrightarrow \boxed{k = \frac{E}{3(1-2\nu)}}$$

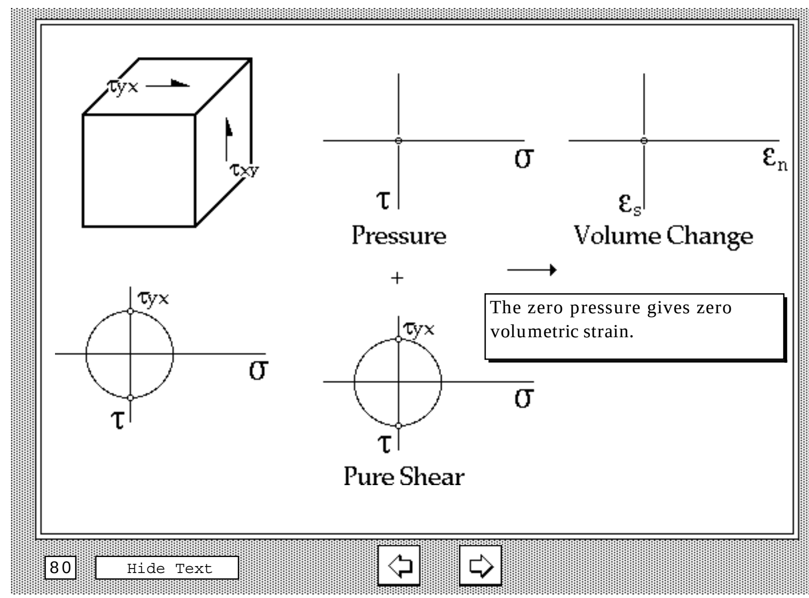
Eliminating p gives the desired result: k expressed in terms of E and ν.

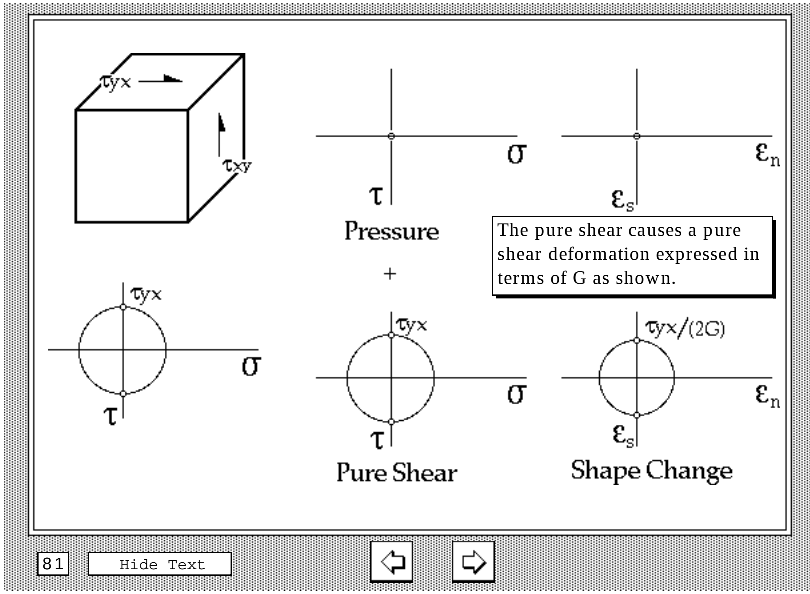
76 Hide Text

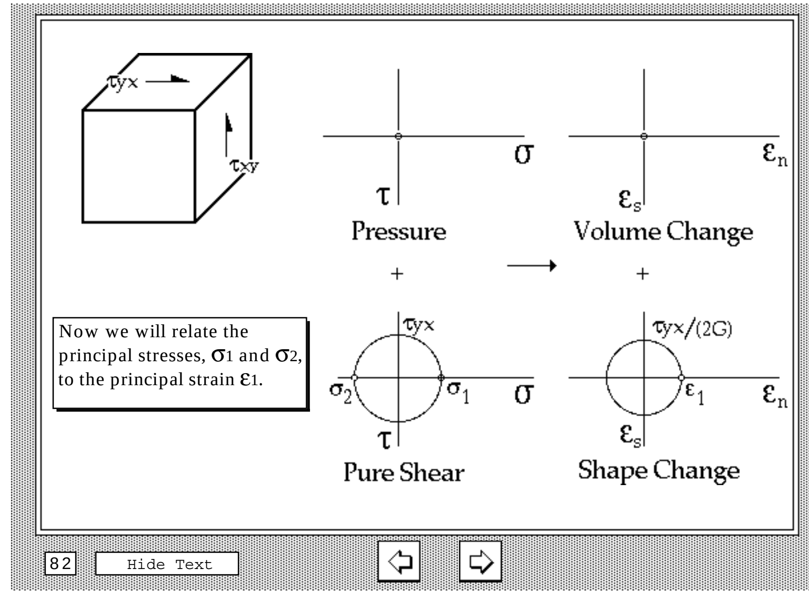

Relating G to E and ν
τyx
τxy
To relate the shear modulus, G, to E and ν, we consider the case of pure shear.
77
Hide Text

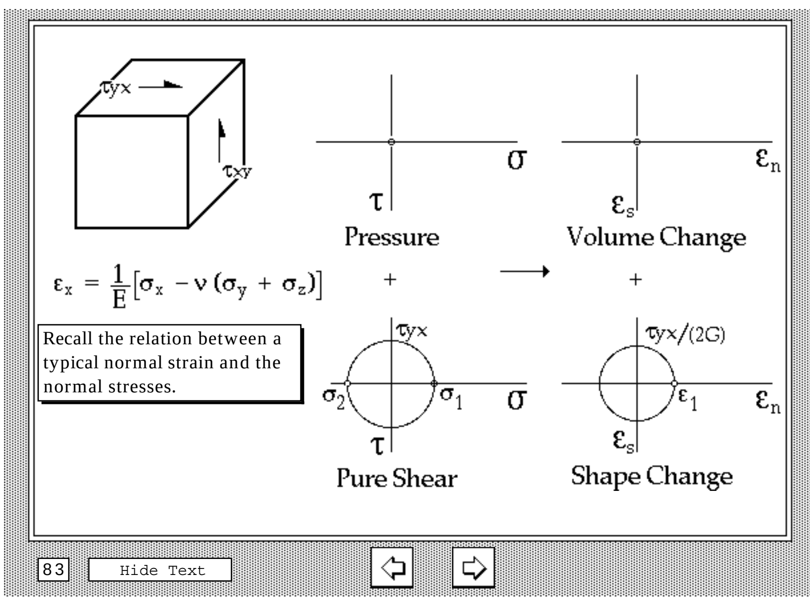

τyx
τxy
τyx
σ
τ
The corresponding Mohr's circle is as shown.
78
Hide Text

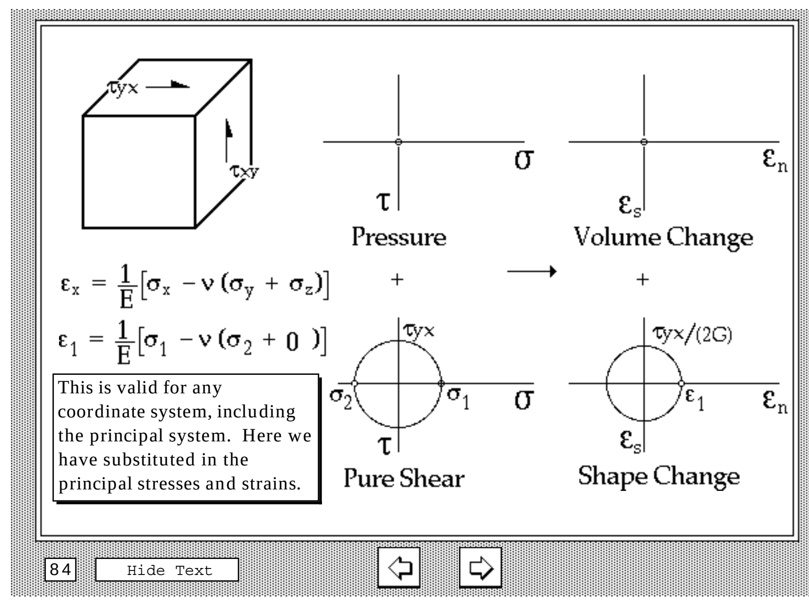

τyx
τxy
σ
τ
Pressure
+
τyx
σ
τ
τyx
σ
τ
Pure Shear
We can break this stress into a (zero) pressure and a pure shear.
79
Hide Text

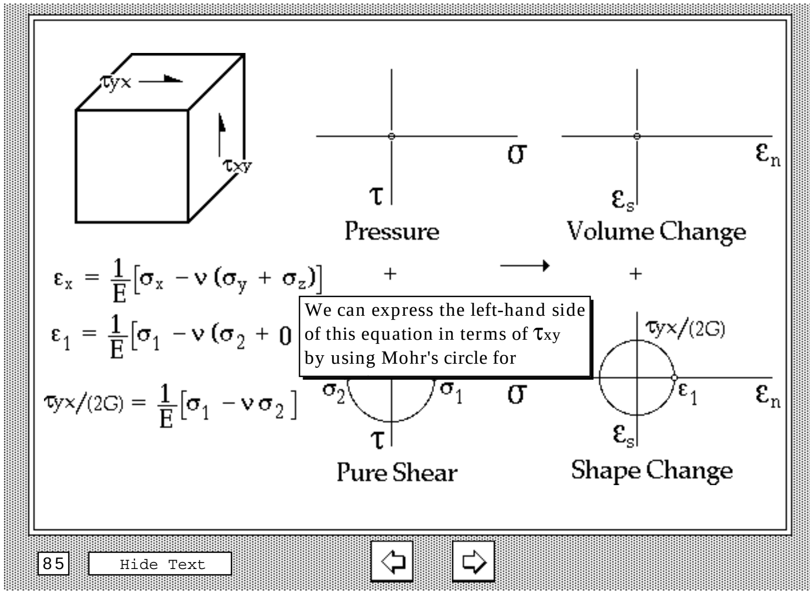

τyx
τxy
σ
τ
Pressure
εn
εs
Volume Change
+
τyx
σ
τ
τyx
σ
τ
Pure Shear
The zero pressure gives zero volumetric strain.
80
Hide Text

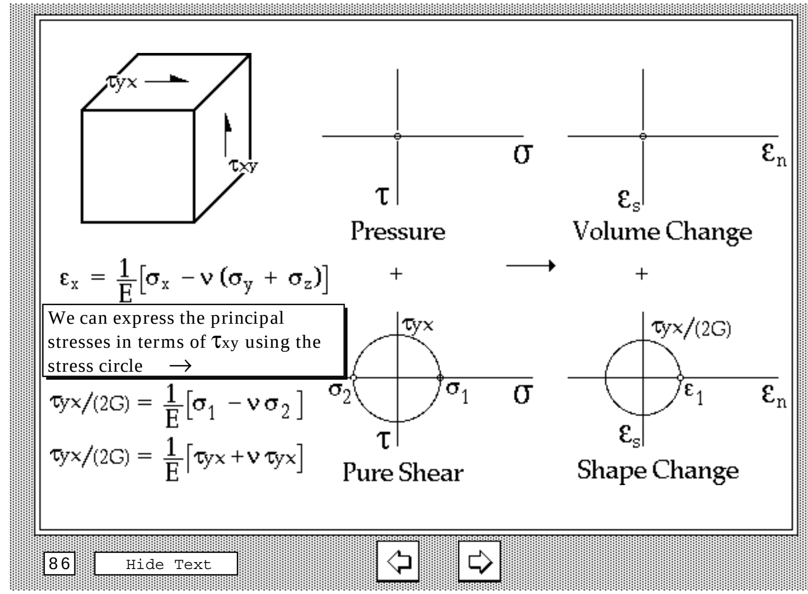

τyx
τxy
σ
τ
Pressure
+
εn
εs
The pure shear causes a pure shear deformation expressed in terms of G as shown.
τyx
σ
τ
τyx
σ
τ
Pure Shear
τyx/(2G)
εn
εs
Shape Change
81
Hide Text

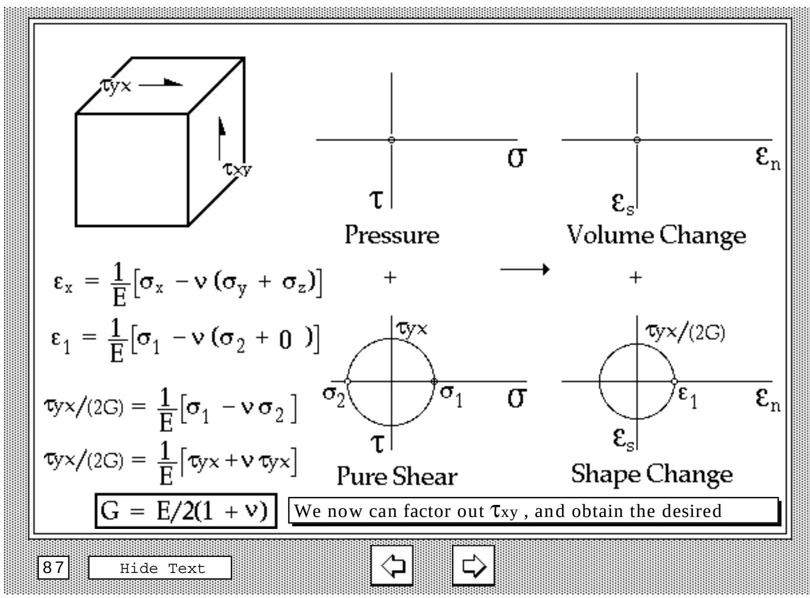

τyx
τxy
σ
τ
Pressure
+
εn
εs
Volume Change
+
Now we will relate the principal stresses, σ1 and σ2, to the principal strain ε1.
τyx
σ2
σ1
σ
τ
Pure Shear
τyx/(2G)
ε1
εn
εs
Shape Change
82
Hide Text


τyx
τxy
σ
τ
Pressure
εn
εs
Volume Change
$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$
+
+
Recall the relation between a typical normal strain and the normal stresses.
τyx
σ2
σ1
σ
τ
Pure Shear
τyx/(2G)
ε1
εn
εs
Shape Change
83
Hide Text


τyx
τxy
σ
τ
Pressure
εn
εs
Volume Change
$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$
+
+
$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + 0)]$
This is valid for any coordinate system, including the principal system. Here we have substituted in the principal stresses and strains.
τyx
σ2
σ1
σ
τ
Pure Shear
τyx/(2G)
ε1
εn
εs
Shape Change
84
Hide Text







$$k = \frac{E}{3(1 - 2\nu)}$$

$$G = E/2(1 + \nu)$$

These relations show that we can always express k and G in terms of E and ν, and *vice-versa*..

We can make two other observations based on these equations. Consider first the relation for k. Note that if $\nu = 1/2$, then $k \longrightarrow \infty$, and if $\nu > 1/2$ then $k<0$. Negative stiffness is a no-no, since it implies creation of energy. Thus ν must be less than or equal to 1/2, and if $\nu = 1/2$, the material can not be squished, since it has infinite stiffness with respect to volume change. Such a material is deemed ***incompressible*** , and common rubber is an example of a material that is nearly so.

A similar argument for the G relation shows the ν must be greater than –1. It is interesting that there is no theoretical reason why ν can not be negative, but no one has yet found a homogeneous material with negative ν. You will generally find Poisson's ratio on the order of 0.2-0.4.

88 Hide Text Detailed Derivation

Summary

Here is a summary of all our results. These equations are both useful and common.

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz}$$

$$(\sigma_x + \sigma_y + \sigma_z) = 3k(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$k = \frac{E}{3(1 - 2\nu)}$$

$$G = E/2(1 + \nu)$$

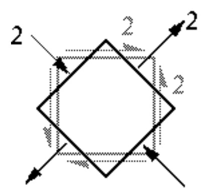
$$-1 < \nu < 1/2$$

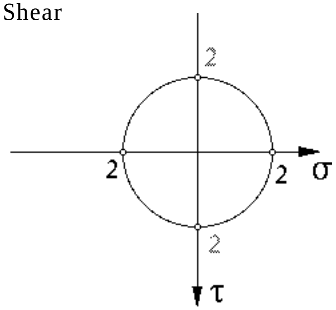
k = Bulk Modulus
G = Shear Modulus
E = Modulus of Elasticity or Young's Modulus
ν = Poisson's Ratio

89 Hide Text

The End

Pure Shear





Recall that the state of pure shear is equivalent to compressive and tensile normal stresses at a relative orientation of 45°. This can perhaps be seen best by looking at the Mohr's circle for the state of pure shear.

91 Hide Text Return