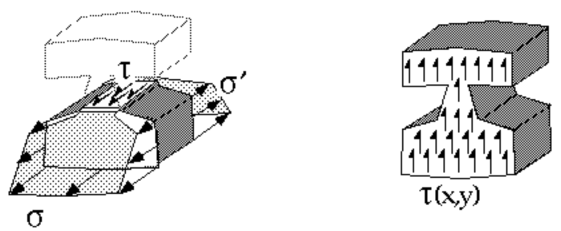
Shear in Beams



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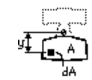

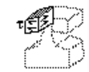

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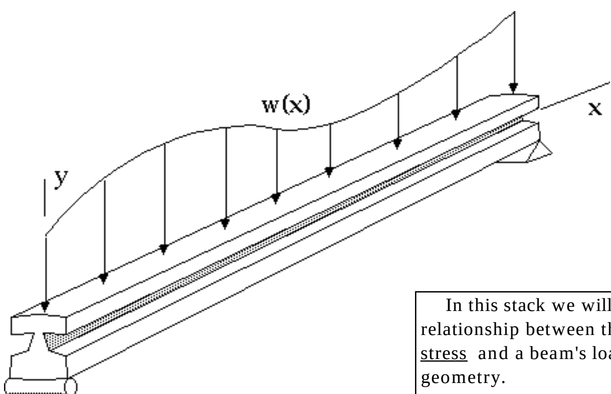

In this stack we will derive a relationship between the shear stress and a beam's load and geometry.

We begin by considering our old friend the arbitrarily loaded generic beam.

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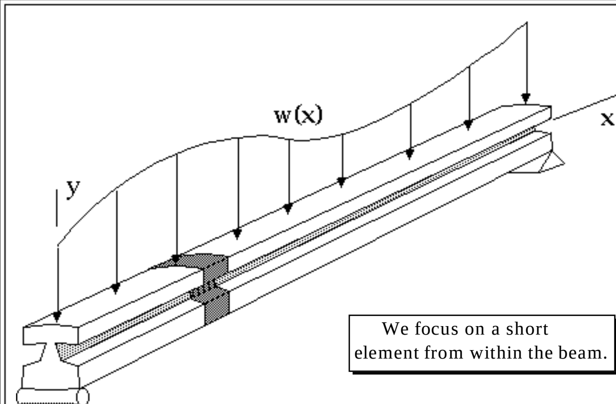

We focus on a short element from within the beam.

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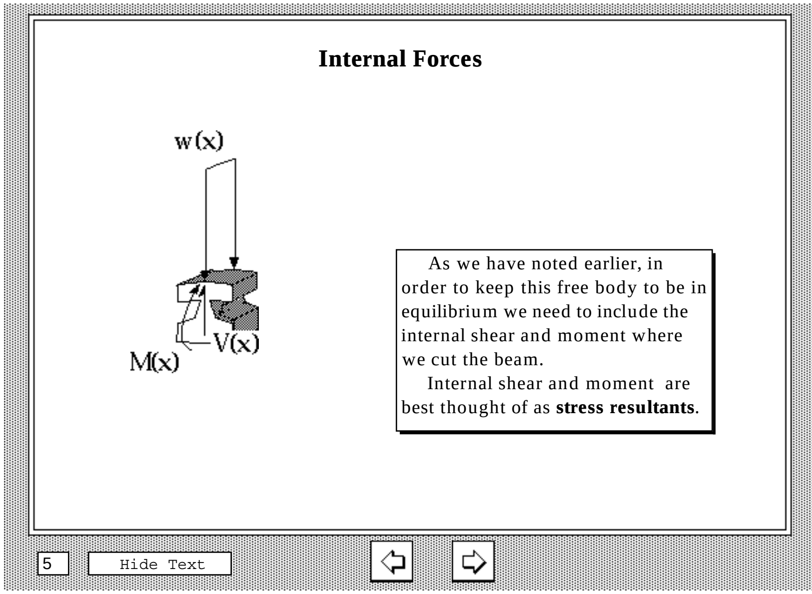
Internal Forces
w(x)
V(x)
M(x)
As we have noted earlier, in order to keep this free body to be in equilibrium we need to include the internal shear and moment where we cut the beam.
Internal shear and moment are best thought of as **stress resultants**.
5
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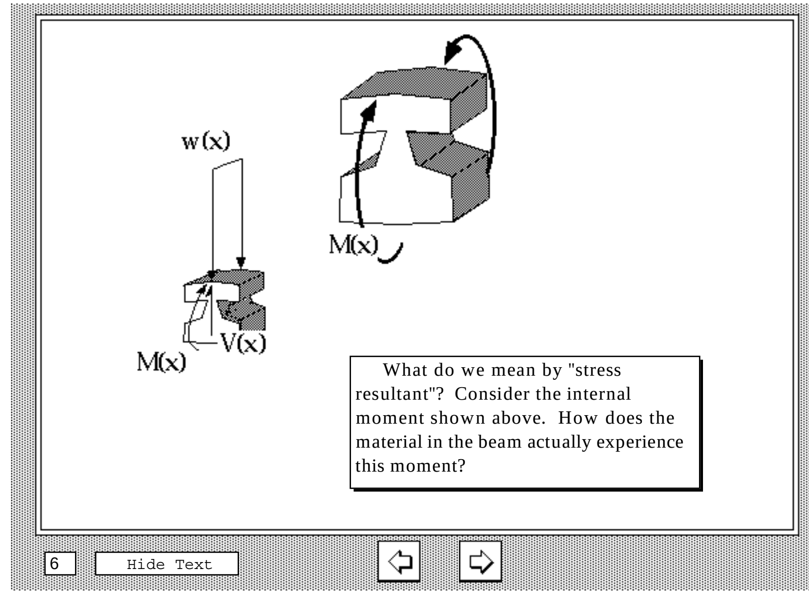
w(x)
M(x)
V(x)
M(x)
What do we mean by "stress resultant"? Consider the internal moment shown above. How does the material in the beam actually experience this moment?
6
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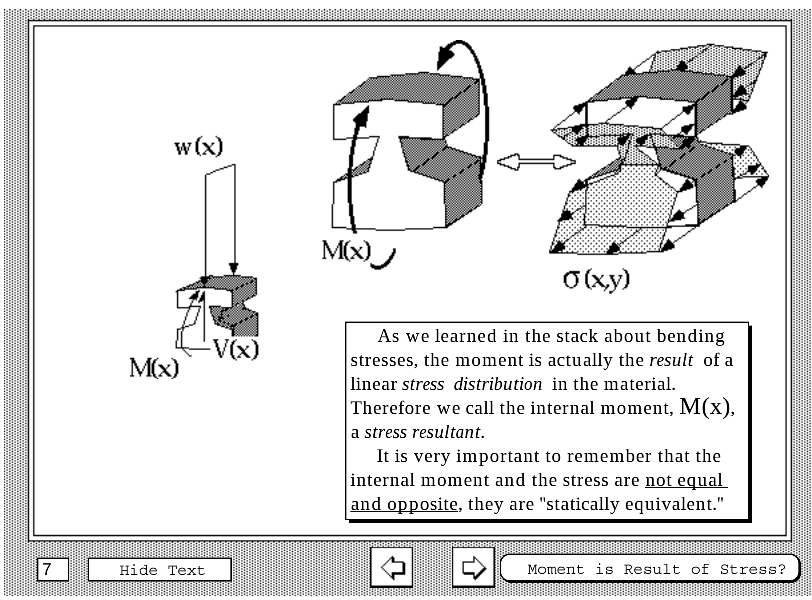
w(x)
M(x)
σ (x,y)
V(x)
M(x)
As we learned in the stack about bending stresses, the moment is actually the *result* of a linear *stress distribution* in the material. Therefore we call the internal moment, M(x), a *stress resultant*.
It is very important to remember that the internal moment and the stress are not equal and opposite, they are "statically equivalent."
7
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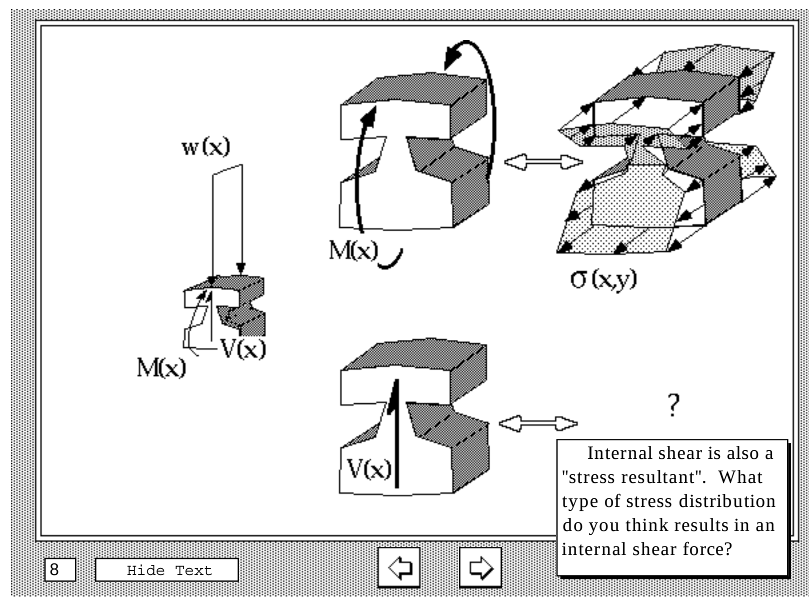
Moment is Result of Stress?


w(x)
M(x)
σ (x,y)
V(x)
M(x)
V(x)
?
Internal shear is also a "stress resultant". What type of stress distribution do you think results in an internal shear force?
8
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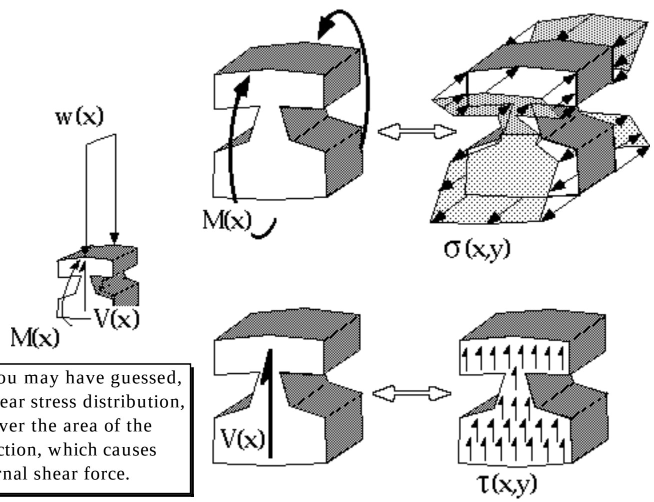

As you may have guessed, it is a shear stress distribution, acting over the area of the cross-section, which causes the internal shear force.

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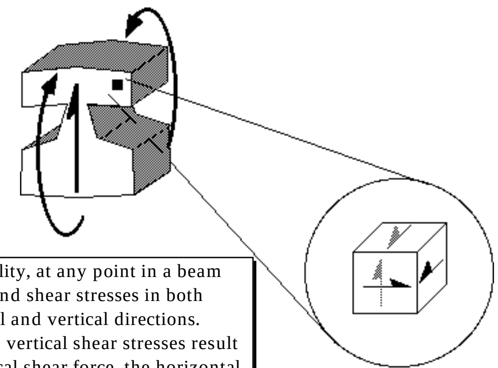
Vertical and Horizontal Shear Stress



In reality, at any point in a beam we will find shear stresses in both horizontal and vertical directions. While the vertical shear stresses result in a vertical shear force, the horizontal shear stresses are "self equilibrating" and result in no net force or moment.

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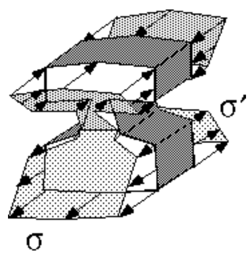

Now that we are convinced that shear stresses exist when we bend a beam, let's pursue our goal of how to calculate the shear stresses from the beam's loading.

Recall the normal stress distribution due to bending in the beam. The normal stress is a function of moment ($\sigma = My/I$). If the moment varies along the beam then the normal stress will also vary along the beam. In general the stress distribution σ , shown above, will not equal the stress distribution σ '.

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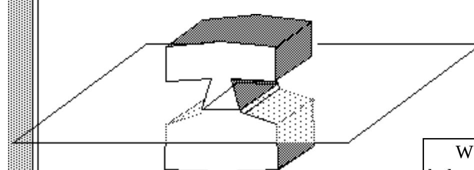
Calculating the Shear Stress



We now take a cut of the beam below the neutral axis.

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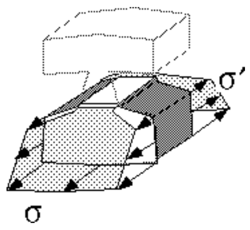

As shown here, the section of the beam which remains is still acted on by normal stresses.

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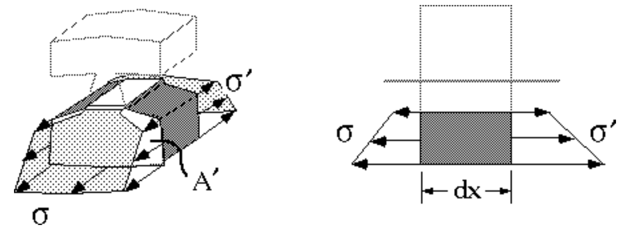

Looking at the element from a side view, we note that the length of the element is dx. The area acted on by the normal stress is denoted as A'.

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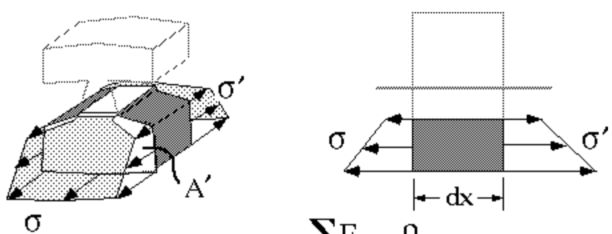

$$\sum F_x = 0$$

Is the free body in equilibrium as we have drawn it?
Recall that in general the two stress distributions σ and σ ' are not equal. Since they both act over the same area, they create unequal forces in the x direction. The free body does not satisfy horizontal force

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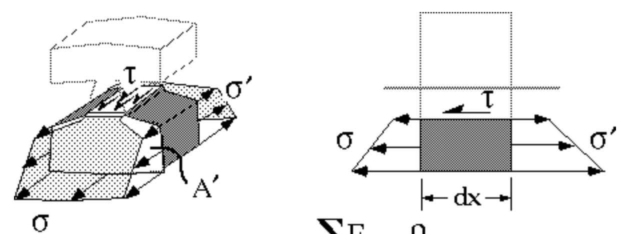

$$\sum F_x = 0$$

The only stress on the element which can remedy our equilibrium dilemma is a shear stress acting on the top face of the element as shown above.

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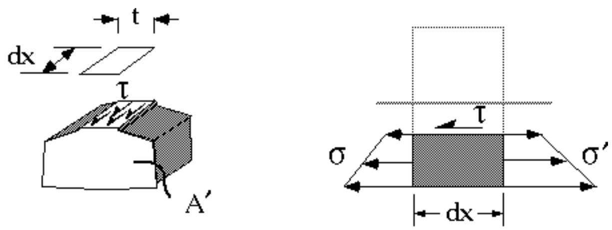


$$\sum F_x = 0$$

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

The contribution of each stress distribution to the horizontal equilibrium equation is shown above. The shear stress acts on a face which is dx long and t wide. Remember that the dimension t is used to determine the width of the area on which the shear stress acts. Also, notice that we have assumed that the shear stress is constant on the top cut. This is valid because we assume that the length of the cut, dx, is very small.

17 Hide Text

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

As you might have expected, the screen has been cleaned up in order to give us room to jump through some algebraic hoops. On with the show.

18 Hide Text

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0 \qquad \boxed{\sigma = \frac{M\,y}{I}}$$

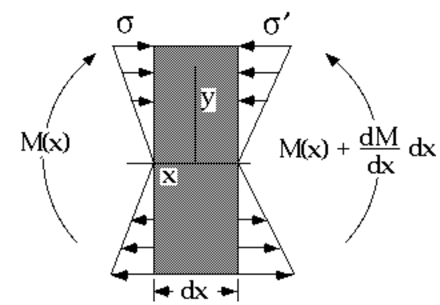
First we recall our equation for normal stress in a beam. You should be so familiar with this equation that you call out its name at night.

19 Hide Text $\sigma = \frac{My}{I}$

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0 \qquad \sigma = \frac{M\,y}{I}$$

$$\int_{A'} \frac{(M + dM)\,y}{I} \, dA - \int_{A'} \frac{M\,y}{I} \, dA - \tau \, (t \, dx) = 0$$

My/I is substituted for σ in our equilibrium equation. On the face where σ' acts, the moment is M + dM, as shown in the figure at the right.



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$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

$$\int_{A'} \frac{(\cancel{M} + dM)\, y}{I} \, dA - \int_{A'} \frac{\cancel{M}\, y}{I} \, dA - \tau \, (t \, dx) = 0$$

At this point we cancel like terms from the equation.

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

$$\int_{A'} \frac{(\cancel{M} + dM)\, y}{I} \, dA - \int_{A'} \frac{\cancel{M}\, y}{I} \, dA - \tau \, (t \, dx) = 0$$

$$\left(\frac{dM}{I}\right) \int_{A'} y \, dA = \tau \, (t \, dx)$$

Terms which do not vary over the area of the cut we pull outside the integral.

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

$$\int_{A'} \frac{(\cancel{M} + dM)\, y}{I} \, dA - \int_{A'} \frac{\cancel{M}\, y}{I} \, dA - \tau \, (t \, dx) = 0$$

$$\left(\frac{dM}{I}\right) \int_{A'} y \, dA = \tau \, (t \, dx)$$

$$\tau = \left(\frac{dM/dx}{I\, t} \right) \int_{A'} y \, dA$$

We then solve the expression for the unknown shear stress, τ.

$$\int_{A'} \sigma' \, dA - \int_{A'} \sigma \, dA - \tau \, (t \, dx) = 0$$

$$\int_{A'} \frac{(\cancel{M} + dM)\, y}{I} \, dA - \int_{A'} \frac{\cancel{M}\, y}{I} \, dA - \tau \, (t \, dx) = 0$$

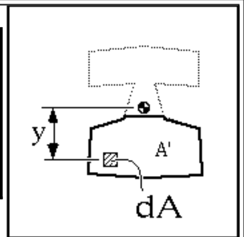
$$\left(\frac{dM}{I}\right) \int_{A'} y \, dA = \tau \, (t \, dx)$$

$$\boxed{dM/dx = V}$$

$$\tau = \left(\frac{dM/dx}{I\, t} \right) \int_{A'} y \, dA$$

At this point we need to recall the relationship between internal shear force, V, and the change in the internal moment along the beam.

We also need to define Q, the first moment of the area A' about the neutral axis. Note that the value of Q depends upon where we cut the section. If we take our cut at the very top or bottom of the beam, Q will be zero since dA will be zero. On the other hand, if we cut the beam at the centroid, Q will be a maximum.



Calculating Q

$$\left(\frac{dM}{I}\right)\int_{A'} y\, dA = \tau\,(t\,dx)$$

$$\tau = \left(\frac{dM/dx}{I\,t}\right)\int_{A'} y\, dA$$

$$dM/dx = V$$

$$\boxed{\int_{A'} y\, dA = Q}$$

$$\int_{A'} \sigma'\, dA - \int_{A'} \sigma\, dA - \tau\,(t\,dx) = 0$$

Substituting V for dM/dx and Q for the integral, we arrive at the expression used to calculate shear stress in beams. Remember, this formula assumes that the shear stress is constant across the beam where we take the cut.

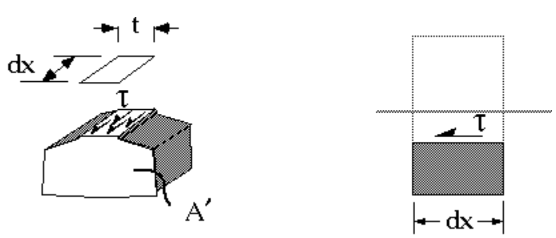
$$\left(\frac{dM}{I}\right)\int_{A'} y\, dA = \tau\,(t\,dx)$$

$$\tau = \left(\frac{dM/dx}{I\,t}\right)\int_{A'} y\, dA$$

$$dM/dx = V$$

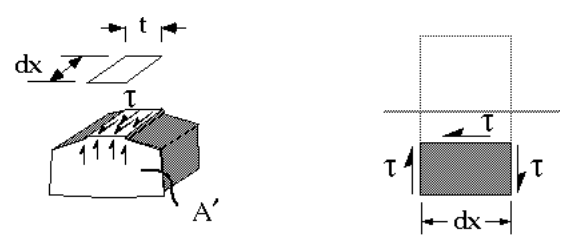
$$\int_{A'} y\, dA = Q$$

$$\boxed{\tau = \frac{V\,Q}{I\,t}}$$



Referring back to the element we have cut from the beam, we see that the shear stress we have just calculated is oriented longitudinally along the beam.

$$\tau = \frac{V\,Q}{I\,t}$$

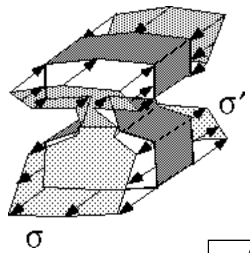


Our knowledge about the complementary nature of shear stresses tells us that shear stress must also exist on the other cut surfaces as shown above.

The vertical shear stresses shown on the front and back cuts of the element are the stresses which result in the internal shear force in the beam.

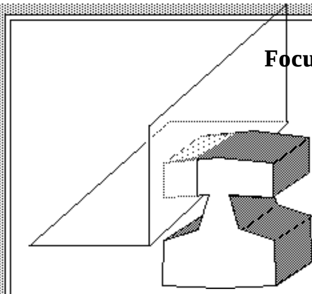
$$\tau = \frac{V\,Q}{I\,t}$$

Let's Take A Second Cut



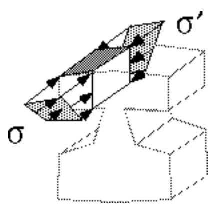
At the beginning of this stack we asserted that both horizontal and vertical shear stresses are present in a loaded beam. In order to confirm that the horizontal shear stresses exist we need to take a different cut of the beam element.

Focus on Upper-Left



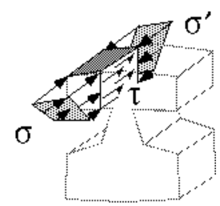
This time we take our cut of the section at the upper left corner.

Normal Stress Varies Along Length



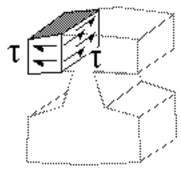
Again, we have different normal stress distributions on the front and back faces of the element.

Shear Stress Must Equate Change in Normal Stress



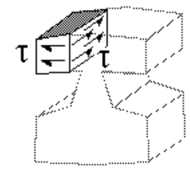
In order to satisfy horizontal force equilibrium, shear stress, τ, must exist on the side face of the free body diagram.

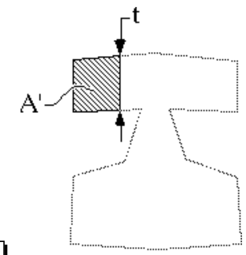
Complementary Component of Shear Stress



The complementary component of this shear stress is a horizontal shear stress distribution on the front face of the element. This is the horizontal shear stress we are looking for.

Complementary Component of Shear Stress

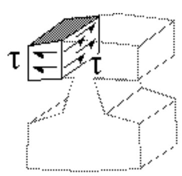


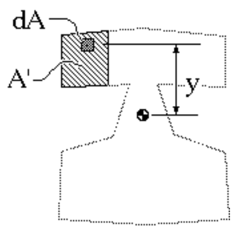


The value of the horizontal shear stress is calculated using the same equation as the vertical shear stress. Be careful when you use the shear stress equation that your value for t is consistent with the width of the cut you have made. In this case the width of the cut, t, is vertical !!

$$\tau = \frac{V\,Q}{I\,t}$$

Complementary Component of Shear Stress

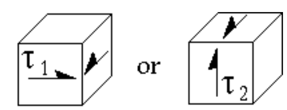


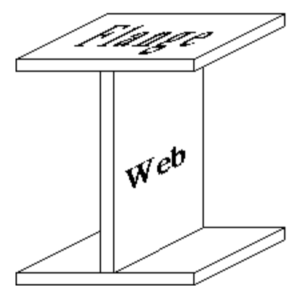


Also be careful to note that even though we have made a vertical cut, we still calculate Q using y as shown. Remember, our expression for τ was the result of equilibrating bending stresses about the x axis with the longitudinal shear stresses.

$$\boxed{Q = \int_{A'} y\,dA} \qquad \tau = \frac{V\,Q}{I\,t}$$

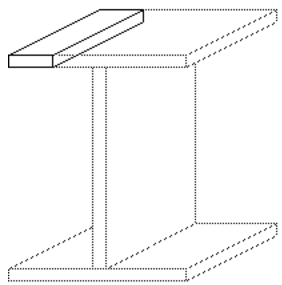
Which Dominates ?

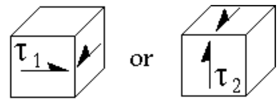




We have stated that both horizontal and vertical shear stresses are present when we bend a beam. Are both of these stresses always present? Are they equal in magnitude? If not, which component dominates?

Take a Cut





It turns out the question of which shear stress component dominates depends on the geometry of the cross section and the exact location in the cross section.

Let's begin by looking at a point in the flange of the I-beam shown

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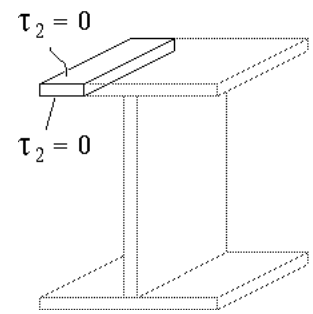
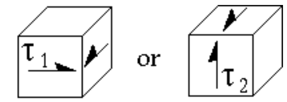
Shear Stress is Zero on Free Surface





$\boxed{\tau_2 \approx 0}$

The longitudinal shear stress must be zero on the top and bottom of this free body since these are free surfaces. From this we can conclude that the vertical shear stress is zero at the top and bottom of the front face. We can argue that the since the distance between the top and bottom of this element is very small, the vertical shear stress on this cut will never be much more

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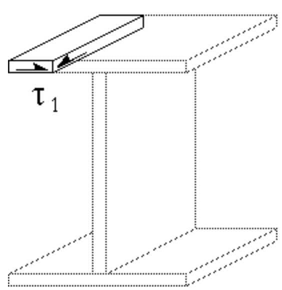
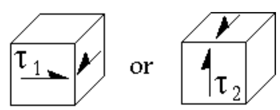
Non-zero Shear Stress





$\boxed{\tau_1 \neq 0}$ $\tau_2 \approx 0$

If we look at the longitudinal shear stress acting on the side of the element, we see that it can have a non-zero value. We conclude that the complementary shear, the horizontal shear stress on the front face, may also be non-zero. At this point in the cross section the horizontal shear stress dominates.

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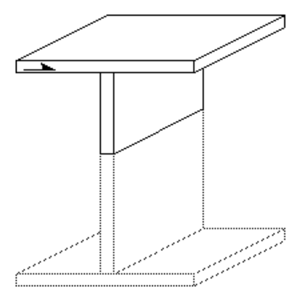
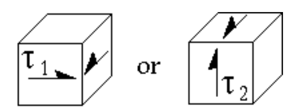
A Second Cut





Let's now look at a point in the web of the I-beam.

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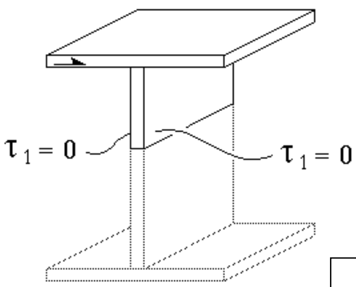
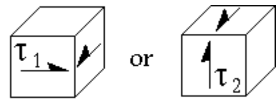
No Shear Stress on Free Surface





$\boxed{\tau_1 \approx 0}$

Using an argument similar to that for vertical shears in the flange, we can conclude that the horizontal shear stress in the web is never much greater than zero.

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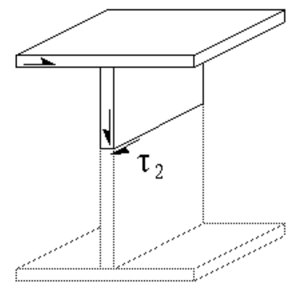
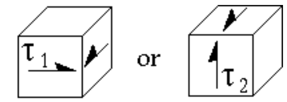
Non-zero Shear Stress





$\tau_1 \approx 0$ $\boxed{\tau_2 \neq 0}$

On the other hand, the vertical shear stress in the web can have non-zero values. This follows from the face that the longitudinal shear stress on bottom of the cut can be non-zero.

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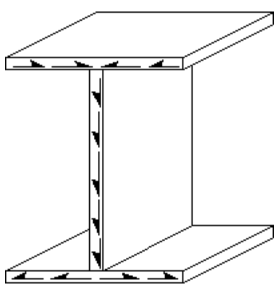
General Direction of Shear Stress in X-Section



If we calculated the dominant shear stress component for the entire cross section, we would find the shear stresses to be oriented as shown in this figure.

Notice that the shear stresses in the flanges are symmetric and result in no net horizontal force. The vertical shear stresses in the flange result in the internal shear force in the section.

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Summary

$$\tau = \frac{VQ}{It}$$

Where:

V = the internal shear force in the beam

Q = the first moment of area above or below the cut

$$= \int_{A'} y \, dA$$

I = the second moment of area (moment of inertia) for the <u>entire</u> section.

t = the width of the cut, in the direction of the cut

In summary, the shear stress in a beam can be calculated as VQ/It.

Common mistakes include using a horizontal measurement for t when taking a vertical cut and miscalculating Q. A clear picture indicating the location of the neutral axis, the area A', and the width of the cut, t, will help prevent such mistakes.

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