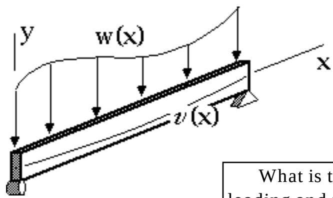
Governing Equations for Beams



What is the relationship between a beam's loading and it's deflection?

In this stack we will derive this relationship, and also establish relationships between the load, shear and moment at any point along the beam.

This stack concludes with a brief outline of two methods used to solve for deflections in beams.

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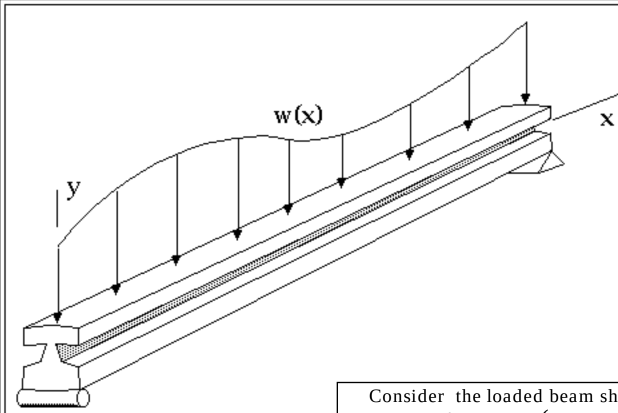

Consider the loaded beam shown above. The function w(x) denotes the value of the distributed load at any point x.

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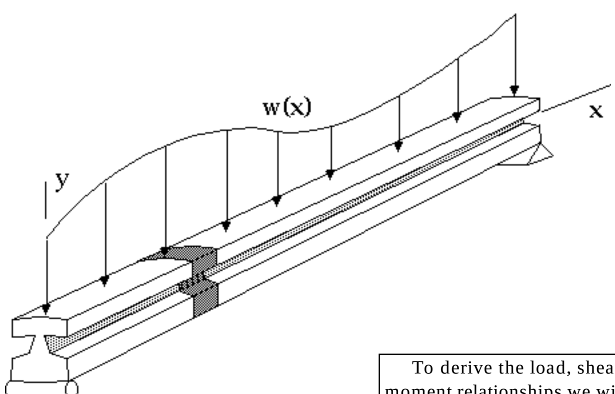

To derive the load, shear, moment relationships we will focus on a short length of the beam taken from anywhere along the beam.

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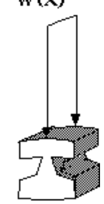
A Section of the Beam

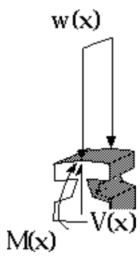


Here is that short element of the beam shown with the portion of the load that is acting on it.

Does this figure appear to satisfy static equilibrium?

4 Hide Text

Internal Forces on the Section



In order to keep the short length of beam in equilibrium, we must include the internal forces acting on the cut of the beam.

For beam bending we will ignore the axial force acting on this cut, and focus on the internal shear and moment.

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Something on sign convention for internal moments and shears!

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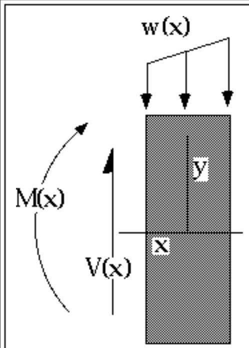
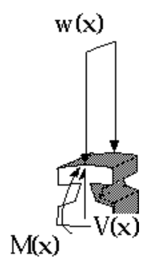
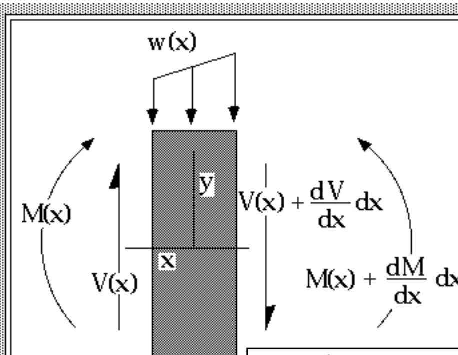




Taking a closer look at the short length of the beam, we now define a local coordinate system for the beam. The x coordinate starts at one end of the beam (usually the left end) and runs along the "neutral axis" of the beam. The y coordinate measures the perpendicular distance from the neutral axis.

7 Hide Text Neutral Axis ?



Hopefully the picture on the previous card appeared incomplete. This figure shows the internal forces which act on the right face of the cut.

In general the shear and moment on the right face will not be the same as that on the left face. The shear will have changed by some amount equal to the change in shear with respect to x (dV/dx) times the length of the element,

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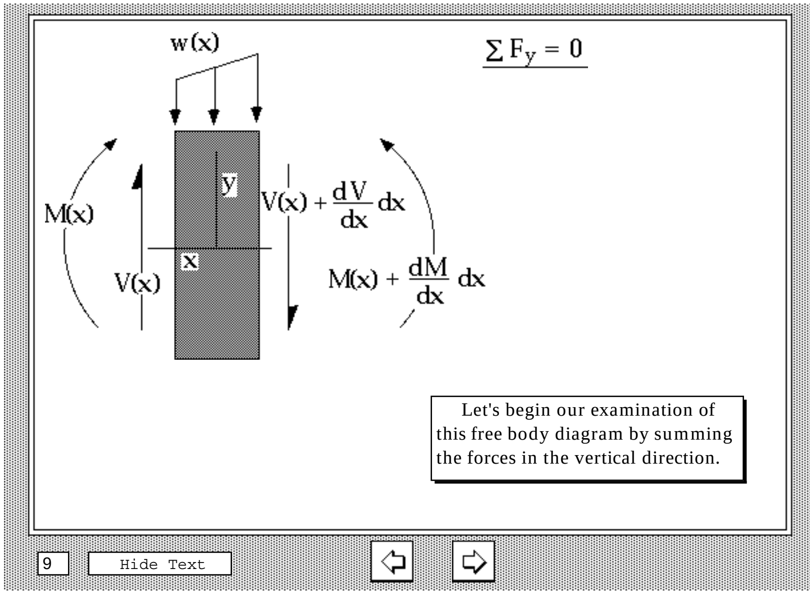
w(x)
$\Sigma F_y = 0$
M(x)
V(x)
y
x
$V(x) + \frac{dV}{dx} dx$
$M(x) + \frac{dM}{dx} dx$
Let's begin our examination of this free body diagram by summing the forces in the vertical direction.
9
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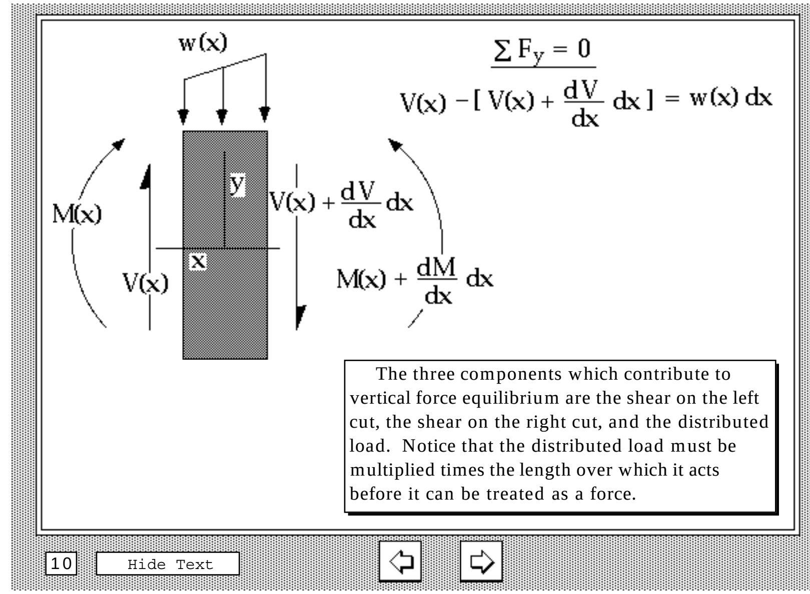
w(x)
$\Sigma F_y = 0$
$V(x) - [\, V(x) + \frac{dV}{dx} dx \,] = w(x)\, dx$
M(x)
V(x)
y
x
$V(x) + \frac{dV}{dx} dx$
$M(x) + \frac{dM}{dx} dx$
The three components which contribute to vertical force equilibrium are the shear on the left cut, the shear on the right cut, and the distributed load. Notice that the distributed load must be multiplied times the length over which it acts before it can be treated as a force.
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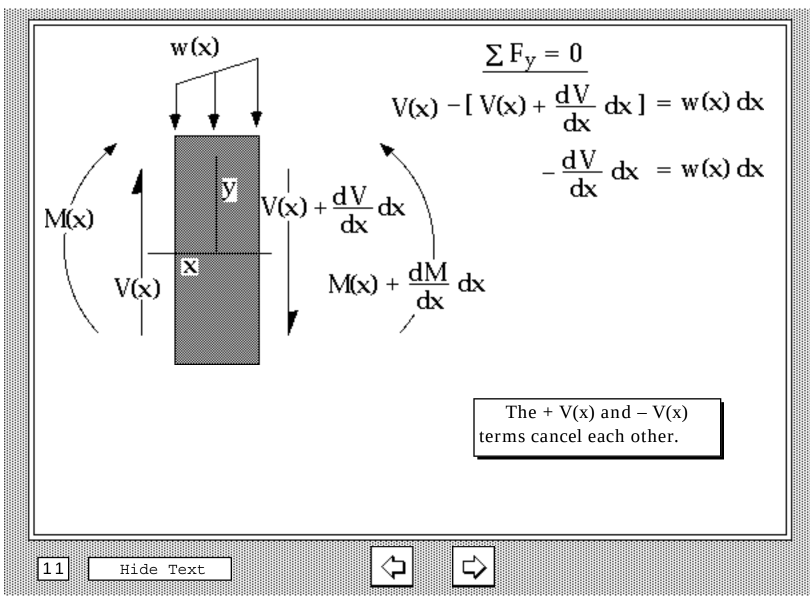
w(x)
$\Sigma F_y = 0$
$V(x) - [\, V(x) + \frac{dV}{dx} dx \,] = w(x)\, dx$
$- \frac{dV}{dx} dx = w(x)\, dx$
M(x)
V(x)
y
x
$V(x) + \frac{dV}{dx} dx$
$M(x) + \frac{dM}{dx} dx$
The + V(x) and – V(x) terms cancel each other.
11
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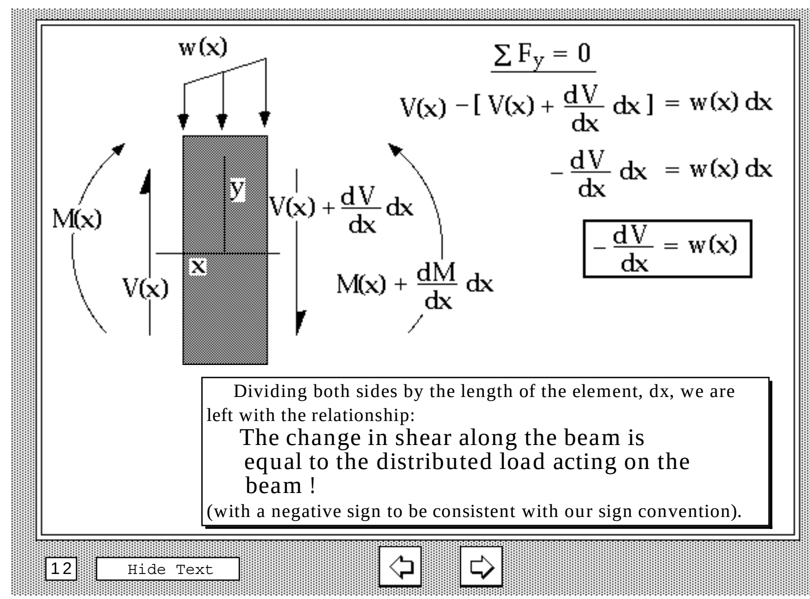
w(x)
$\Sigma F_y = 0$
$V(x) - [\, V(x) + \frac{dV}{dx} dx \,] = w(x)\, dx$
$- \frac{dV}{dx} dx = w(x)\, dx$
$- \frac{dV}{dx} = w(x)$
M(x)
V(x)
y
x
$V(x) + \frac{dV}{dx} dx$
$M(x) + \frac{dM}{dx} dx$
Dividing both sides by the length of the element, dx, we are left with the relationship:
The change in shear along the beam is equal to the distributed load acting on the beam !
(with a negative sign to be consistent with our sign convention).
12
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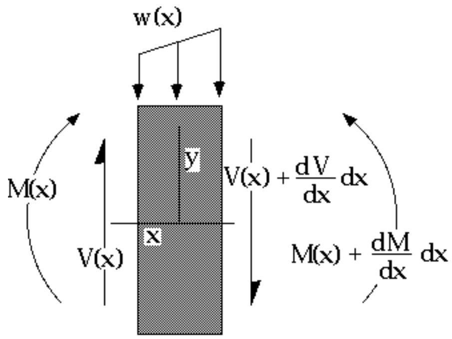
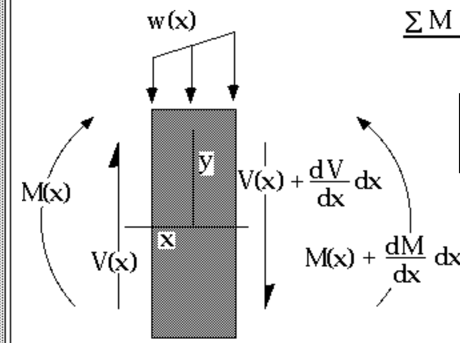


This is an important result, so let's store it at the bottom of the screen.

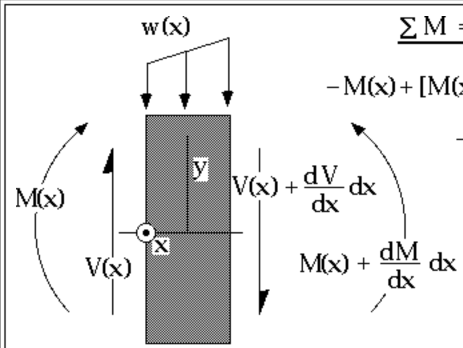
$$-\frac{dV}{dx} = w(x)$$



$$\underline{\sum M = 0}$$

We continue our quest by applying moment equilibrium to the free body diagram.

$$-\frac{dV}{dx} = w(x)$$

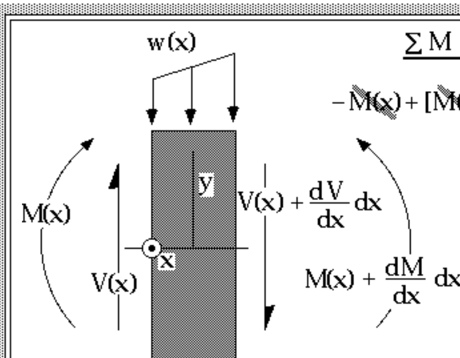


$$\underline{\sum M = 0}$$

$$-M(x) + [M(x) + \frac{dM}{dx}\,dx] - [\,V(x) + \frac{dV}{dx}\,dx\,]\,dx - w(x)\,dx\,(dx/2) = 0$$

We sum the moments about a point located on the neutral axis at the left cut. See if you can determine where each component in the equilibrium equation above has come from.

$$-\frac{dV}{dx} = w(x)$$

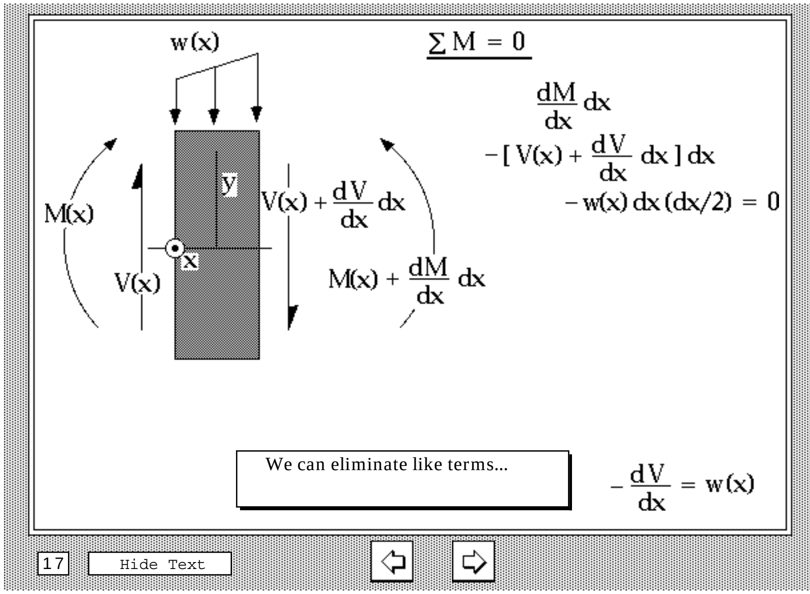


$$\underline{\sum M = 0}$$

$$-M(x) + [M(x) + \frac{dM}{dx}\,dx] - [\,V(x) + \frac{dV}{dx}\,dx\,]\,dx - w(x)\,dx\,(dx/2) = 0$$

We can eliminate like terms...

$$-\frac{dV}{dx} = w(x)$$


w(x)
∑ M = 0
dM/dx dx
− [V(x) + dV/dx dx] dx
− w(x) dx (dx/2) = 0
M(x)
V(x)
y
x
V(x) + dV/dx dx
M(x) + dM/dx dx
We can eliminate like terms...
− dV/dx = w(x)
17
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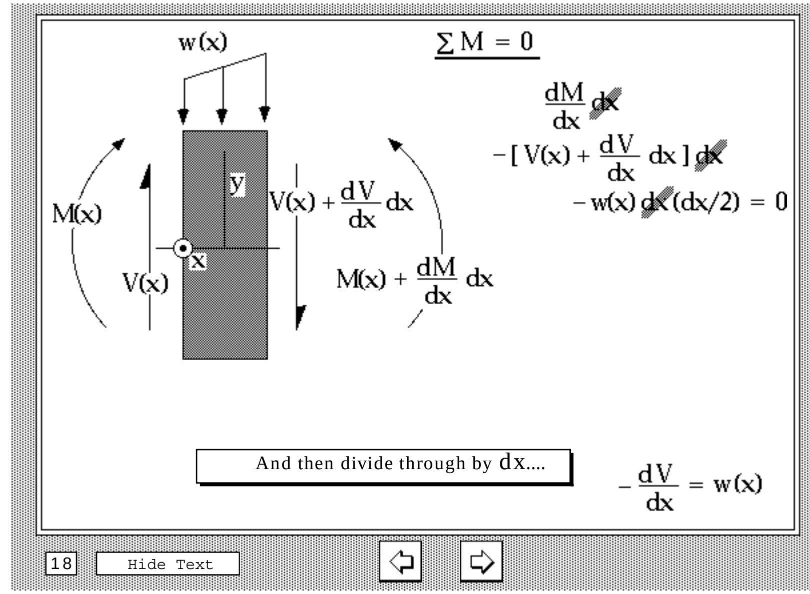
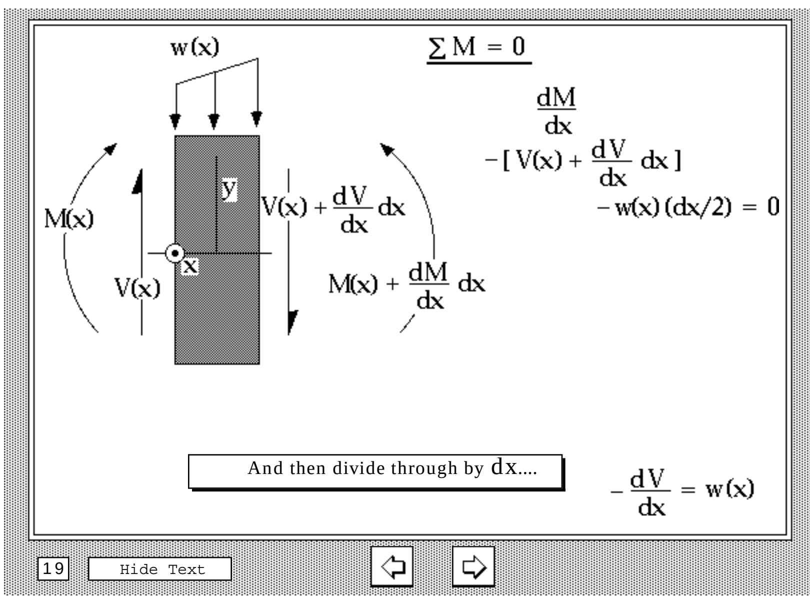

w(x)
∑ M = 0
dM/dx dx
− [V(x) + dV/dx dx] dx
− w(x) dx (dx/2) = 0
M(x)
V(x)
y
x
V(x) + dV/dx dx
M(x) + dM/dx dx
And then divide through by dx....
− dV/dx = w(x)
18
Hide Text


w(x)
∑ M = 0
dM/dx
− [V(x) + dV/dx dx]
− w(x) (dx/2) = 0
M(x)
V(x)
y
x
V(x) + dV/dx dx
M(x) + dM/dx dx
And then divide through by dx....
− dV/dx = w(x)
19
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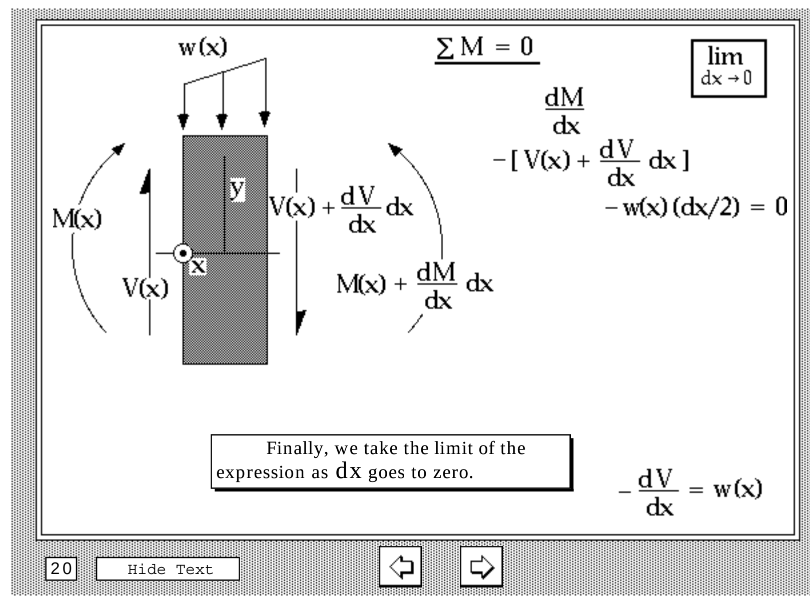
w(x)
∑ M = 0
lim dx → 0
dM/dx
− [V(x) + dV/dx dx]
− w(x) (dx/2) = 0
M(x)
V(x)
y
x
V(x) + dV/dx dx
M(x) + dM/dx dx
Finally, we take the limit of the expression as dx goes to zero.
− dV/dx = w(x)
20
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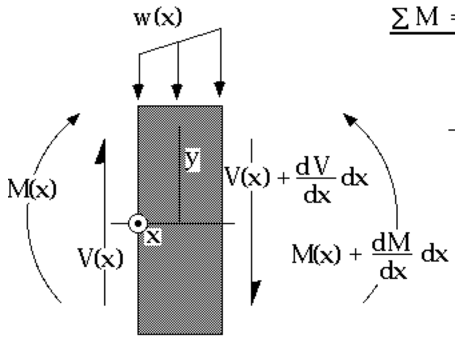

$\sum M = 0$

$\lim_{dx \to 0}$

$$\frac{dM}{dx} - \left[V(x) + \frac{dV}{dx} dx \right] - w(x)(dx/2) = 0$$

Finally, we take the limit of the expression as dx goes to zero.

$$-\frac{dV}{dx} = w(x)$$

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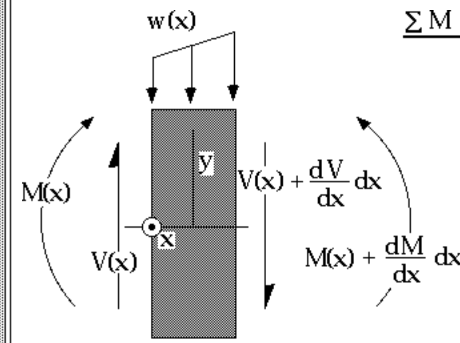


$\sum M = 0$

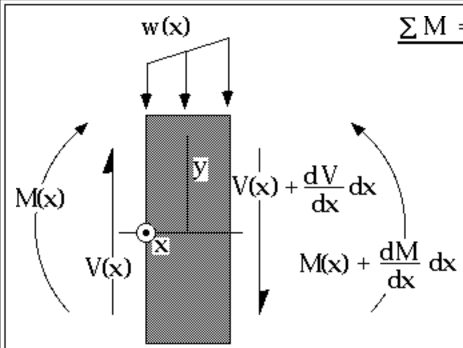
$\lim_{dx \to 0}$

$$\frac{dM}{dx} - V(x) = 0$$

And we are left with a relationship between shear force and change in moment.

$$-\frac{dV}{dx} = w(x)$$

22 Hide Text



$\sum M = 0$

$$\frac{dM}{dx} = V(x)$$

Formally, we say that the shear in a beam is equal to the change in moment along the length of the beam.

$$-\frac{dV}{dx} = w(x)$$

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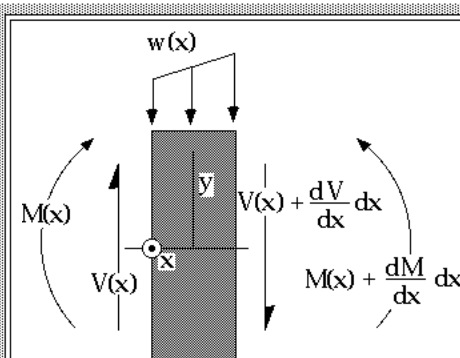

These are two important relationships between shear, moment, and external load. We will refer back to them often in our discussion of stresses in beams.

We arrived at these relationships by applying equilibrium to the free body diagram shown above.

$$\frac{dM}{dx} = V(x)$$

$$-\frac{dV}{dx} = w(x)$$

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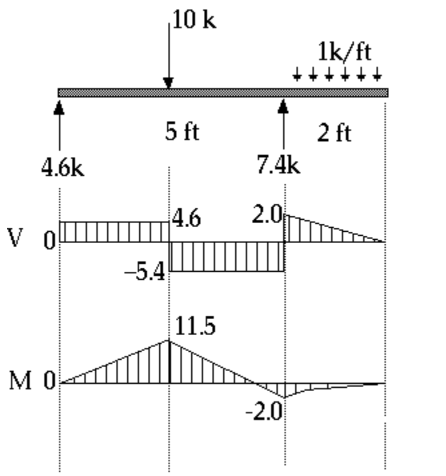

Here are the shear and moment diagrams from a previous example. Confirm for yourself that the change in the shear equals the loading, and that the change in the moment equals the shear.

$$\frac{dM}{dx} = V(x)$$

$$-\frac{dV}{dx} = w(x)$$

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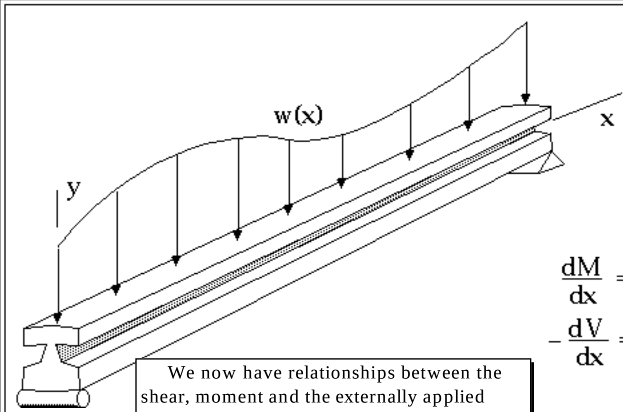

$$\frac{dM}{dx} = V(x)$$

$$-\frac{dV}{dx} = w(x)$$

We now have relationships between the shear, moment and the externally applied load. However, we are not yet done. Remember, that our goal was to establish a link between externally applied load and the displaced shape of the beam.

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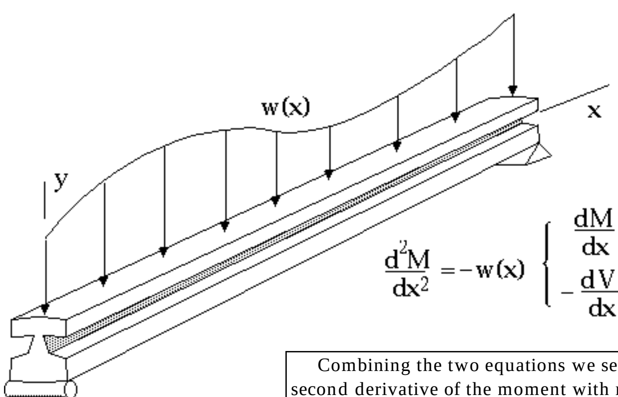

$$\frac{d^2M}{dx^2} = -w(x) \quad \begin{cases} \dfrac{dM}{dx} = V(x) \\ -\dfrac{dV}{dx} = w(x) \end{cases}$$

Combining the two equations we see that the second derivative of the moment with respect to x equals minus the externally applied load.

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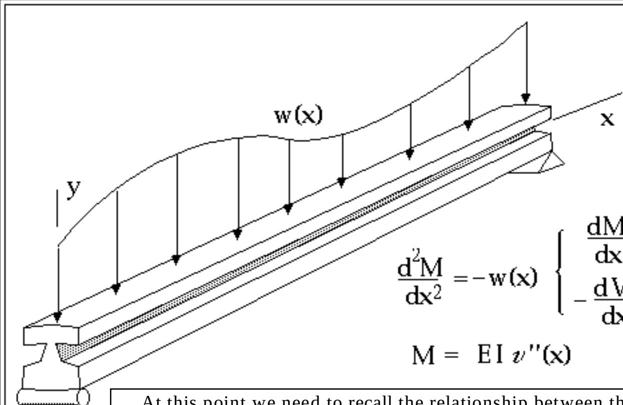


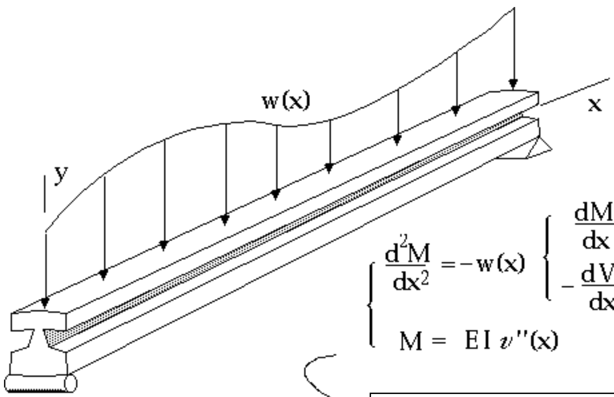
$$\frac{d^2M}{dx^2} = -w(x) \quad \begin{cases} \dfrac{dM}{dx} = V(x) \\ -\dfrac{dV}{dx} = w(x) \end{cases}$$

$$M = EI\,v''(x)$$

At this point we need to recall the relationship between the moment in a beam and the displaced shape, $\upsilon(x)$.

If you don't remember how we arrived at this relationship, you may wish to review this information by clicking the M = EI v" button.

28 Hide Text M = EI v"



$$\frac{d^2M}{dx^2} = -w(x) \quad \left\{ \begin{array}{l} \frac{dM}{dx} = V(x) \\ -\frac{dV}{dx} = w(x) \end{array} \right.$$

$$M = EI\,v''(x)$$

$$\frac{d^2}{dx^2}[EI\,v''(x)] = -w(x)$$

We can combine these two equations to arrive at the coveted load-displacement relationship for beams.

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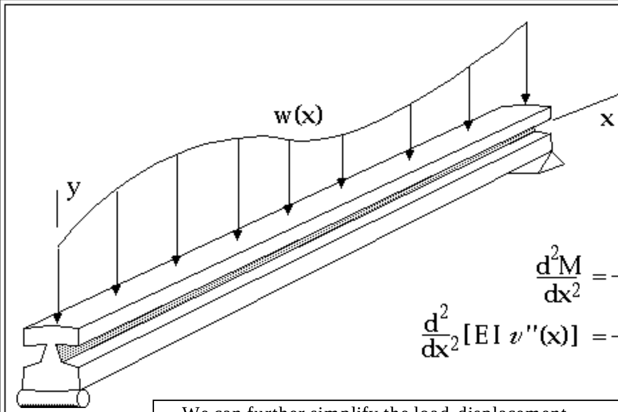

$$\frac{d^2M}{dx^2} = -w(x)$$

$$\frac{d^2}{dx^2}[EI\,v''(x)] = -w(x)$$

We can further simplify the load-displacement relationship if we know that E and I are constant along the length of the beam.

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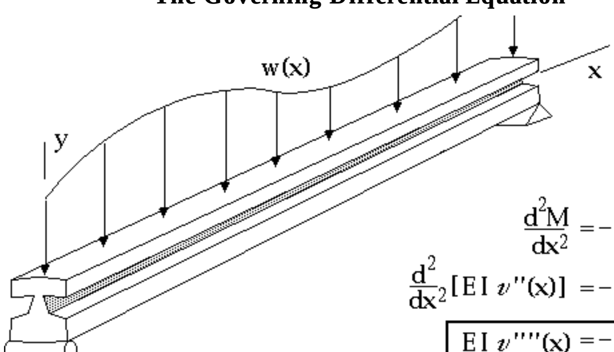
The Governing Differential Equation



$$\frac{d^2M}{dx^2} = -w(x)$$

$$\frac{d^2}{dx^2}[EI\,v''(x)] = -w(x)$$

$$EI\,v''''(x) = -w(x) \quad \text{for } EI = \text{const}$$

This last equation is the one most commonly referred to as the governing equation for beams. Never use it blindly. Always confirm that I for the beam is constant.

31 Hide Text

The Whole Story

$EI\,v''''(x) = -w(x)$	Distributed Load
$EI\,v'''(x) = V(x)$	Shear
$EI\,v''(x) = M(x)$	Moment
$v'(x) =$	Slope
$v(x) =$	Deflection

The governing equation for beam deflections, shown at the top, is a fourth order differential equation. The four integrations needed to calculate the deflections of the beam are shown below the governing equation. Note the result of each integration is related to a particular property of the beam's internal loading or shape. Refer back to this figure if you are unsure at what step the beam equation must satisfy a certain boundary condition.

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Determining Displaced Shapes: 1

1. Determine M(x) $$\frac{d^2M}{dx^2} = -w(x)$$

2. Integrate twice to determine $v(x)$
   (Two constants of integration/boundary conditions)

There are two ways we can use the previously derived relationships to calculate a beams displaced shape from its loading. The first method is outlined here.

The procedure begins by determining the function which defines moment in the beam as a function of position, M(x). To do this, use your favorite (or the easiest) method to calculate the moment diagram for the beam. Note that (1) this method is only appropriate for statically determinate beams, and (2) if you have point loads on the beam, the function M(x) will have kinks.

Once you establish M(x), integrate the function twice. Don't forget the integration constants that come with each indefinite integration!

Finally apply the two displacement boundary conditions for the beam. For

Determining Displaced Shapes: 2

$$EI\, v''''(x) = -w(x)$$

1. Integrate four times
   (Four constants of integration/boundary conditions)

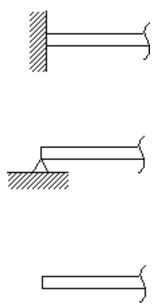

= 2 conditions × 2 ends

The second method for determining beam deflections involves integrating the "beam equation" four times. The four integrations will result in four unknown integration constants. To solve for the constants, apply the four boundary conditions for the beam.

The four integration constants come from the fact that every beam has two ends, and each end has two boundary conditions.

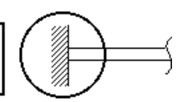
Boundary Conditions

It was just stated that we can read two boundary conditions from any end of a beam. Shown at the right are the three common end conditions for beams; fixed, simple,

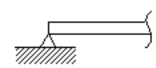


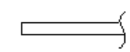
Boundary Conditions

$$v = 0$$
$$v' = 0$$



At a fixed support we know that the deflection of the beam is zero and the slope of the beam is zero.

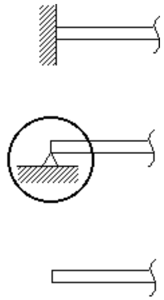




Boundary Conditions

$v = 0$
$v' = 0$

$v = 0$
$EI\,v'' = M = 0$



At a simple support, the deflection of the beam is zero and the moment in the beam is zero.

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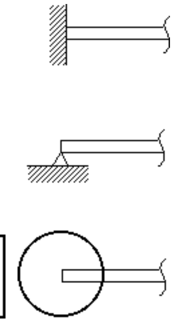
Boundary Conditions

$v = 0$
$v' = 0$

$v = 0$
$EI\,v'' = M = 0$

$EI\,v'' = M = 0$
$EI\,v''' = V = 0$



At a free end, the moment in the beam is zero, and the shear in the beam is zero.

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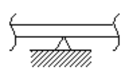
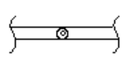
Other Supports/Connections



$v = 0$



$EI\,v'' = M = 0$

Other supports which you might encounter when analyzing a beam are the pin support in the middle of a beam, and the internal hinge.

At an internal pin support the deflection must be zero. At an internal hinge, the moment must be zero.

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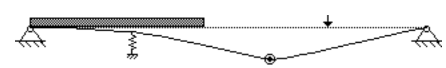
Summary

The key to determining displacements in beams lies in integrating simple differential equations: the tedious part is handling boundary conditions and non-continuous loads. In subsequent courses you will learn many methods for finding displacements more conveniently, but you already have the complete theory under your belt: EIv'''' = w(x).



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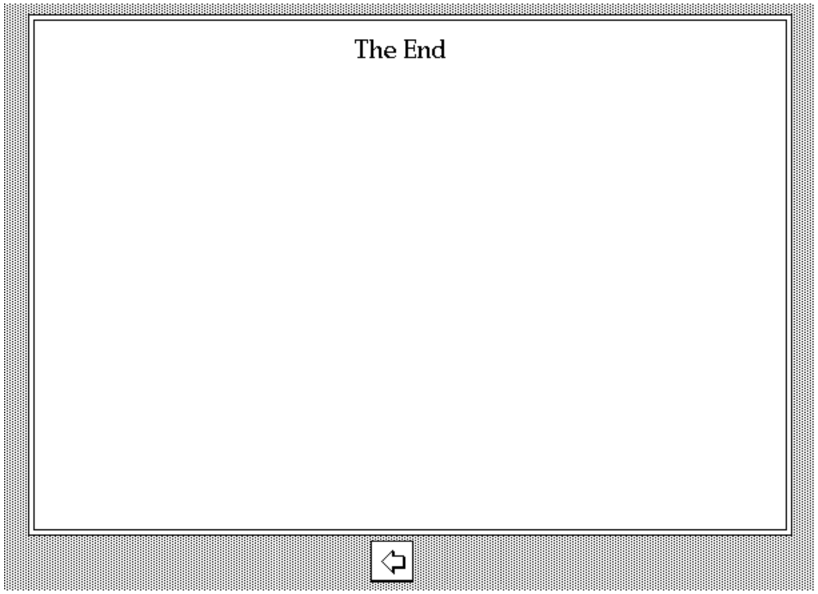
The End