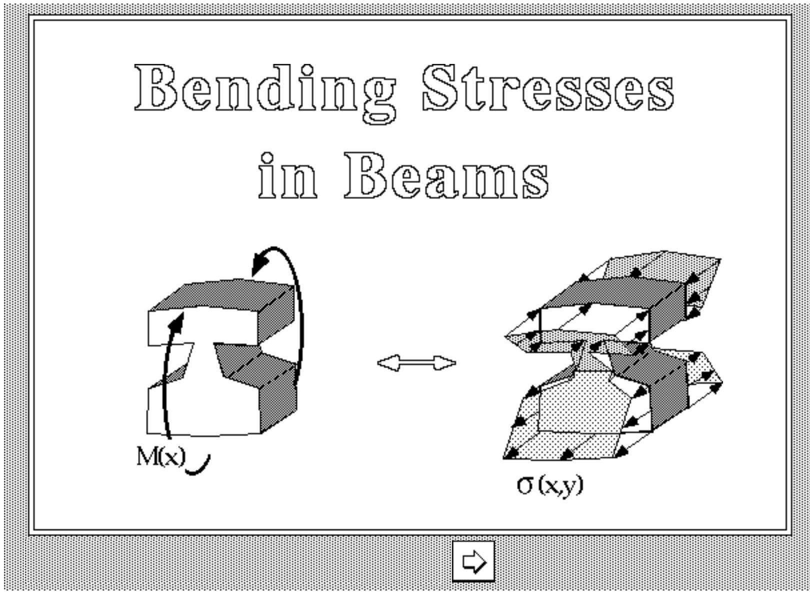
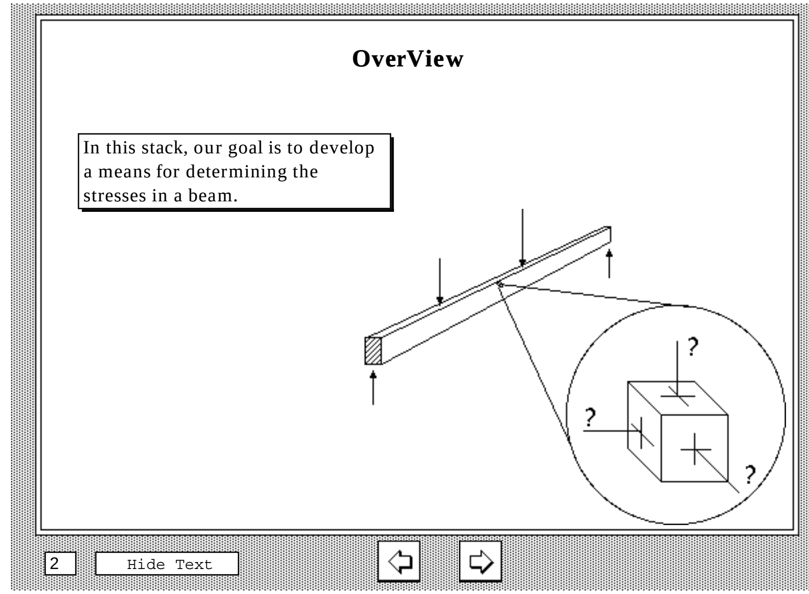

Bending Stresses
in Beams
M(x)
σ(x,y)


OverView
In this stack, our goal is to develop a means for determining the stresses in a beam.
?
?
?
2
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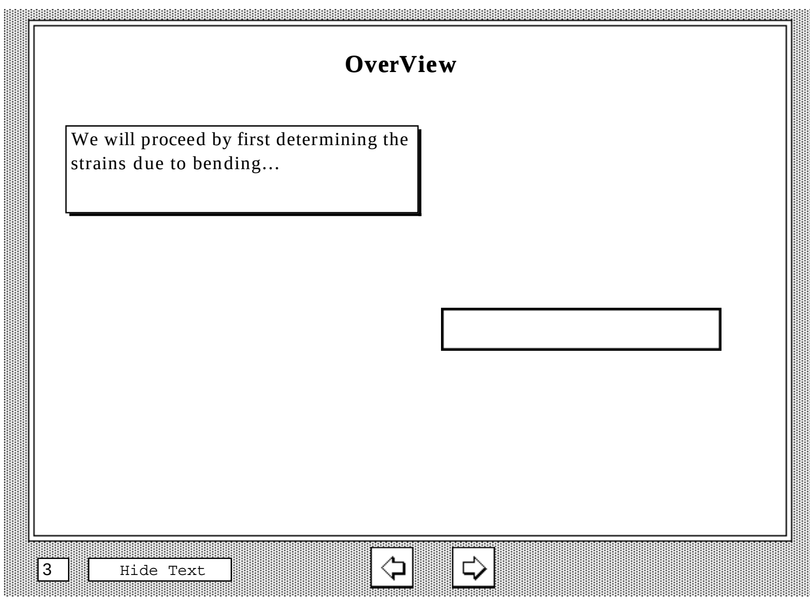
OverView
We will proceed by first determining the strains due to bending...
3
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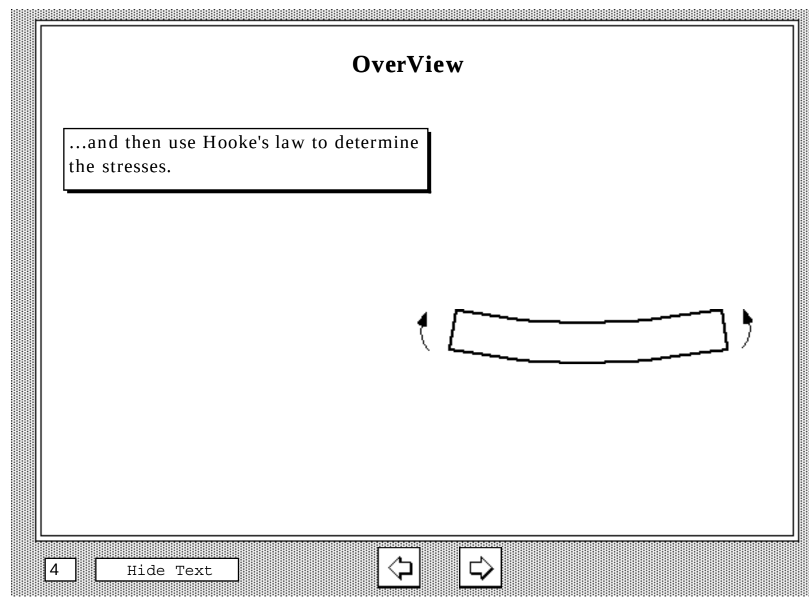
OverView
...and then use Hooke's law to determine the stresses.
4
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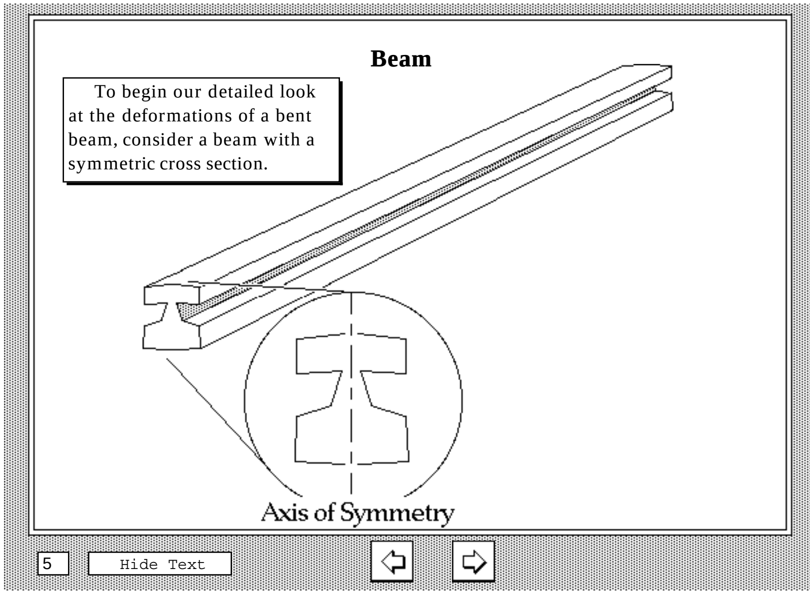
Beam
To begin our detailed look at the deformations of a bent beam, consider a beam with a symmetric cross section.
Axis of Symmetry
5
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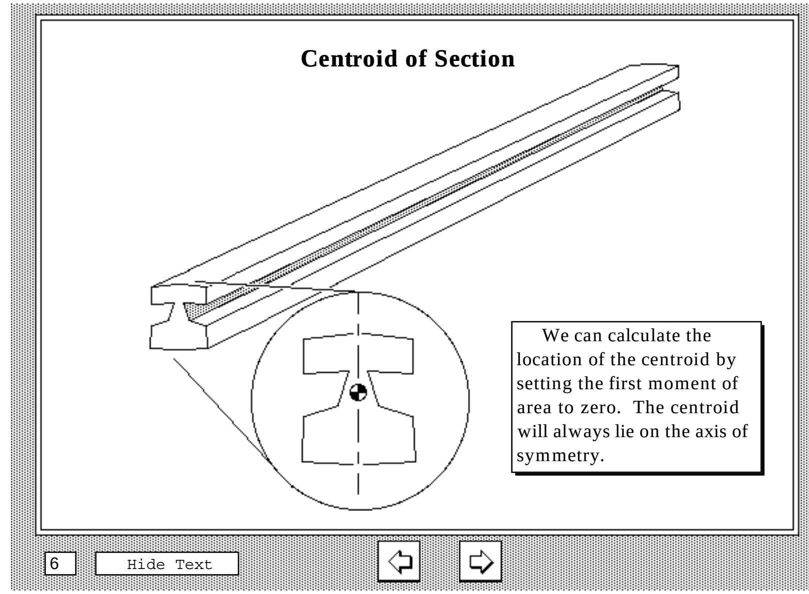
Centroid of Section
We can calculate the location of the centroid by setting the first moment of area to zero. The centroid will always lie on the axis of symmetry.
6
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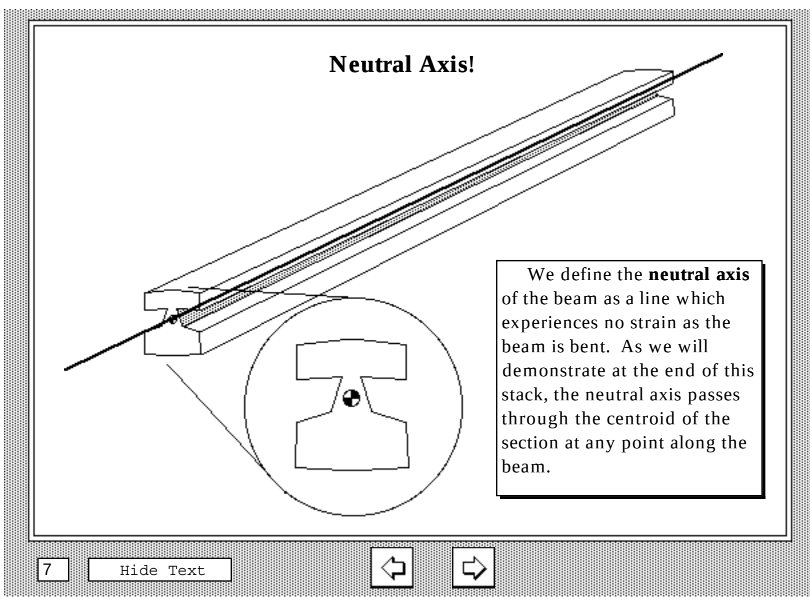
Neutral Axis!
We define the **neutral axis** of the beam as a line which experiences no strain as the beam is bent. As we will demonstrate at the end of this stack, the neutral axis passes through the centroid of the section at any point along the beam.
7
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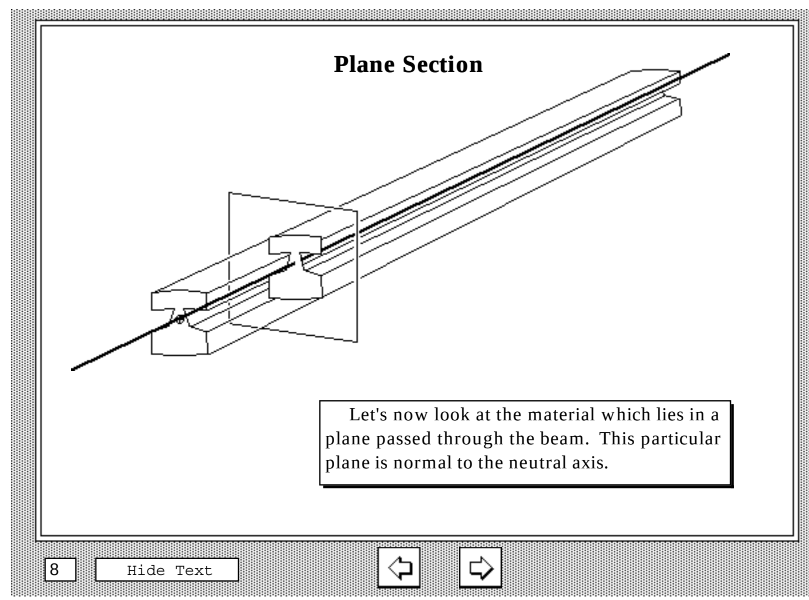
Plane Section
Let's now look at the material which lies in a plane passed through the beam. This particular plane is normal to the neutral axis.
8
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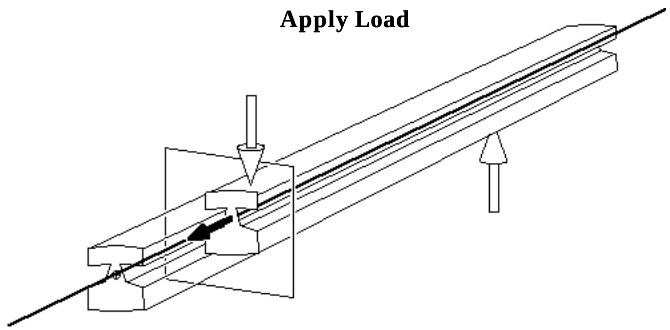
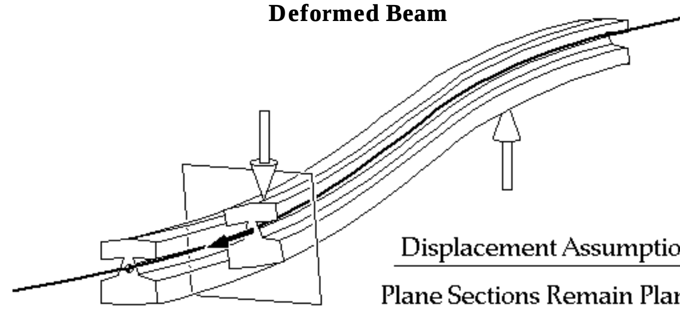
Apply Load



We now load the beam and allow it to deflect.

9 Hide Text

Deformed Beam



Displacement Assumption

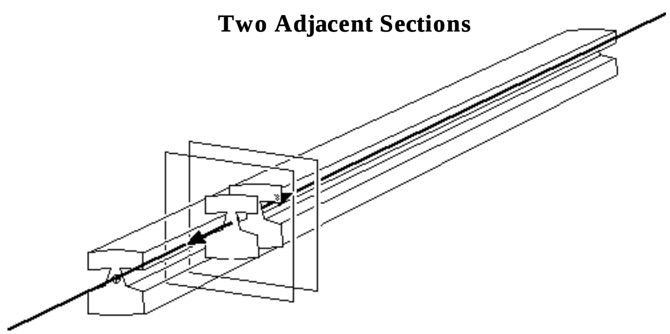
Plane Sections Remain Plane and Normal to the Neutral Axis

After deformations, we observe that the plane section we were viewing remains plane, and further, it remains normal to the neutral axis.

This observation is the fundamental assumption in the derivation of the beam bending equations. Note that we will not observe this same behavior for very large deformations.

10 Hide Text

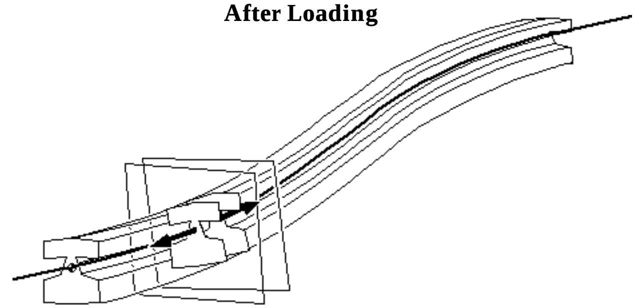
Two Adjacent Sections



Let's take another look at the deformation of the loaded beam, only this time we will look at the material which lies between two adjacent planes.

11 Hide Text

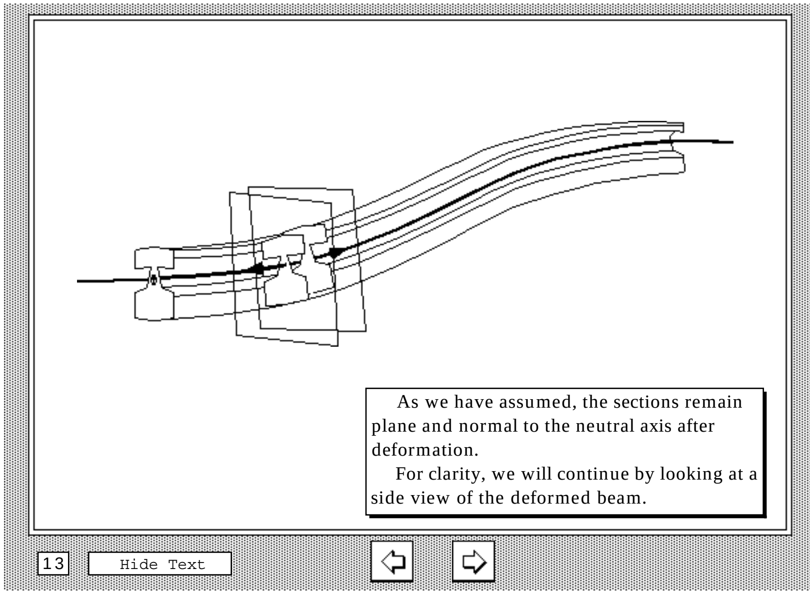
After Loading



As we have observed before, the sections remain plane and normal to the neutral axis after deformation.

For clarity, we will continue by looking at a side view of the deformed beam.

12 Hide Text


As we have assumed, the sections remain plane and normal to the neutral axis after deformation.
For clarity, we will continue by looking at a side view of the deformed beam.
13
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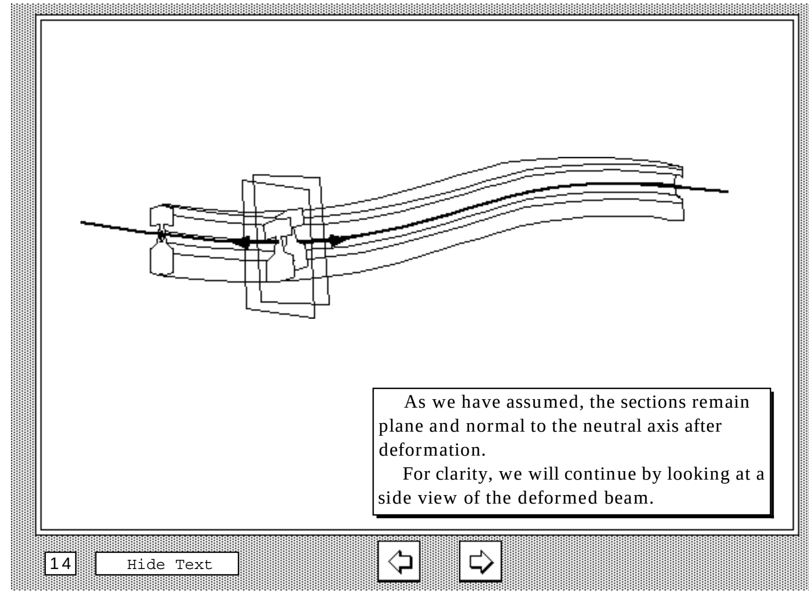
As we have assumed, the sections remain plane and normal to the neutral axis after deformation.
For clarity, we will continue by looking at a side view of the deformed beam.
14
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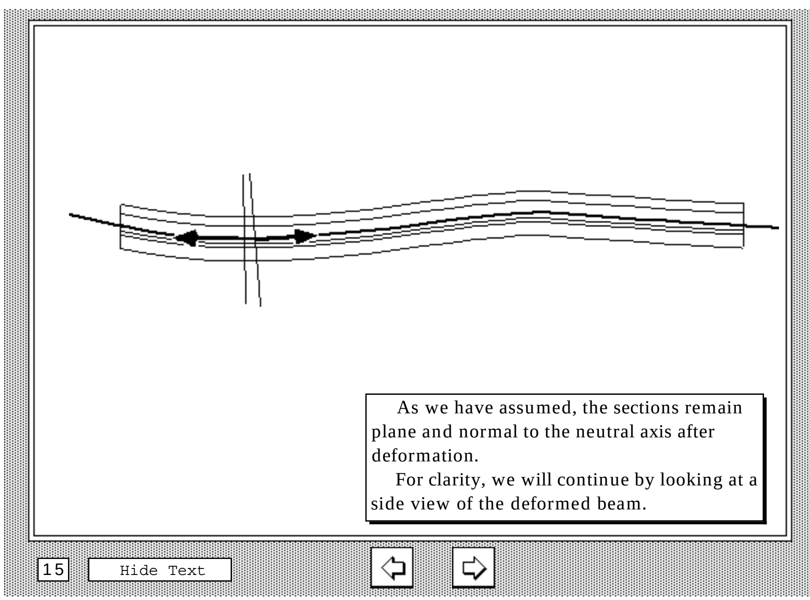
As we have assumed, the sections remain plane and normal to the neutral axis after deformation.
For clarity, we will continue by looking at a side view of the deformed beam.
15
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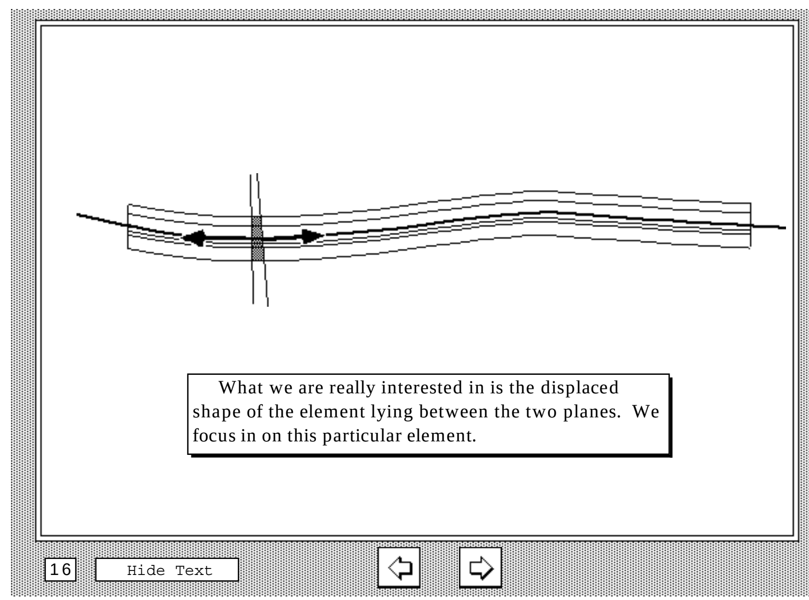
What we are really interested in is the displaced shape of the element lying between the two planes. We focus in on this particular element.
16
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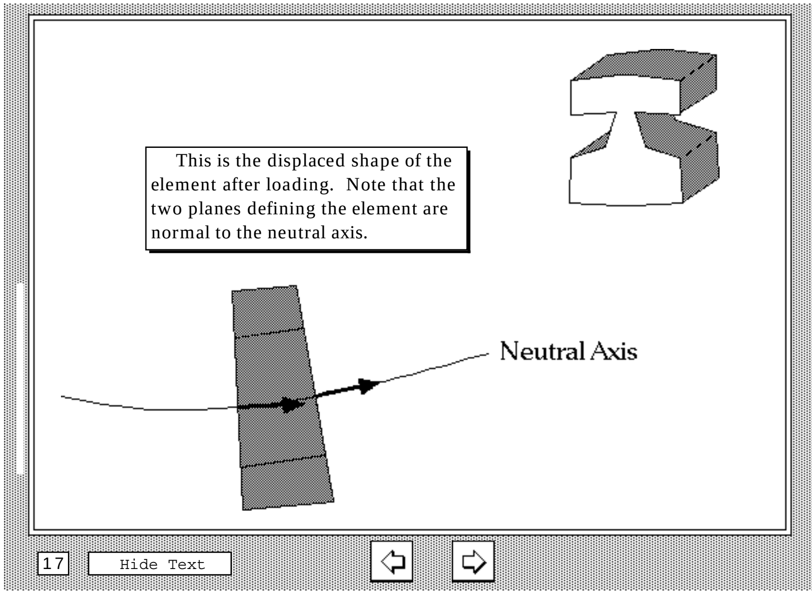
This is the displaced shape of the element after loading. Note that the two planes defining the element are normal to the neutral axis.
Neutral Axis
17
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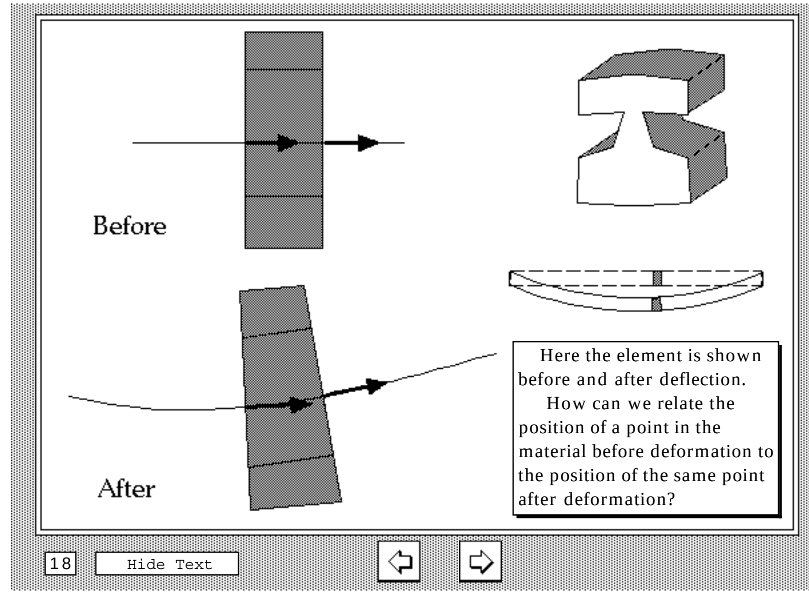
Before
After
Here the element is shown before and after deflection.
How can we relate the position of a point in the material before deformation to the position of the same point after deformation?
18
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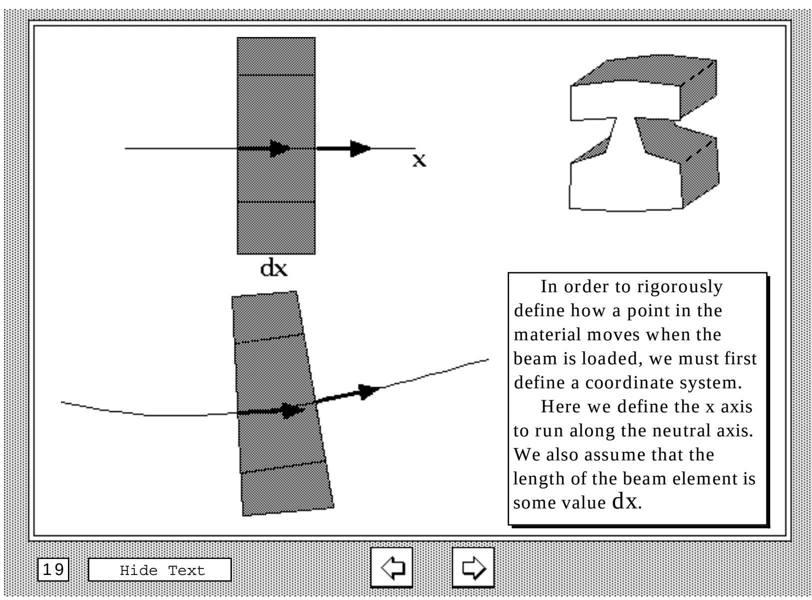
x
dx
In order to rigorously define how a point in the material moves when the beam is loaded, we must first define a coordinate system.
Here we define the x axis to run along the neutral axis. We also assume that the length of the beam element is some value dx.
19
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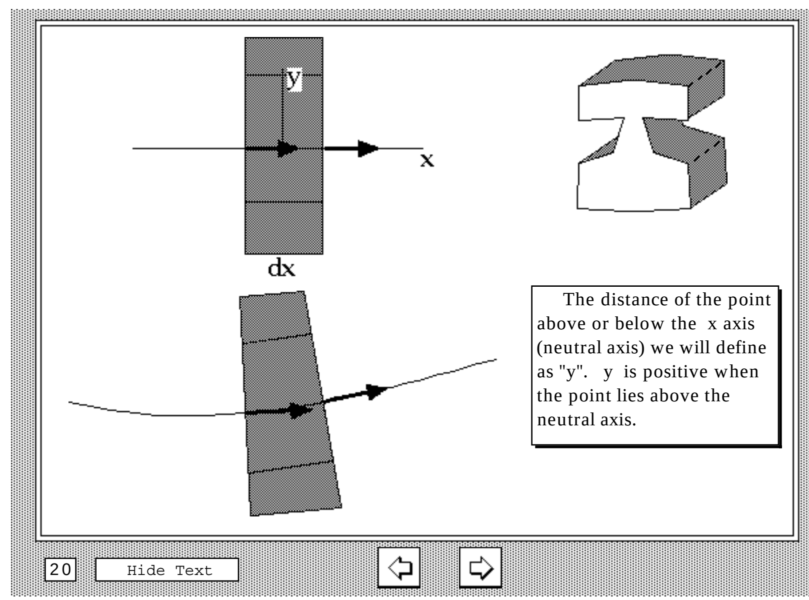
y
x
dx
The distance of the point above or below the x axis (neutral axis) we will define as "y". y is positive when the point lies above the neutral axis.
20
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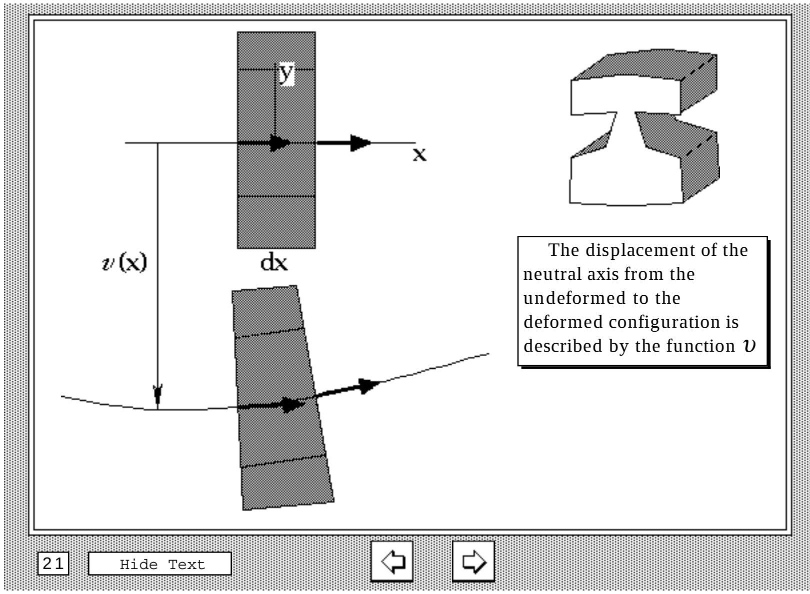
y
x
v(x)
dx
The displacement of the neutral axis from the undeformed to the deformed configuration is described by the function v
21
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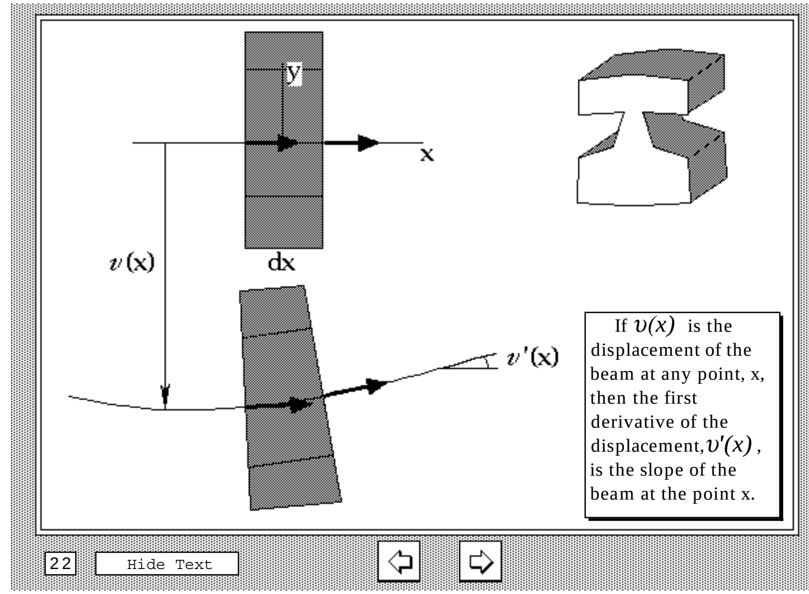
y
x
v(x)
dx
v'(x)
If v(x) is the displacement of the beam at any point, x, then the first derivative of the displacement, v'(x) , is the slope of the beam at the point x.
22
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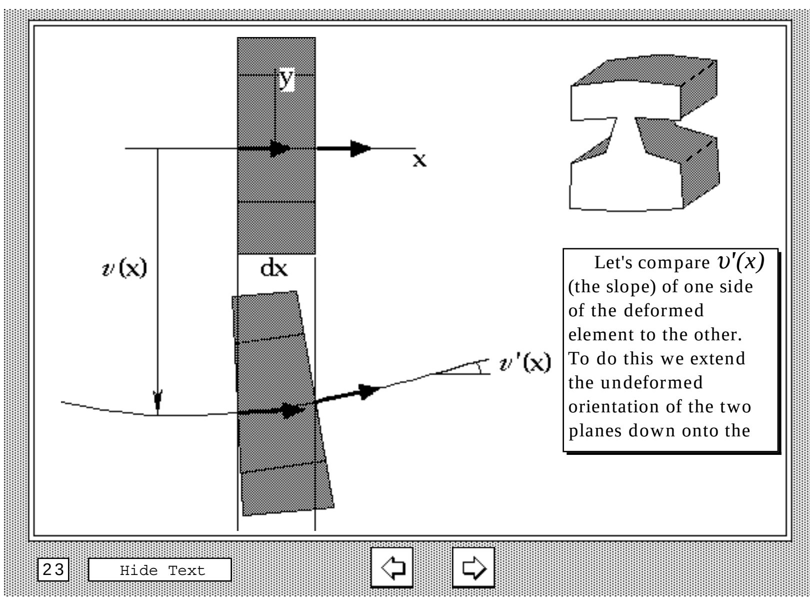
y
x
v(x)
dx
v'(x)
Let's compare v'(x) (the slope) of one side of the deformed element to the other. To do this we extend the undeformed orientation of the two planes down onto the
23
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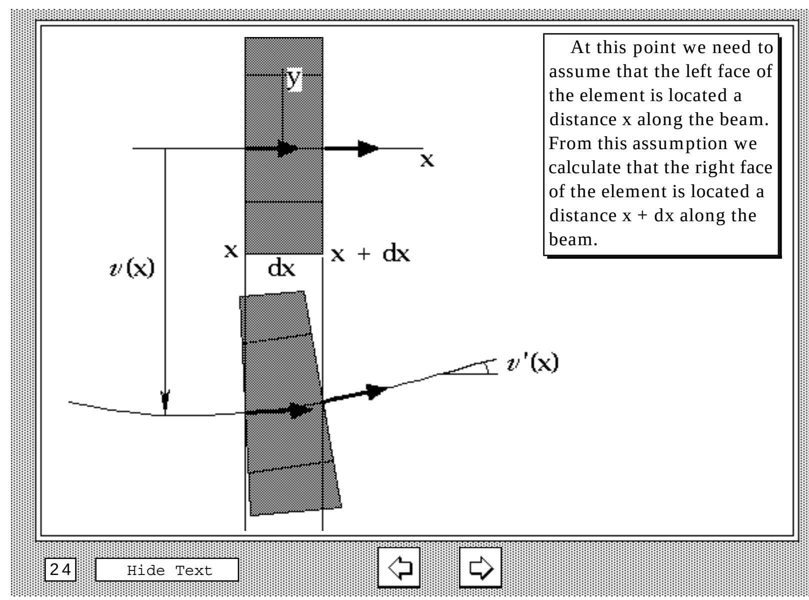
y
x
v(x)
x
dx
x + dx
v'(x)
At this point we need to assume that the left face of the element is located a distance x along the beam. From this assumption we calculate that the right face of the element is located a distance x + dx along the beam.
24
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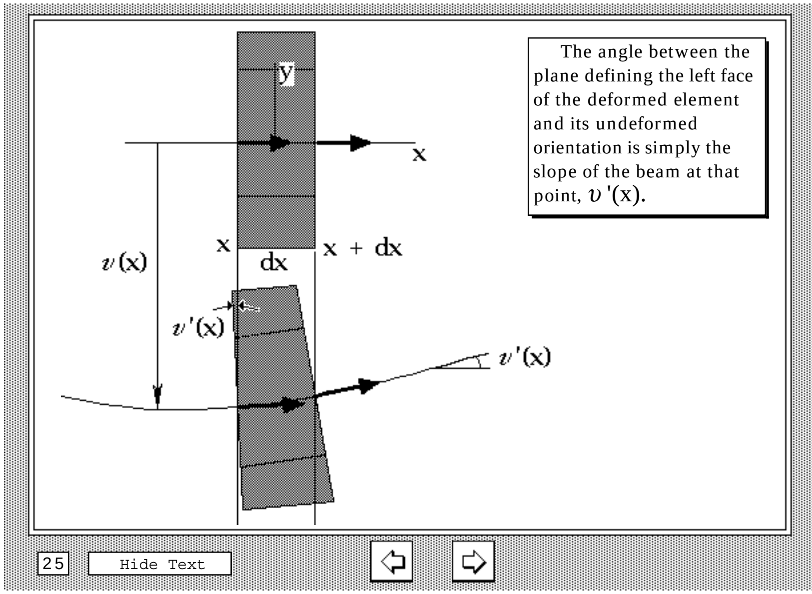
The angle between the plane defining the left face of the deformed element and its undeformed orientation is simply the slope of the beam at that point, v '(x).
y
x
v(x)
x
dx
x + dx
v'(x)
v'(x)
25
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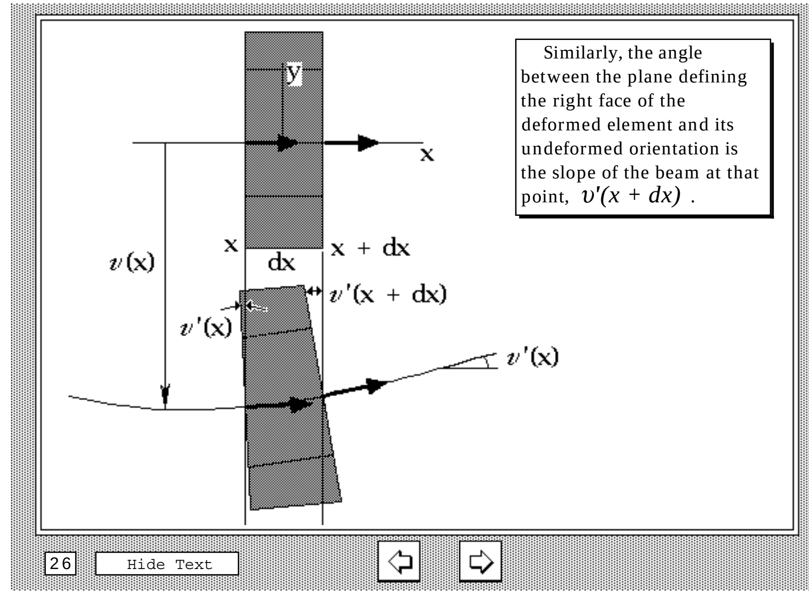
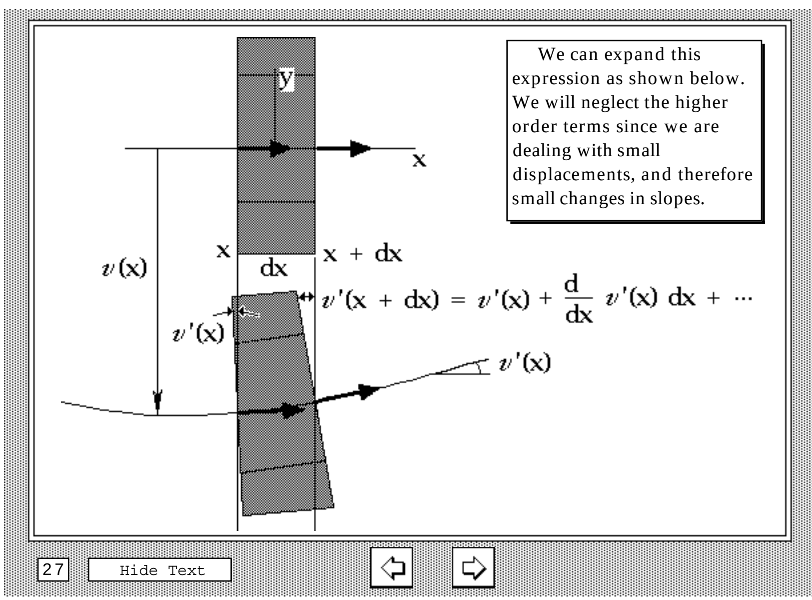

Similarly, the angle between the plane defining the right face of the deformed element and its undeformed orientation is the slope of the beam at that point, v'(x + dx) .
y
x
v(x)
x
dx
x + dx
v'(x + dx)
v'(x)
v'(x)
26
Hide Text


We can expand this expression as shown below. We will neglect the higher order terms since we are dealing with small displacements, and therefore small changes in slopes.
y
x
v(x)
x
dx
x + dx
v'(x + dx) = v'(x) + d/dx v'(x) dx + ···
v'(x)
v'(x)
27
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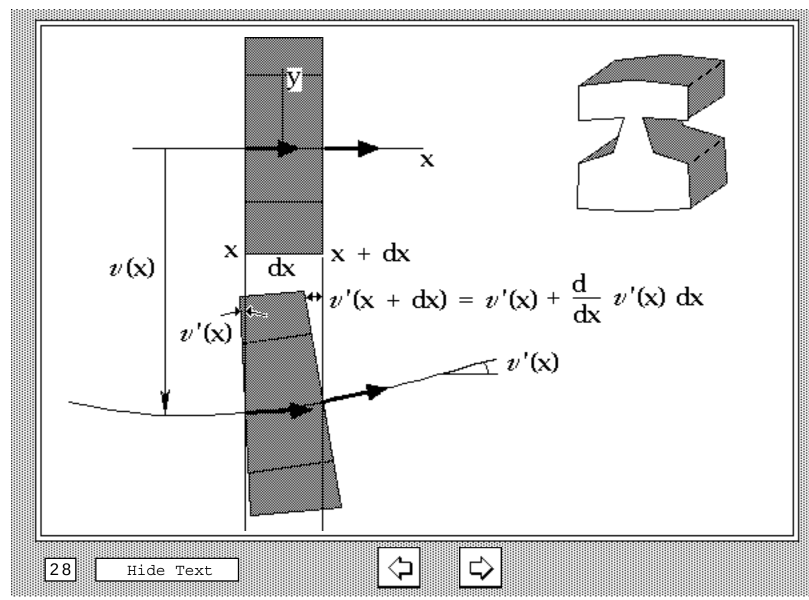
y
x
v(x)
x
dx
x + dx
v'(x + dx) = v'(x) + d/dx v'(x) dx
v'(x)
v'(x)
28
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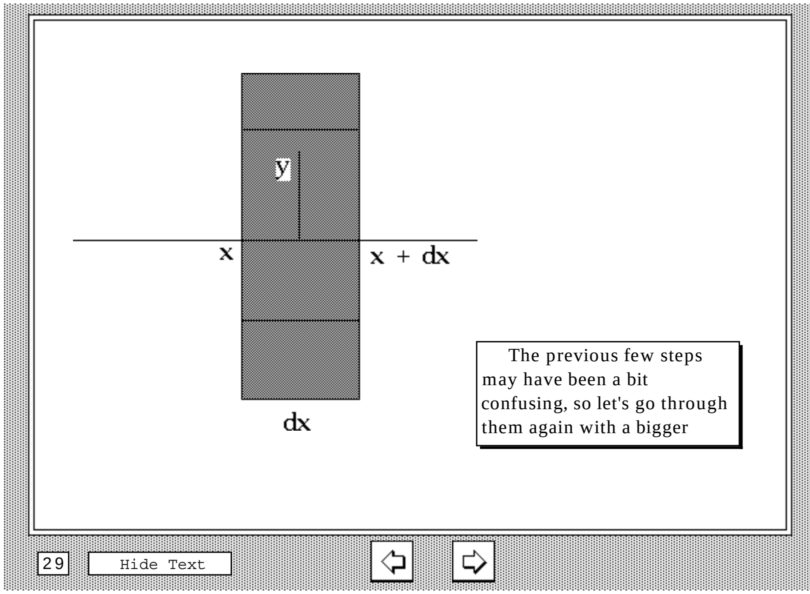
y
x
x + dx
dx
The previous few steps may have been a bit confusing, so let's go through them again with a bigger
29
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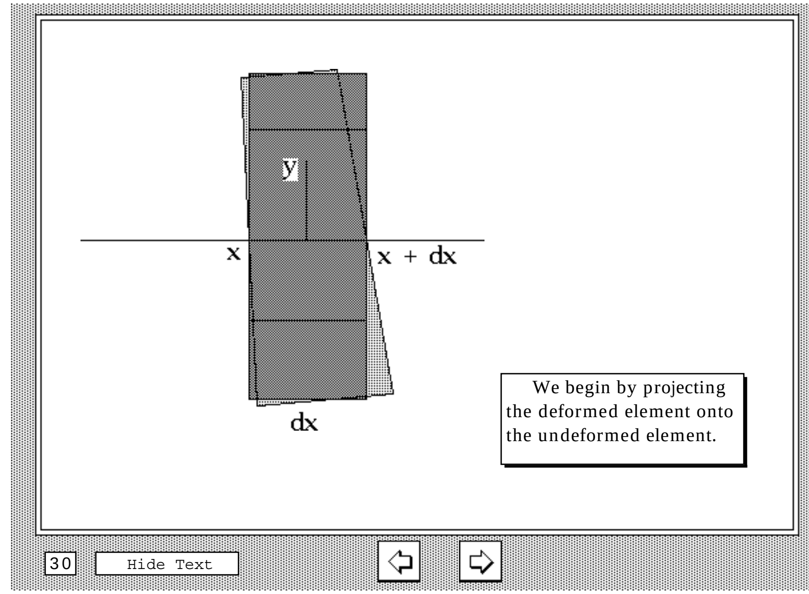
y
x
x + dx
dx
We begin by projecting the deformed element onto the undeformed element.
30
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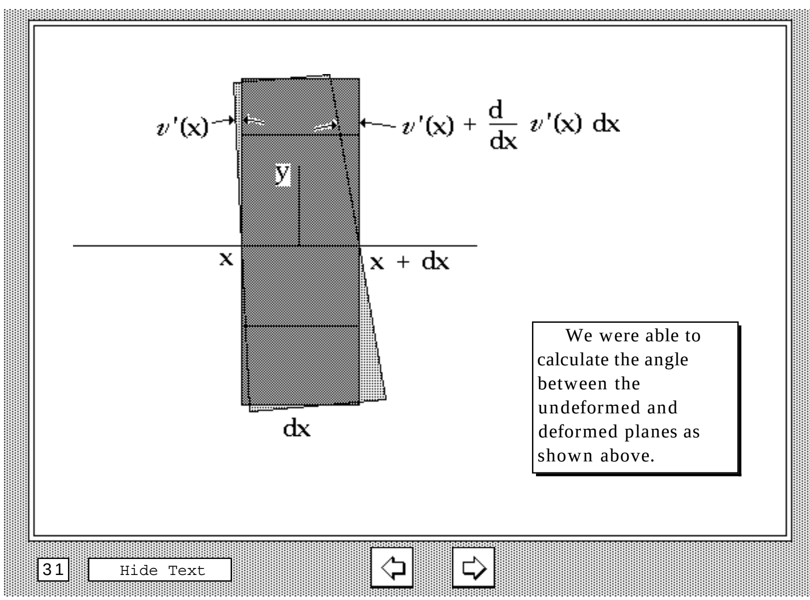
$v'(x)$
$v'(x) + \frac{d}{dx} v'(x)\, dx$
y
x
x + dx
dx
We were able to calculate the angle between the undeformed and deformed planes as shown above.
31
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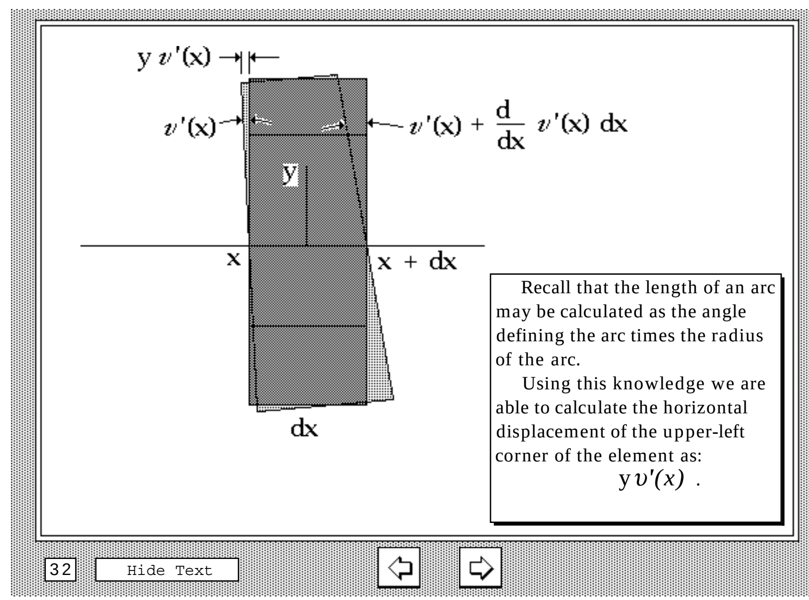
$y\, v'(x)$
$v'(x)$
$v'(x) + \frac{d}{dx} v'(x)\, dx$
y
x
x + dx
dx
Recall that the length of an arc may be calculated as the angle defining the arc times the radius of the arc.
Using this knowledge we are able to calculate the horizontal displacement of the upper-left corner of the element as:
$y\, v'(x)$.
32
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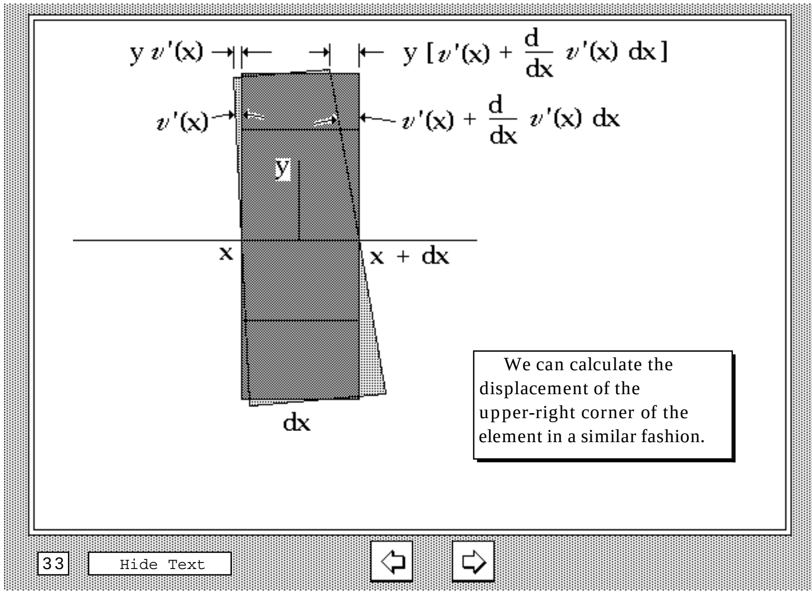
y v'(x)
y [v'(x) + d/dx v'(x) dx]
v'(x)
v'(x) + d/dx v'(x) dx
y
x
x + dx
dx
We can calculate the displacement of the upper-right corner of the element in a similar fashion.
33
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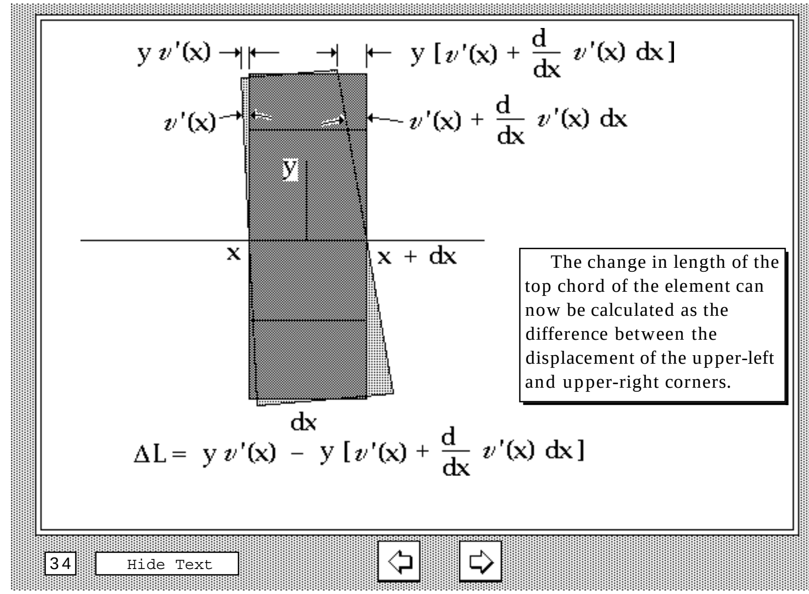
y v'(x)
y [v'(x) + d/dx v'(x) dx]
v'(x)
v'(x) + d/dx v'(x) dx
y
x
x + dx
dx
The change in length of the top chord of the element can now be calculated as the difference between the displacement of the upper-left and upper-right corners.
ΔL = y v'(x) − y [v'(x) + d/dx v'(x) dx]
34
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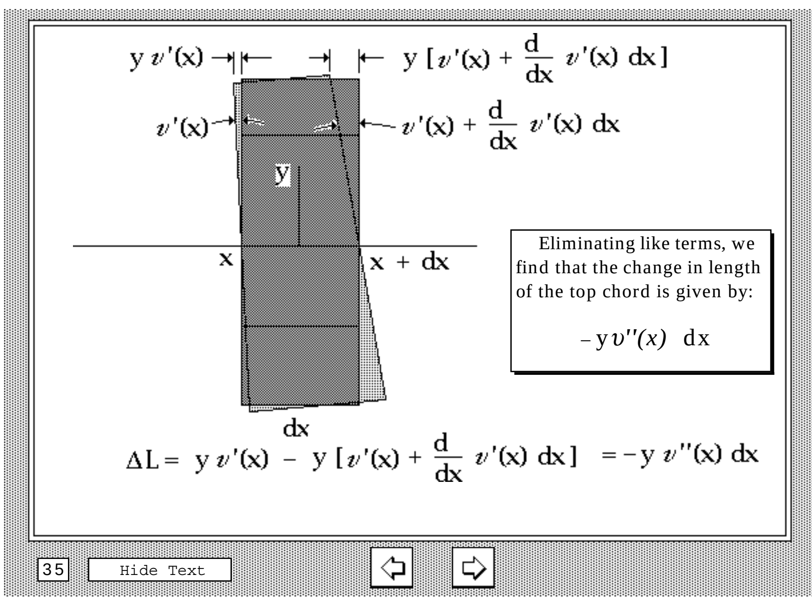
y v'(x)
y [v'(x) + d/dx v'(x) dx]
v'(x)
v'(x) + d/dx v'(x) dx
y
x
x + dx
dx
Eliminating like terms, we find that the change in length of the top chord is given by:
− y v''(x) dx
ΔL = y v'(x) − y [v'(x) + d/dx v'(x) dx] = − y v''(x) dx
35
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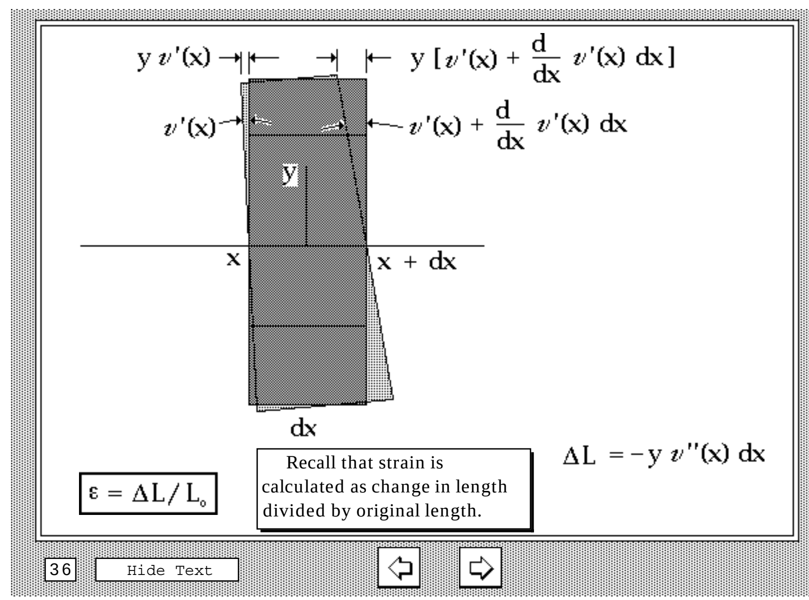
y v'(x)
y [v'(x) + d/dx v'(x) dx]
v'(x)
v'(x) + d/dx v'(x) dx
y
x
x + dx
dx
ε = ΔL / Lo
Recall that strain is calculated as change in length divided by original length.
ΔL = − y v''(x) dx
36
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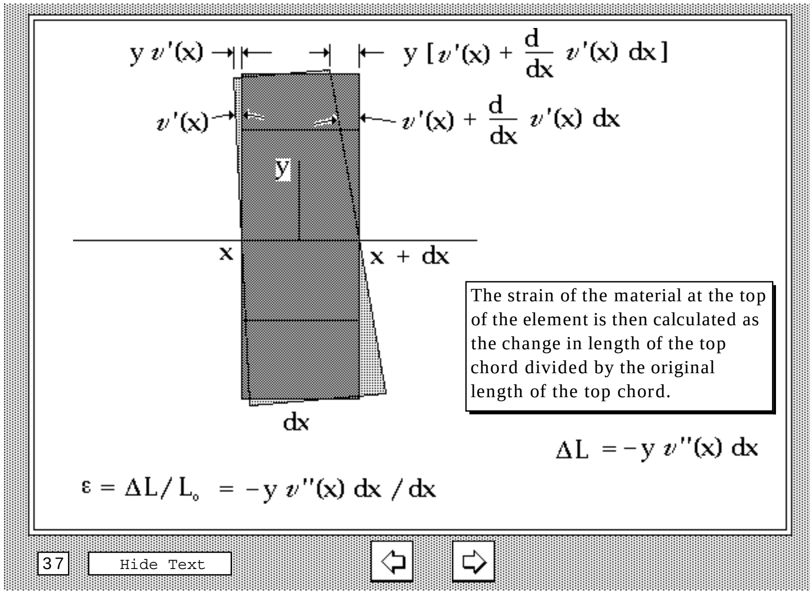
$y\,v'(x)$
$y\,[v'(x) + \frac{d}{dx}\,v'(x)\,dx]$
$v'(x)$
$v'(x) + \frac{d}{dx}\,v'(x)\,dx$
y
x
x + dx
The strain of the material at the top of the element is then calculated as the change in length of the top chord divided by the original length of the top chord.
dx
$\Delta L = -y\,v''(x)\,dx$
$\varepsilon = \Delta L / L_o = -y\,v''(x)\,dx\,/\,dx$
37
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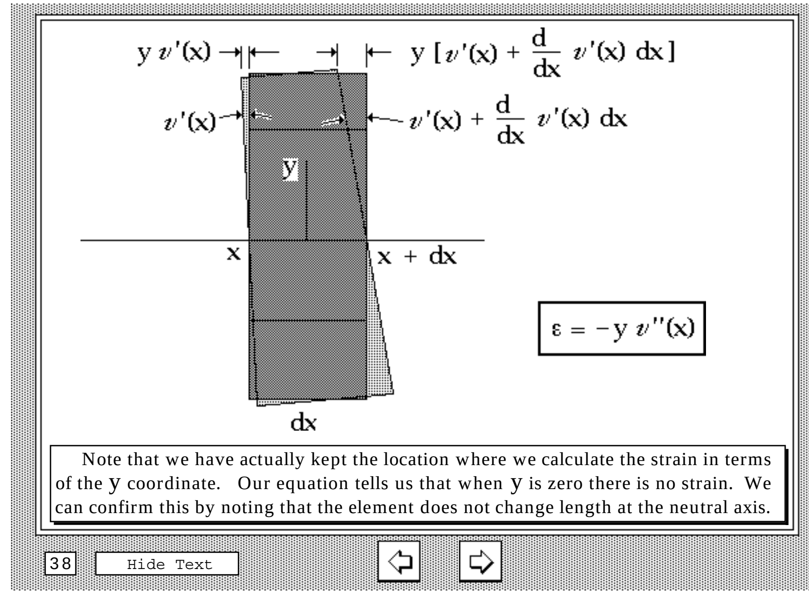
$y\,v'(x)$
$y\,[v'(x) + \frac{d}{dx}\,v'(x)\,dx]$
$v'(x)$
$v'(x) + \frac{d}{dx}\,v'(x)\,dx$
y
x
x + dx
$\varepsilon = -y\,v''(x)$
dx
Note that we have actually kept the location where we calculate the strain in terms of the y coordinate. Our equation tells us that when y is zero there is no strain. We can confirm this by noting that the element does not change length at the neutral axis.
38
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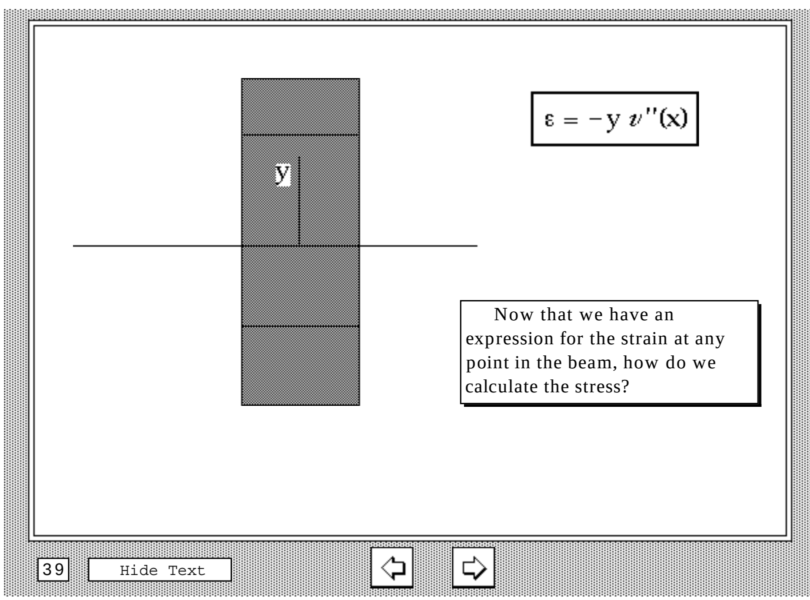
$\varepsilon = -y\,v''(x)$
y
Now that we have an expression for the strain at any point in the beam, how do we calculate the stress?
39
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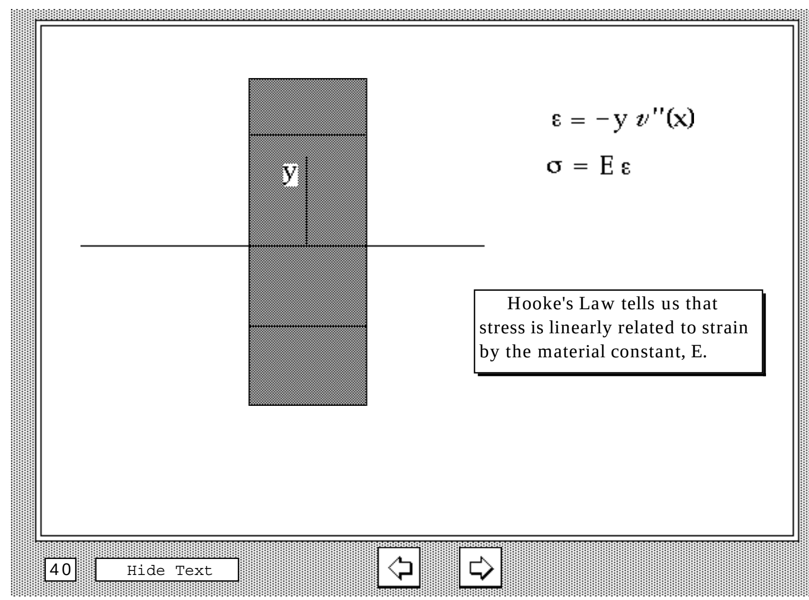
$\varepsilon = -y\,v''(x)$
$\sigma = E\,\varepsilon$
y
Hooke's Law tells us that stress is linearly related to strain by the material constant, E.
40
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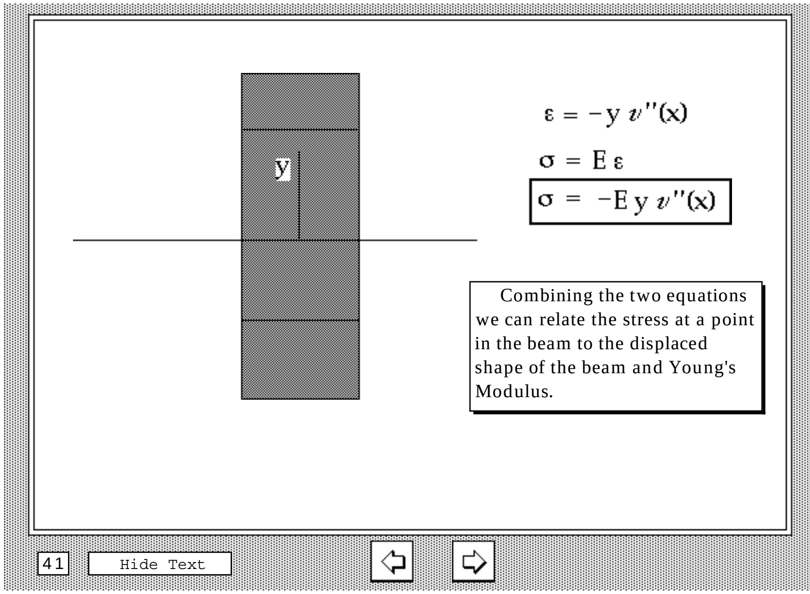
y
ε = −y v''(x)
σ = E ε
σ = −E y v''(x)
Combining the two equations we can relate the stress at a point in the beam to the displaced shape of the beam and Young's Modulus.
41
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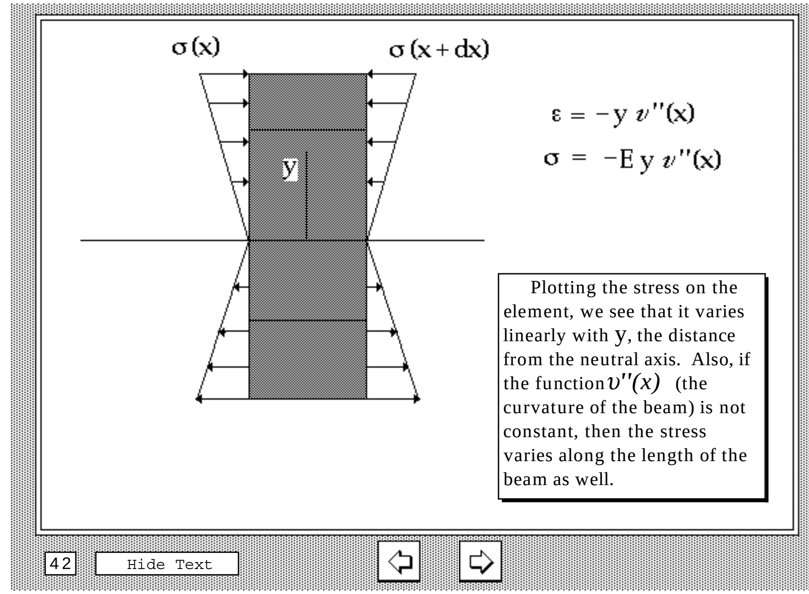
σ (x)
σ (x + dx)
y
ε = −y v''(x)
σ = −E y v''(x)
Plotting the stress on the element, we see that it varies linearly with y, the distance from the neutral axis. Also, if the function v''(x) (the curvature of the beam) is not constant, then the stress varies along the length of the beam as well.
42
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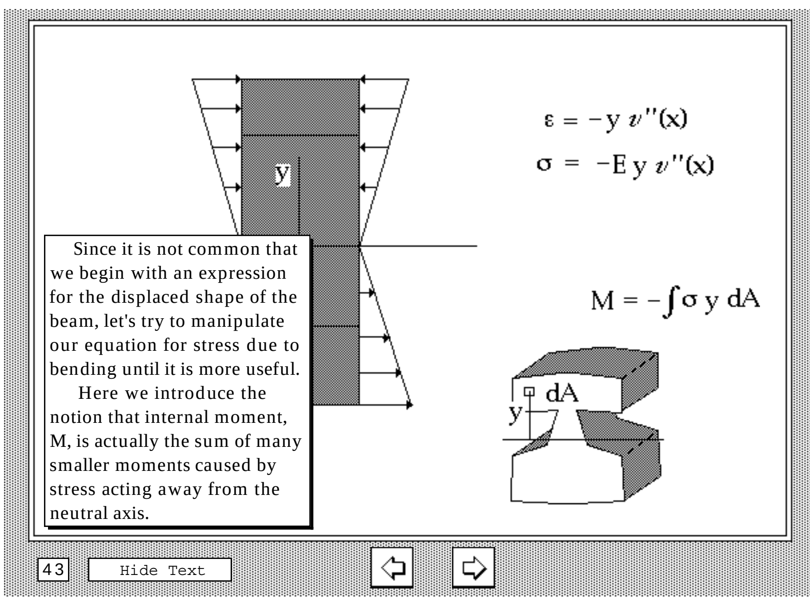
y
ε = −y v''(x)
σ = −E y v''(x)
M = −∫σ y dA
dA
y
Since it is not common that we begin with an expression for the displaced shape of the beam, let's try to manipulate our equation for stress due to bending until it is more useful.
Here we introduce the notion that internal moment, M, is actually the sum of many smaller moments caused by stress acting away from the neutral axis.
43
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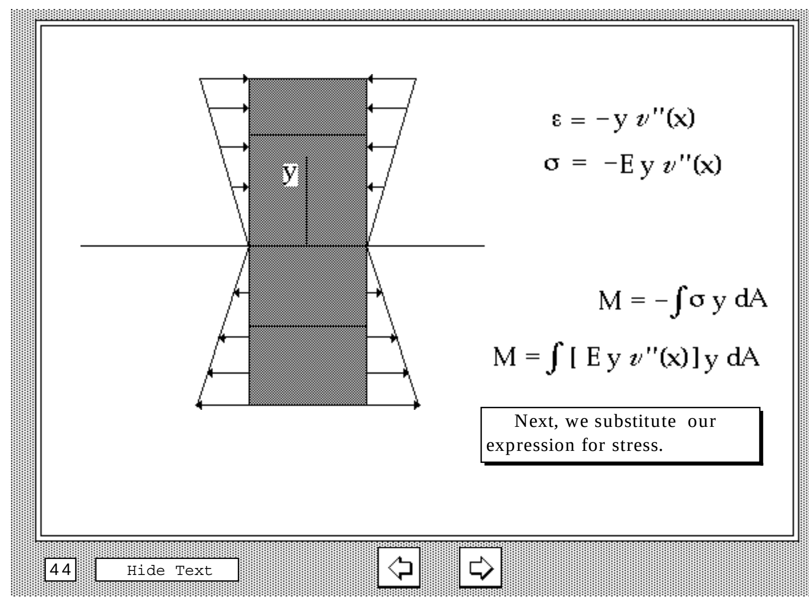
y
ε = −y v''(x)
σ = −E y v''(x)
M = −∫σ y dA
M = ∫ [E y v''(x)] y dA
Next, we substitute our expression for stress.
44
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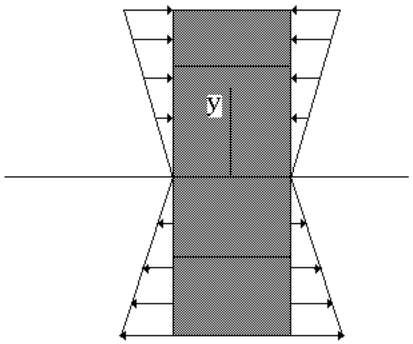


$$\varepsilon = -y\, v''(x)$$

$$\sigma = -E\, y\, v''(x)$$

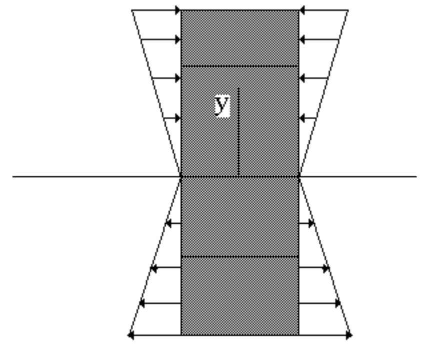
$$M = -\int \sigma\, y\, dA$$

$$M = \int [\, E\, y\, v''(x)\,]\, y\, dA$$

$$= E\,\{\int y^2\, dA\,\}\, v''(x)$$

Rearranging terms...

45 Hide Text



$$\varepsilon = -y\, v''(x)$$

$$\sigma = -E\, y\, v''(x)$$

$$M = -\int \sigma\, y\, dA$$

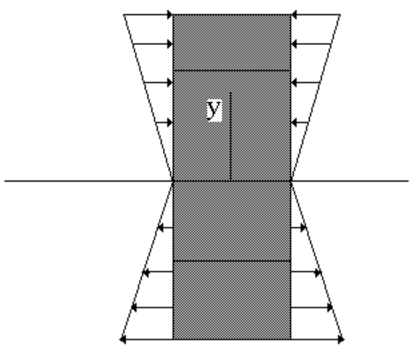
$$M = \int [\, E\, y\, v''(x)\,]\, y\, dA$$

$$= E\,\{\int y^2\, dA\,\}\, v''(x)$$

$$I = \int y^2\, dA$$

Recall the expression for moment of inertia.

46 Hide Text



$$\varepsilon = -y\, v''(x)$$

$$\sigma = -E\, y\, v''(x)$$

$$M = -\int \sigma\, y\, dA$$

$$M = \int [\, E\, y\, v''(x)\,]\, y\, dA$$

$$= E\,\{\int y^2\, dA\,\}\, v''(x)$$

Performing the substitution we find that internal moment is related to the curvature of the beam.

$$I = \int y^2\, dA$$

$$M = E I\, v''(x)$$

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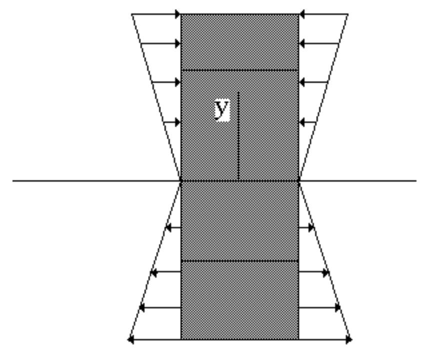

$$\varepsilon = -y\, v''(x)$$

$$\sigma = -E\, y\, v''(x)$$

$$M = E I\, v''(x)$$

How can we use this result to simplify our expression for stress in the beam?

$$I = \int y^2\, dA$$

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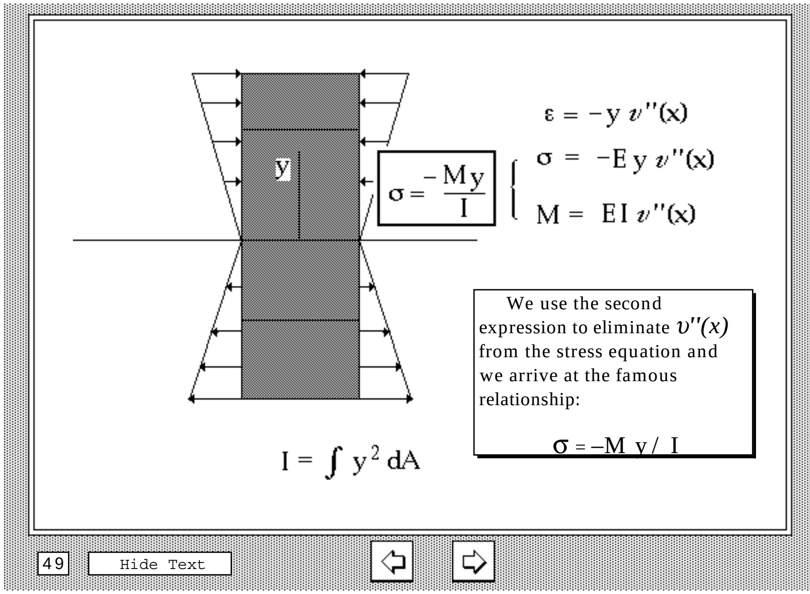
y
$\sigma = \frac{-My}{I}$
$\varepsilon = -y\, v''(x)$
$\sigma = -E y\, v''(x)$
$M = EI\, v''(x)$
We use the second expression to eliminate $v''(x)$ from the stress equation and we arrive at the famous relationship:
$\sigma = -M y / I$
$I = \int y^2\, dA$
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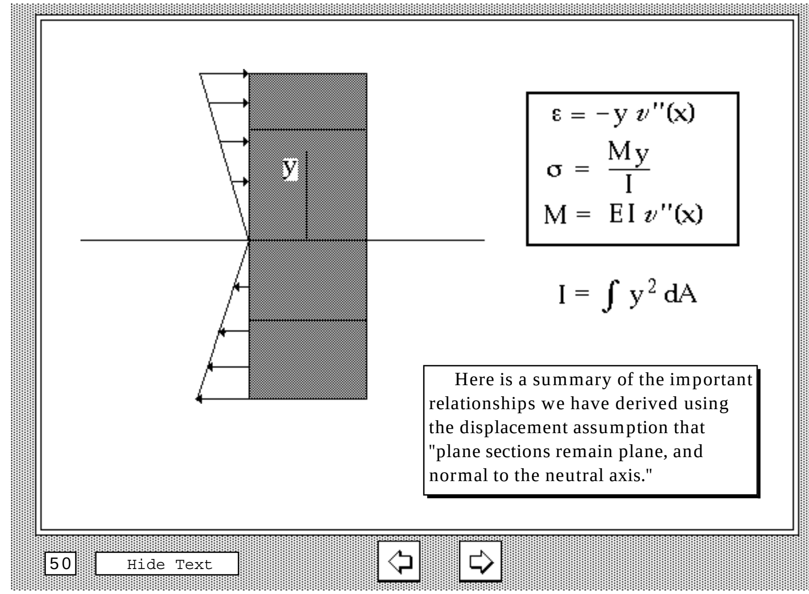
y
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
Here is a summary of the important relationships we have derived using the displacement assumption that "plane sections remain plane, and normal to the neutral axis."
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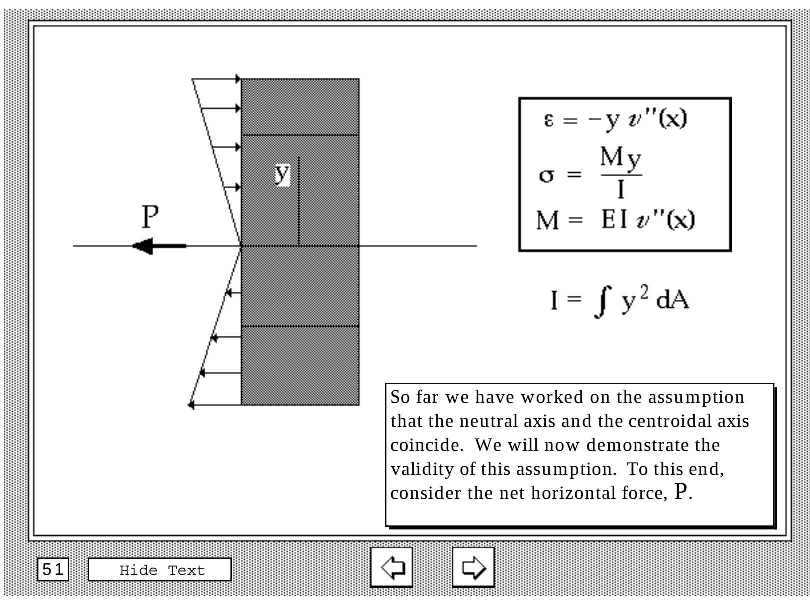
y
P
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
So far we have worked on the assumption that the neutral axis and the centroidal axis coincide. We will now demonstrate the validity of this assumption. To this end, consider the net horizontal force, P.
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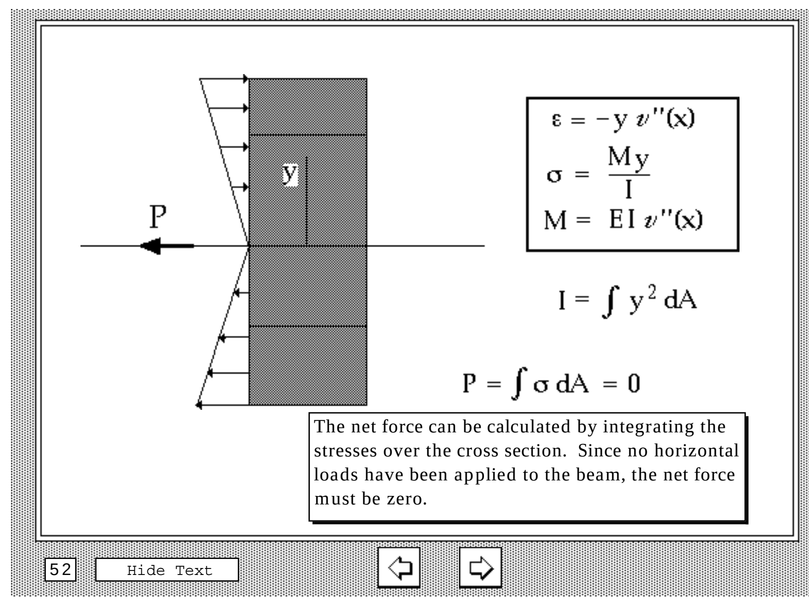
y
P
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
$P = \int \sigma\, dA = 0$
The net force can be calculated by integrating the stresses over the cross section. Since no horizontal loads have been applied to the beam, the net force must be zero.
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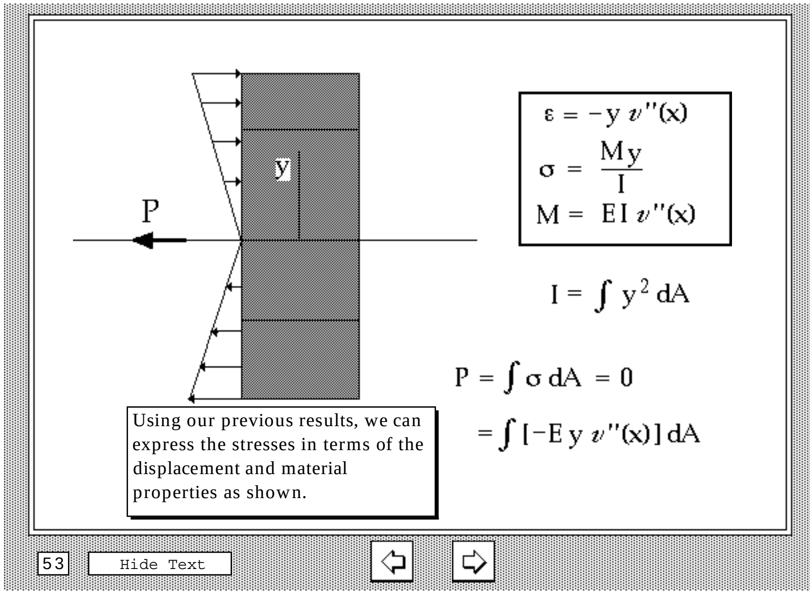

P
y
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
$P = \int \sigma\, dA = 0$
$= \int [-E\, y\, v''(x)]\, dA$
Using our previous results, we can express the stresses in terms of the displacement and material properties as shown.
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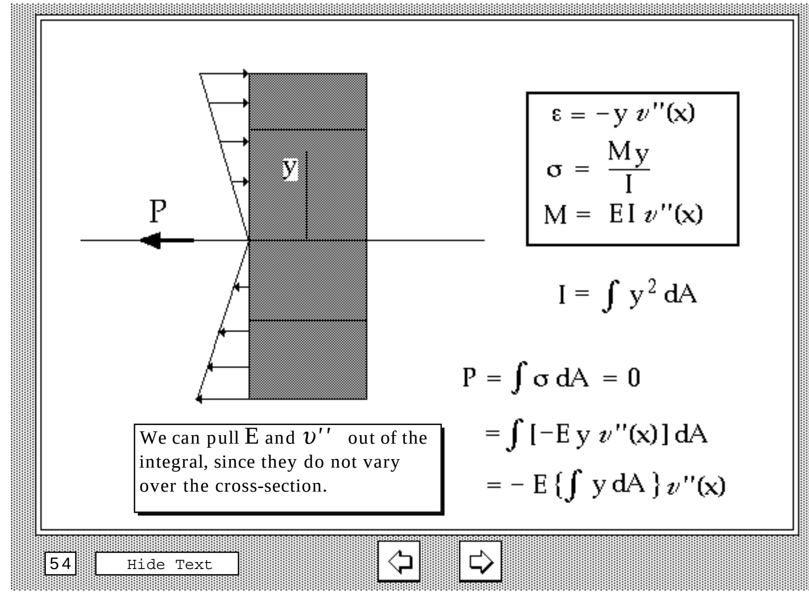

P
y
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
$P = \int \sigma\, dA = 0$
$= \int [-E\, y\, v''(x)]\, dA$
$= -E\{\int y\, dA\}\, v''(x)$
We can pull E and v'' out of the integral, since they do not vary over the cross-section.
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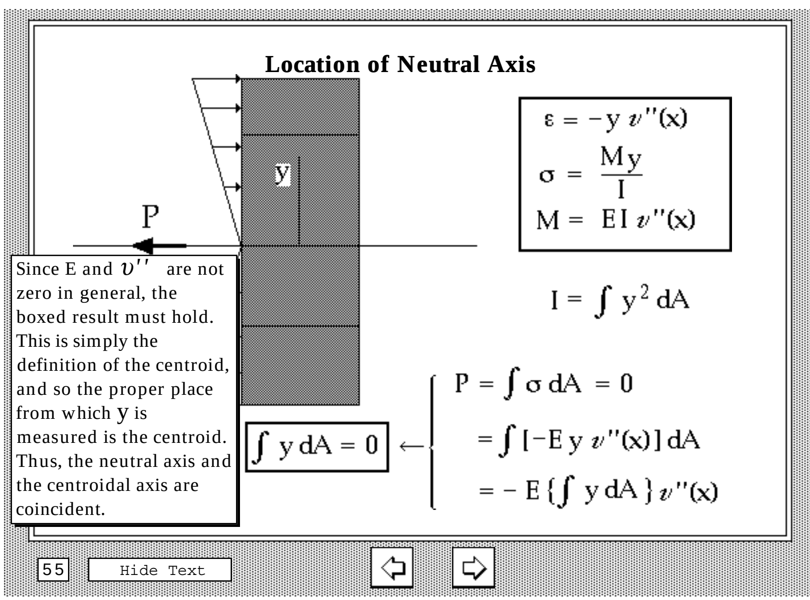

Location of Neutral Axis
P
y
$\varepsilon = -y\, v''(x)$
$\sigma = \frac{My}{I}$
$M = EI\, v''(x)$
$I = \int y^2\, dA$
$P = \int \sigma\, dA = 0$
$= \int [-E\, y\, v''(x)]\, dA$
$= -E\{\int y\, dA\}\, v''(x)$
$\int y\, dA = 0 \leftarrow$
Since E and v'' are not zero in general, the boxed result must hold. This is simply the definition of the centroid, and so the proper place from which y is measured is the centroid. Thus, the neutral axis and the centroidal axis are coincident.
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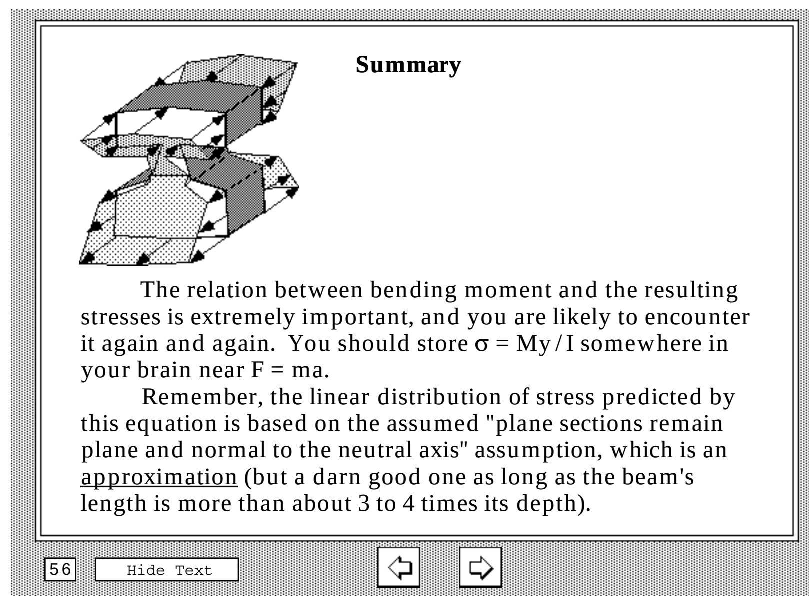

Summary
The relation between bending moment and the resulting stresses is extremely important, and you are likely to encounter it again and again. You should store $\sigma = My/I$ somewhere in your brain near F = ma.
Remember, the linear distribution of stress predicted by this equation is based on the assumed "plane sections remain plane and normal to the neutral axis" assumption, which is an approximation (but a darn good one as long as the beam's length is more than about 3 to 4 times its depth).
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End