

University of Washington
Department of Chemistry
Chemistry 453
Winter Quarter 2015

Lecture 24 3/13/15

A. Diffusion-Controlled Reactions

- As we learned activated kinetics depends on the rate at which reactant encounters cross an activation barrier to form products, i.e. $k = Ae^{-E_a/k_B T}$
- In contrast in a diffusion-controlled reaction $A + B \rightarrow P$ every encounter between A and B results in a reaction so the reaction rate depends on the length of time required for this encounter to occur.
- To model diffusion-controlled kinetics between A and B, we, assume the B molecule is stationary. The B molecule is assumed spherical and around this molecule is a sphere of radius R_B . If the center of a spherical A molecule enters this sphere it encounters and reacts with the B molecule.

- The flux of A molecules toward this sphere is given by

$$J_A = -D_A \frac{\partial C_A}{\partial r} \quad (24.5)$$

- Now the total number of A molecules entering the sphere of radius r ...also called the particle current, is

$$I_A = -J_A 4\pi r^2 = 4\pi D_A r^2 \frac{\partial C_A}{\partial r} \quad (24.6)$$

where the negative sign in the current expression designates motion toward the sphere. Now we separate variables:

$$I_A \frac{dr}{r^2} = 4\pi D_A dC_A \quad (24.7)$$

- The integration limits are for r from R_0 to ∞ . For C_A at $r=R_0$ that concentration is $C_A=0$. At $r=\infty$ that concentration of C_A is \bar{C}_A . Then we can integrate equation 24.7:

$$I_A \int_{R_0}^{\infty} \frac{dr}{r^2} = 4\pi D_A \int_0^{\bar{C}_A} dC_A \quad (24.8)$$

$$\therefore I_A = 4\pi D_A R_0 \bar{C}_A$$

- To correct for the fact that both molecules A and B move we replace in equation 24.8 D_A with D_A+D_B . To obtain the overall rate we also multiply by the concentration of B molecules \bar{C}_B :

$$rate = 4\pi (D_A + D_B) R_0 \bar{C}_A \bar{C}_B \quad (24.9)$$

- Because we assumed both A and B are spherical with radii R_A and R_B , we can refine equation 24.9 to

$$rate = 4\pi (D_A + D_B) (R_A + R_B) \bar{C}_A \bar{C}_B \quad (24.10)$$

- Equation 24.10 gives the kinetic constant for diffusion-controlled kinetics:

$$k_{diffusion} = 4\pi(D_A + D_B)(R_A + R_B) \quad (24.11)$$

- Note in equation 24.11 the kinetic constant is in units of $\text{m}^3 \text{s}^{-1}$ (per molecule). If the concentrations are in units of moles m^{-3} the kinetic constant can be expressed as a molar quantity by multiplying by Avogadro's number:

$$k_{diffusion,m} = 4\pi N_A (D_A + D_B)(R_A + R_B) \quad (24.12)$$

- If the A and B concentrations are in units of moles per liter but the diffusion coefficient and radii are in units of meters, we express the overall rate in units of moles per liter per second by multiplying by 1000Lm^{-3} , where

$$rate = 4\pi N_A (1000)(D_A + D_B)(R_A + R_B) \bar{C}_A \bar{C}_B \quad (24.13)$$

- Equations 24.11-24.13 indicate several differences between activated and diffusion-controlled kinetics.
- The temperature dependence for the Arrhenius kinetic constant is in the argument of the exponential $k = Ae^{-E_a/k_B T}$. Therefore, activated kinetic constants are strongly dependent on temperature. In contrast the temperature dependence in diffusion-controlled reactions resides in the diffusion coefficient. Recall for spherical molecules the diffusion coefficient is:

$$D = \frac{k_B T}{6\pi\eta R} \quad (24.14)$$

- Substitute 24.14 into 24.13 to get the temperature dependence of the diffusion-controlled reaction rate...

$$rate = 4\pi N_A (1000) \frac{k_B T}{6\pi\eta} \left(\frac{1}{R_A} + \frac{1}{R_B} \right) (R_A + R_B) \bar{C}_A \bar{C}_B \quad (24.15)$$

- Therefore unlike activated kinetics where the temperature dependence is exponential, the temperature dependence of the rate in equation 24.15 is linear and also inversely dependent on the solution viscosity.

B. Interpretation of Translational Diffusion Coefficients: Hydration of Globular Proteins and Aggregation

- The diffusion coefficient can yield information on molecular shape and degree of hydration.
- Interest in the diffusion coefficient derives from the fact that the coefficient of friction reflects molecular shape and size in the hydrated solution state. For globular macromolecules the friction coefficient is given by the Stokes-Einstein equation:

$$f = 6\pi\eta R \quad (24.16)$$

where η is the viscosity of the solvent, expressed in units of Poise= $0.100 \text{ kg m}^{-1} \text{ s}^{-1}$. and R is the radius of the molecule in units of m.

- The radius of a spherical protein can be calculated from the diffusion coefficient (i.e. the Stokes radius) reflects the dimension of the hydrated molecule. Attachment of a layer of waters of hydration to the macromolecule surface increases the size of the globular molecule and increases R.
- The size of the molecule can be estimated also from the specific volume \bar{V}_2 of the macromolecule. This is the volume in mL per gram of dry (i.e. unhydrated) macromolecule and usually lies in the range 0.70-0.76 mL g⁻¹ for proteins.
- From the specific volume the volume per molecule can be calculated:

$$V_{molecule} = \frac{M_2 \bar{V}_2}{N_A} \quad (24.17)$$

- The volume of the unhydrated spherical molecule can be calculated from

$$\frac{4\pi R_0^3}{3} = \frac{M_2 \bar{V}_2}{N_A} \quad (24.18)$$

- From the radius of the unhydrated molecule the hypothetical friction coefficient can be calculated:

$$f_0 = 6\pi\eta R_0 \quad (24.19)$$

- The two numbers f and f₀ often differ because the Stokes radius R is a hydrodynamic radius, i.e. the radius of a sphere composed of a protein covered with tightly attached water molecules (waters of hydration). The degree of hydration of the protein (i.e. grams of hydrated waters per gram of protein) may be determined by comparison of the Stokes radius with the radius calculated assuming an unhydrated protein sphere according to equation 24.3.
- Example: The diffusion coefficient and specific volume of bovine pancreas ribonuclease (an enzyme that digests RNA) have been measured in dilute buffer at T=293K and are found to be:

$$D = 13.1 \times 10^{-11} \text{ m}^2/\text{s}$$

$$\bar{V}_2 = 0.707 \text{ cm}^3 / \text{g}$$

The molecular weight of the protein is 13,690 g/mole.

- a) Calculate the frictional coefficient f.

$$f = \frac{k_B T}{D} = \frac{(1.38 \times 10^{-16} \text{ g} \cdot \text{cm}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1})(298 \text{ K})}{13.1 \times 10^{-7} \text{ cm}^2 / \text{s}} = 3.08 \times 10^{-8} \text{ g} / \text{s}$$

- b) Assuming the protein is an unhydrated sphere, calculate its frictional coefficient and its radius.

$$\frac{4}{3} \pi r_0^3 = \frac{M_2 \bar{V}_2}{N_A} \Rightarrow r^3 = \frac{3}{4\pi} \frac{M_2 \bar{V}_2}{N_A} = \frac{3}{4\pi} \frac{(13,690 \text{ g} / \text{mole})(0.707 \text{ cm}^3 / \text{g})}{6.02 \times 10^{23} \text{ molecules} / \text{mole}}$$

$$\therefore r_0 = 1.57 \times 10^{-7} \text{ cm} = 15.7 \text{ Angstroms} \text{ and}$$

$$f_0 = 6\pi\eta r_0 = 6\pi(0.01 \text{ g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1})(1.57 \times 10^{-7} \text{ cm}) = 2.95 \times 10^{-8} \text{ g} / \text{s}$$

$$c) \frac{f}{f_0} = \frac{r}{r_0} = \frac{3.09 \times 10^{-8}}{2.95 \times 10^{-8}} = 1.05.$$

$$\text{Therefore...} \left(\frac{f}{f_0} \right)^3 = 1.15 = \left(\frac{r}{r_0} \right)^3 = \frac{\bar{V}_2 + \delta_1 \bar{V}_1}{\bar{V}_2}$$

$$\delta_1 = \frac{1}{1.0 \text{ cm}^3 / \text{g}} \left[(1.15)(0.707 \text{ cm}^3 / \text{g}) - 0.707 \text{ cm}^3 / \text{g} \right] = 0.105 \text{ grams water/gram protein}$$

d) How many water molecules hydrate each protein molecule

$$\text{moles } H_2O = 0.105 \text{ g} / 18 \text{ g} \cdot \text{mole}^{-1} = 0.0058 \text{ moles}$$

$$\text{moles protein} = 1 \text{ g} / 13,690 \text{ g} \cdot \text{mole}^{-1} = 0.000073 \text{ moles}$$

$$H_2O / \text{protein} = 0.0058 / 0.000073 = 79.5 \approx 80$$

- Non-spherical molecules also have coefficients of friction For the purpose of calculating the translational friction f , many rod-like polymers can be approximated in solution as a chain of beads, each bead of diameter d . For such a polymer the frictional coefficient is

$$f_{tr} = \frac{3\pi\eta Nd}{\ln N} \quad (24.20)$$

where N is the number of beads in the polymer chain and η is the solvent viscosity.

Example 1: Suppose a spherical macromolecule with radius $3.55 \times 10^{-9} \text{ m}$ aggregates as a linear, rod-like hexamer. Calculate the coefficient of translational friction and the coefficient of translational diffusion. Calculate the rms displacement after 1 millisecond. Assume $\eta = 0.891 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$

$$f_{chain} = \frac{3\pi\eta Nd}{\ln N} = \frac{(3\pi)(0.891 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1})(6)(2 \times 3.55 \times 10^{-9} \text{ m})}{\ln(6)}$$

$$= \frac{3.58 \times 10^{-10}}{1.79} \text{ kg s}^{-1} = 2.00 \times 10^{-10} \text{ kg s}^{-1}$$

$$D = \frac{k_B T}{f_{chain}} = \frac{(1.38 \times 10^{-23} \text{ JK}^{-1})(298 \text{ K})}{2.00 \times 10^{-10} \text{ kg s}^{-1}} = 2.06 \times 10^{-11} \text{ m}^2 \text{ s}^{-1}$$

$$\langle \Delta r^2 \rangle = 6D\Delta\tau = 6 \times 2.06 \times 10^{-11} \text{ m}^2 \text{ s}^{-1} \times 0.001 \text{ s} = 1.24 \times 10^{-13} \text{ m}^2$$

$$\therefore r_{rms} = \sqrt{\langle \Delta r^2 \rangle} = (1.24 \times 10^{-13} \text{ m}^2)^{1/2} = 3.52 \times 10^{-7} \text{ m}$$

Example 2: Suppose the same spherical macromolecule aggregates into a spherical hexamer with radius 2.5 times the radius of the monomer. Calculate the friction coefficient of the spherical hexamer, its diffusion coefficient, and the rms displacement after 1 ms.

$$f_{sphere} = 6\pi\eta R_{hex} = (18.84)(0.891 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1})(2.5)(3.55 \times 10^{-9} \text{ m}) = 1.49 \times 10^{-10} \text{ kg s}^{-1}$$

$$D = \frac{k_B T}{f_{sphere}} = \frac{(1.38 \times 10^{-23} \text{ JK}^{-1})(298 \text{ K})}{1.49 \times 10^{-10} \text{ kg s}^{-1}} = 2.76 \times 10^{-11} \text{ m}^2 \text{ s}^{-1}$$

$$\langle \Delta r^2 \rangle = 6D\Delta\tau = 6 \times 2.76 \times 10^{-11} \text{ m}^2 \text{ s}^{-1} \times 0.001 \text{ s} = 1.66 \times 10^{-13} \text{ m}^2$$

$$\therefore r_{rms} = \sqrt{\langle \Delta r^2 \rangle} = (1.66 \times 10^{-13} \text{ m}^2)^{1/2} = 4.07 \times 10^{-7} \text{ m}$$

The two different aggregation geometries can be distinguished based on diffusion coefficient measurements.