Confidence Bounds and Intervals for Parameters of the Poisson and (Negative) Binomial Distributions

With Applications to Rare Event Probabilities

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1 Introduction and Overview

We present here by direct argument the classical Clopper-Pearson (1934) "exact" confidence bounds and corresponding intervals for the binomial or negative binomial parameter p, for the Poisson parameter λ, and for the ratio of two Poisson parameters, $\rho = \lambda_1/\lambda_2$. The bounds presented here are exact in the sense that their confidence level of covering the unknown parameters is at least the specified and targeted value γ, $0 < \gamma < 1$. The qualifier "at least" means that the minimum coverage probability of these bounds equals the desired confidence level, i.e., for some parameters the coverage probability of these bounds is equal to the desired confidence level. Because of the discrete nature of the underlying distributions the actual confidence varies with the unknown parameter and can, for some parameter ranges, be considerably higher than the stated value γ. In that sense these bounds are conservative in their coverage.

Agresti and Coull (1998) have recently discussed advantages of alternate methods, where the actual coverage oscillates more or less around the target value γ, and not above it. The advantage of such intervals is that they are somewhat shorter than the Clopper-Pearson intervals. Given that we often deal with confidence bounds concerning rare events we take the conservative approach of Clopper and Pearson. A recent discussion for the Poisson parameter can found in Barker (2002).

It is shown how such "exact" bounds can be computed quite easily using either the Excel spread sheet or the statistical packages R or S-Plus. However, the `GAMMAINV` function in the Excel spread sheet is not always stable in older versions of Excel. Thus care needs to be exercised.

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We first give the argument for confidence bounds for a Poisson parameter λ. The arguments for lower and upper bounds are completely parallel and it suffices to get a complete grasp of only one such derivation. This is followed by the corresponding argument for the binomial or negative binomial parameter p. For very small p it is pointed out how to use the very effective Poisson approximation to get bounds on p. Finally we give the classical method for constructing confidence bounds on the ratio $\rho = \lambda_1/\lambda_2$ based on two independent Poisson counts X and Y from Poisson distributions with parameters λ_1 and λ_2, respectively. This latter method nicely ties in with our earlier binomial confidence bounds and is quite useful in assessing relative accident or incident rates.

In the next to last section we show how lower and upper bounds can be combined into confidence intervals with the desired confidence level. We do this here in a generic sense that applies to all previous situations. This makes it unnecessary to repeat the same argument each time. However, for these intervals it is no longer clear whether the minimum coverage probability is in fact equal to the desired level but it is at least as large as the targeted level.

The last section deals with the topic of inverse probability solving for the binomial and Poisson distributions and shows how to accomplish this in Excel.

2 Poisson Parameter Upper and Lower Bounds

Suppose X is a Poisson random variable with mean λ ($\lambda > 0$), i.e.,

$$P_\lambda(X \le k) = \sum_{i=0}^{k} \frac{\exp(-\lambda)\lambda^i}{i!} \ .$$

2.1 Upper Bounds for λ

Small values of X can be viewed as evidence against the hypothesis

$$H(\lambda_0) : \lambda \ge \lambda_0 \ .$$

Upon observing $X = k$ we ask: is this value k small enough to reject $H(\lambda_0)$ with sufficient confidence? That depends on λ_0 and the desired confidence level γ, $0 < \gamma < 1$. Suppose we choose λ_0 such that

$$P_{\lambda_0}(X \le k) = \sum_{i=0}^{k} \frac{\exp(-\lambda_0)\lambda_0^i}{i!} = 1 - \gamma \ . \tag{1}$$

Then for all $\lambda \ge \lambda_0$ we have

$$P_\lambda(X \le k) \le P_{\lambda_0}(X \le k) = 1 - \gamma \ .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $X \leq k$ with probability at most $1-\gamma$) or to reject the hypothesis with confidence γ and thus state $\lambda < \lambda_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(\lambda_0)$ when $H(\lambda_0)$ is true.

Thus we can treat λ_0 as a $100\gamma\%$ upper confidence bound for λ. As such we also denote it by $\hat{\lambda}_U(\gamma, k)$. It is found by solving equation (1) for λ_0.

This value λ_0 solves

$$\gamma = 1 - \sum_{i=0}^{k} \frac{\exp(-\lambda_0)\lambda_0^i}{i!} = \int_0^{\lambda_0} \frac{t^k \exp(-t)}{k!}\, dt$$

and can thus be obtained using the inverse of the incomplete gamma function.

In Excel one gets λ_0 by invoking `GAMMAINV`$(\gamma, k+1, 1)$ and in R or S-Plus by the command `qgamma(`γ`,`$k+1$`)`. As a check use the case $k = 2$ with $\gamma = .95$, then one gets $\hat{\lambda}_U(.95, 2) = 6.295794$ as 95% upper bound for λ.

For the special case $k = 0$ one can give an explicit formula for the upper bound, namely $\hat{\lambda}_U(\gamma, 0) = -\log(1-\gamma)$. For $\gamma = .95$ this amounts to $\hat{\lambda}_U(.95, 0) = 2.995732 \approx 3$, which is sometimes referred to as the **Rule of Three**, because of its mnemonic simplicity[2].

2.2 Lower Bounds for λ

Large values of X can be viewed as evidence against the hypothesis

$$H(\lambda_0) : \lambda \leq \lambda_0 \ .$$

Upon observing $X = k$ we ask: is this value k large enough to reject $H(\lambda_0)$ with sufficient confidence? That depends on λ_0 and the desired confidence level γ, $0 < \gamma < 1$. Suppose we choose λ_0 such that

$$P_{\lambda_0}(X \geq k) = \sum_{i=k}^{\infty} \frac{\exp(-\lambda_0)\lambda_0^i}{i!} = 1 - \gamma \ . \tag{2}$$

Then for all $\lambda \leq \lambda_0$ we have

$$P_\lambda(X \geq k) \leq P_{\lambda_0}(X \geq k) = 1 - \gamma \ .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $X \geq k$ with probability at most $1-\gamma$) or to reject the hypothesis

[2] so called by Gerald van Belle in *Statistical Rules of Thumb* 2002, John Wiley & Sons, New York

with confidence γ and thus state $\lambda > \lambda_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(\lambda_0)$ when $H(\lambda_0)$ is true.
Thus we can treat λ_0 as a $100\gamma\%$ lower confidence bound for λ. As such we also denote it by $\hat{\lambda}_L(\gamma, k)$. It is found by solving equation (2) for λ_0. Of course, for $\gamma > 0$ this equation can only have a solution when $k > 0$. When $k = 0$ one cannot give a positive lower bound for λ. It is quite natural to state zero as the natural and trivial[3] 100% lower bound in the case $k = 0$.
Equation(2) can be rewritten as

$$P_{\lambda_0}(X \le k-1) = \sum_{i=0}^{k-1} \frac{\exp(-\lambda_0)\lambda_0^i}{i!} = \gamma \ .$$

This value λ_0 can be obtained from Excel by invoking GAMMAINV($1-\gamma, k, 1$) and in R or S-Plus by the command `qgamma(`$1-\gamma$`,`k`)`. For $k = 1$ one can give the formula for the lower bound explicitly as $\hat{\lambda}_L(\gamma, 1) = -\log(\gamma)$.
As a check use the case $k = 30$ with $\gamma = .95$, then one gets $\hat{\lambda}_L(.95, 30) = 21.59399$ as 95% lower bound for λ.

3 Binomial Parameter Upper and Lower Bounds

Suppose X is a binomial random variable, i.e., X counts successes in n independent trials with success probability p ($0 \le p \le 1$) in each trial. Then we have

$$P_p(X \le k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \ .$$

3.1 Upper Bounds for p

Small values of X can be viewed as evidence against the hypothesis

$$H(p_0) : p \ge p_0 \ .$$

Upon observing $X = k$ we ask: is this value k small enough to reject $H(p_0)$ with sufficient confidence? That depends on p_0 and the desired confidence level γ, $0 < \gamma < 1$. Suppose we choose p_0 such that

$$P_{p_0}(X \le k) = \sum_{i=0}^{k} \binom{n}{i} p_0^i (1-p_0)^{n-i} = 1-\gamma \ . \qquad (3)$$

[3]since $\lambda > 0$

Then for all $p \geq p_0$ we have

$$P_p(X \leq k) \leq P_{p_0}(X \leq k) = 1 - \gamma \; .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $X \leq k$ with probability at most $1-\gamma$) or to reject the hypothesis with confidence γ and thus state $p < p_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(p_0)$ when $H(p_0)$ is true.

Thus we can treat p_0 as a $100\gamma\%$ upper confidence bound for p. As such we also denote it by $\hat{p}_U(\gamma, k, n)$. For $k < n$ it is found by solving equation (3) for p_0. This value p_0 can be obtained from Excel by invoking `BETAINV(`$\gamma, k+1, n-k$`)` and in R or S-Plus by the command `qbeta(`γ`,`$k+1$`,`$n-k$`)`. For $k = 0$ one can give the upper bound explicitly as

$$\hat{p}_U(\gamma, 0, n) = 1 - (1-\gamma)^{1/n} \; .$$

For $\gamma = .95$ this becomes

$$\hat{p}_U(.95, 0, n) = 1 - (.05)^{1/n} = 1 - \exp\left[\frac{\log(.05)}{n}\right] \approx 1 - \exp(-3/n) \approx \frac{3}{n} \; ,$$

which can be viewed as another instance of the Rule of Three. Here the last approximation is valid only for large n.

For $\gamma > 0$ and $k = n$ use the natural and trivial[4] upper bound $\hat{p}_U(\gamma, n, n) = 1$ with $100\gamma\%$ confidence, since in that case the equation (3) has no solution.

As a check use the case $k = 12$ and $n = 1600$ with $\gamma = .95$, then one gets $\hat{p}_U(.95, 12, 1600) = .012123$ as 95% upper bound for p.

Side Comment on Treatment of $X = 0$

When one observes $X = 0$ successes in n trials, especially when n is large, one is still not inclined to estimate the success probability p by $\widehat{p}(0) = 0/n = 0$, since that is a very strong statement. When $p = 0$ then we will never see a success in however many trials. Thus one sometimes finds the following argument. Since $p = 0$ is such a strong statement, but one still thinks that p is likely to be very small if $X = 0$ in a large number n of trials, one just gets out of this dilemma by "conservatively" pretending that the first success is just around the corner, i.e., happens on the next trial. With that one would estimate p by $\tilde{p} = 1/(n+1)$ which is small but not zero. There are other (and statistically better) rationales for justifying

[4] since $p \leq 1$

$\tilde{p} = 1/(n+1)$ as an estimate of p but we won't enter into that here, since they have no bearing on the issue of confidence that some construe out of the above "conservative" step.

As an estimate of the true value of p the use of $\tilde{p}$ is not entirely unreasonable but somewhat conservative. It is however quite different from our 95% upper confidence bound of $3/n$, namely by roughly a factor of 3. One could ask: what is the actual confidence associated with $\tilde{p}$?

The same argument that was presented before shows that the hypothesis $H(p_0) : p \geq p_0$ for a stipulated value of $p_0 = 1/(n+1)$ gives us the following maximum probability for the actually observed event

$$P_p(X \leq 0) \leq P_{p_0}(X \leq 0) = (1-p_0)^n = \left(1 - \frac{1}{n+1}\right)^n \approx \exp\left(-\frac{n}{n+1}\right) \approx \exp(-1) \approx .3679$$

with approximations quite good for moderate to large n. Thus there is a good chance (up to .3679) that $X = 0$ when H_0 is true. Rejecting H_0 on the basis of $X = 0$ has therefore an up to 36.79% chance of leading to a wrong decision. In turn this means that we can treat $\tilde{p} = 1/(n+1)$ only as a 63.21% upper confidence bound for p. This is substantially lower than the 95% which led to the factor 3 in $3/n$ as upper bound.

3.2 Lower Bounds for p

Large values of X can be viewed as evidence against the hypothesis

$$H(p_0) : p \leq p_0 \ .$$

Upon observing $X = k$ we ask: is this value k large enough to reject $H(p_0)$ with sufficient confidence? That depends on p_0 and the desired confidence level γ. Suppose we choose p_0 such that

$$P_{p_0}(X \geq k) = \sum_{i=k}^{n} \binom{n}{i} p_0^i (1-p_0)^{n-i} = 1 - \gamma \ . \qquad (4)$$

Then for all $p \leq p_0$ we have

$$P_p(X \geq k) \leq P_{p_0}(X \geq k) = 1 - \gamma \ .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $X \geq k$ with probability at most $1-\gamma$) or to reject the hypothesis with confidence γ and thus state $p > p_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(p_0)$ when $H(p_0)$ is true.

Thus we can treat p_0 as a $100\gamma\%$ lower confidence bound for p. As such we also denote it by $\hat{p}_L(\gamma, k, n)$. For $k > 0$ it is found by solving equation (4) for p_0. This equation is equivalent to

$$P_{p_0}(X \le k-1) = \sum_{i=0}^{k-1} \binom{n}{i} p_0^i (1-p_0)^{n-i} = \gamma \ .$$

This value p_0 can be obtained in Excel by invoking `BETAINV(`$1-\gamma, k, n-k+1$`)` and from R or S-Plus by the command `qbeta(`$1-\gamma$`,`k`,`$n-k+1$`)`. For $k = n$ one can give the lower bound explicitly as $\hat{p}_L(\gamma, n, n) = (1-\gamma)^{1/n}$. For $\gamma = .95$ this becomes

$$\hat{p}_L(.95, n, n) = (1-.95)^{1/n} \approx \exp(-3/n) \approx 1 - \frac{3}{n} \ ,$$

another instance of the Rule of Three. Here the last approximation is only valid for large n. For $\gamma > 0$ and $k = 0$ the above equation does not yield a solution and one takes $\hat{p}_L(\gamma, 0, n) = 0$ as natural and trivial[5] 100% lower bound for p.

As a check take $k = 4$ and $n = 500$ with $\gamma = .95$, then one gets $\hat{p}_L(.95, 4, 500) = .002737$ as 95% lower bound for p.

3.3 Poisson Approximation to Binomial

For very small p the binomial distribution of X can be well approximated by the Poisson distribution with mean $\lambda = np$. Thus confidence bounds for $p = \lambda/n$ can be based on those obtained via the Poisson distribution, namely by using $\hat{\lambda}_U(\gamma, k)/n$ and $\hat{\lambda}_L(\gamma, k)/n$.

A typical application would concern the number X of well defined, rare incidents (crashes or part failures) in n flight cycles in a fleet of airplanes. Here p would denote the probability of such an incident during a particular flight cycle. Typically p is very small and n, as accumulated over the whole fleet, is very large.

4 Negative Binomial Parameter Upper and Lower Bounds

Suppose N is a negative binomial random variable, i.e., N counts the number of required independent trials, with success probability p ($0 \le p \le 1$) in each trial, in order to obtain a predetermined number k of successes. Then we have

$$P_p(N = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}, \quad n = k, k+1, \ldots$$

[5]since $0 \le p$

or from a different angle and more useful for our purpose

$$\begin{aligned} P_p(N \geq n) &= P(\text{at most } k-1 \text{ successes in the first } n-1 \text{ independent trials}) \\ &= \sum_{i=0}^{k-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \; . \end{aligned}$$

4.1 Upper Bounds for p

Large values of N can be viewed as evidence against the hypothesis

$$H(p_0) : p \geq p_0 \; .$$

Upon observing $N = n$ we ask: is this value n large enough to reject $H(p_0)$ with sufficient confidence? That depends on p_0 and the desired confidence level γ, $0 < \gamma < 1$. Suppose we choose p_0 such that

$$P_{p_0}(N \geq n) = \sum_{i=0}^{k-1} \binom{n-1}{i} p_0^i (1-p_0)^{n-1-i} = 1 - \gamma \; . \tag{5}$$

Then for all $p \geq p_0$ we have

$$P_p(N \geq n) \leq P_{p_0}(N \geq n) = 1 - \gamma \; .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $N \geq n$ with probability at most $1-\gamma$) or to reject the hypothesis with confidence γ and thus state $p < p_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(p_0)$ when $H(p_0)$ is true.

Thus we can treat p_0 as a $100\gamma\%$ upper confidence bound for p. As such we also denote it by $\tilde{p}_U(\gamma, k, n)$ (the tilde over the p is used to distinguish it from the binomial upper confidence bound). For $k < n$ it is found by solving equation (5) for p_0. This value p_0 can be obtained from Excel by invoking BETAINV($\gamma, k, n-k$) and in R or S-Plus by the command qbeta(γ,k,$n-k$). For $k = 1$ one can give the upper bound explicitly as

$$\tilde{p}_U(\gamma, 1, n) = 1 - (1-\gamma)^{1/(n-1)} \; .$$

For $\gamma = .95$ this becomes

$$\tilde{p}_U(.95, 1, n) = 1 - (.05)^{1/(n-1)} = 1 - \exp\left[\frac{\log(.05)}{n-1}\right] \approx 1 - \exp\left[-\frac{3}{n-1}\right] \approx \frac{3}{n-1} \; ,$$

which can be viewed as another instance of the Rule of Three. The last invoked approximation is only valid for large n.

For $\gamma > 0$ and $k = n$ use the natural and trivial[6] upper bound $\tilde{p}_U(\gamma, n, n) = 1$ with $100\gamma\%$

[6] since $p \leq 1$

confidence, since in that case the equation (5) has no solution.
As a check take $k = 25$ and $n = 1200$ with $\gamma = .95$, then $\tilde{p}_U(.95, 25, 1200) = .028036$ is the 95% upper confidence bound for p.

4.2 Lower Bounds for p

Small values of N can be viewed as evidence against the hypothesis

$$H(p_0) : p \leq p_0 \; .$$

Upon observing $N = n$ we ask: is this value n small enough to reject $H(p_0)$ with sufficient confidence? That depends on p_0 and the desired confidence level γ. Suppose we choose p_0 such that

$$P_{p_0}(N \leq n) = 1 - P_{p_0}(N \geq n+1) = 1 - \sum_{i=0}^{k-1} \binom{n}{i} p_0^i (1-p_0)^{n-i} = 1 - \gamma \; . \qquad (6)$$

Then for all $p \leq p_0$ we have

$$P_p(N \leq n) \leq P_{p_0}(N \leq n) = 1 - \gamma \; .$$

If this value $1-\gamma$ is small we have the choice to either believe that we saw a rare event under the hypothesis (namely $N \leq n$ with probability at most $1 - \gamma$) or to reject the hypothesis with confidence γ and thus state $p > p_0$. This confidence means that there is at most a $1-\gamma$ chance of having made the wrong decision, namely rejecting $H(p_0)$ when $H(p_0)$ is true.
Thus we can treat p_0 as a 100γ% lower confidence bound for p. As such we also denote it by $\tilde{p}_L(\gamma, k, n)$. It is found by solving equation (6) for p_0. This value p_0 can be obtained in Excel by invoking `BETAINV(`$1-\gamma, k, n-k+1$`)` and from R or S-Plus by the command `qbeta(`$1-\gamma$`,`k`,`$n-k+1$`)`. For $k = n$ one can give the lower bound explicitly as $\hat{p}_L(\gamma, n, n) = (1-\gamma)^{1/n}$. For $\gamma = .95$ this becomes

$$\tilde{p}_L(.95, n, n) = (1 - .95)^{1/n} \approx \exp(-3/n) \approx 1 - \frac{3}{n} \; ,$$

another instance of the Rule of Three. Here the last approximation is only valid for large n.
As a check take $k = 5$ and $n = 30$ with $\gamma = .95$, then $\tilde{p}_L(.95, 5, 30) = .068056$ is the 95% lower confidence bound for p.

5 Comparing Two Poisson Means

Suppose X and Y are independent Poisson random variables with respective means λ and μ. We are interested in confidence bounds for $\rho = \lambda/\mu$. If these Poisson distributions represent approximations of binomials for small "success" probabilities π_1 and π_2, i.e., $\lambda = n_1\pi_1$ and $\mu = n_2\pi_2$, then confidence bounds for the ratio $\rho = \lambda/\mu = (n_1\pi_1)/(n_2\pi_2)$ are equivalent to confidence bounds for π_1/π_2, since n_1 and n_2 are typically known.

The classical method for getting confidence bounds for ρ is to consider the conditional distribution of Y given $T = X + Y = t$, which is

$$P(Y = k|X + Y = t) = \binom{t}{k} p^i(1-p)^{t-i} ,$$

where $p = \mu/(\lambda + \mu) = 1/(1 + \rho)$. Hence this is a binomial distribution with parameters t trials and success probability $p = 1/(1 + \rho)$, which is a monotone function of ρ.

Given the previous treatment we can get binomial upper confidence bounds $\hat{p}_U(\gamma, k, t)$ for p. By inverting the monotone relationship $p = 1/(1 + \rho)$ we get from this a lower confidence bound for ρ in

$$\hat{\rho}_L(\gamma, k, t) = \frac{1}{\hat{p}_U(\gamma, k, t)} - 1 .$$

Similarly one gets upper confidence bounds for ρ in

$$\hat{\rho}_U(\gamma, k, t) = \frac{1}{\hat{p}_L(\gamma, k, t)} - 1 .$$

In the context of the above Poisson approximations to binomial distributions one gets a lower confidence bound for $\kappa = \pi_1/\pi_2$ in

$$\hat{\kappa}_L(\gamma, k, t) = \frac{n_2}{n_1} \times \hat{\rho}_L(\gamma, k, t) = \frac{n_2}{n_1} \times \left(\frac{1}{\hat{p}_U(\gamma, k, t)} - 1 \right) .$$

Similarly one gets upper bounds for $\kappa = \pi_1/\pi_2$ in

$$\hat{\kappa}_U(\gamma, k, t) = \frac{n_2}{n_1} \times \hat{\rho}_U(\gamma, k, t) = \frac{n_2}{n_1} \times \left(\frac{1}{\hat{p}_L(\gamma, k, t)} - 1 \right) .$$

As an example consider some accident data. Among Modern Wide Body Airplanes we had 0 accidents (substantial damage, hull loss, or hull loss with fatalities) during 11.128×10^6 flights. Among Modern Narrow Body Airplanes we had 5 such accidents during 55.6×10^6

flights. Thus we have $Y = k = 5$ and $T = X + Y = t = 0 + 5$ and $n_1 = 11.128 \times 10^6$ and $n_2 = 55.6 \times 10^6$. We find

$$\hat{p}_U(.95, 5, 5) = 1$$

and thus

$$\hat{\kappa}_L(.95, 5, 5) = \frac{n_2}{n_1} \times \left(\frac{1}{1} - 1\right) = 0$$

and

$$\hat{p}_L(.95, 5, 5) = \texttt{qbeta}(.05, 5, 1) = (1 - .95)^{1/5} = 0.5492803$$

resulting in

$$\hat{\kappa}_U(.95, 5, 5) = \frac{n_2}{n_1} \times \left(\frac{1}{0.5492803} - 1\right) = 4.099871$$

Thus we can view 4.099871 as a 95% upper confidence bound for π_1/π_2, the ratio of rates for the two groups. Since this bound is above 1, one cannot rule out that the rates in the two groups may be the same. This is not surprising since the group with 5 accidents had about five times the exposure of the other group. Thus for a fifth of the exposures one might have expected to see one such accident in the second group. That is not all that different from zero in the realm of counting rare events.

6 Confidence Intervals

Suppose $\hat{L}(\gamma)$ is a $100\gamma\%$ lower confidence bound for a parameter θ such that the minimum probablity of correct coverage is indeed γ. Similarly, suppose that $\hat{U}(\gamma)$ is a $100\gamma\%$ upper confidence bound for a parameter θ such that the minimum probablity of correct coverage is indeed γ. The probability that either one of these respective bounds falls on the wrong side of θ is at most $1 - \gamma$ in each case. Assuming $\gamma > .5$ we typically have $\hat{L}(\gamma) \le \hat{U}(\gamma)$. It follows that $\hat{U}(\gamma) < \theta$ and $\hat{L}(\gamma) > \theta$ are mutally exclusive events so that the probability of $\theta \notin [\hat{L}(\gamma),\ \hat{U}(\gamma)]$ is at most $2(1 - \gamma)$ and thus the probability of $\theta \in [\hat{L}(\gamma),\ \hat{U}(\gamma)]$ is at least $1 - 2(1 - \gamma) = 2\gamma - 1 > 0$.

If the desired interval coverage probability is $\tilde{\gamma}$ we equate $\tilde{\gamma} = 2\gamma - 1$ and find that we should use $\gamma = (\tilde{\gamma} + 1)/2$ in the construction of the bounds, i.e., use $[\hat{L}((\tilde{\gamma} + 1)/2),\ \hat{U}((\tilde{\gamma} + 1)/2)]$ as the interval with coverage probability at least $\tilde{\gamma}$.

The reason why the minimum coverage probability for this interval may be higher than the targeted value $\tilde{\gamma}$ is that the θ values at which the respective bounds achieve their minimum coverage probability of γ may not be the same for both bounds.

7 Inverse Probability Solving

At times the question is asked: what is the smallest k such that $P(X \le k) \ge \gamma$, where X is a binomial random variable with parameters n and p and γ is a desired probability level. Clearly $P(X \le k)$ increases to one as k reaches n, so there is a smallest such number k. The question was asked[7] whether one can solve this problem quickly in Excel without much iteration.

A similar problem arises when Y is a Poisson random variable with mean λ. What is the smallest k such that $P(Y \le k) \ge \gamma$.

The answer to the binomial problem is simple: yes, there is such a function in Excel. It is called `CRITBINOM` and

$$\texttt{CRITBINOM}(n, p, \gamma)$$

gives the smallest k such that $P(X \le k) \ge \gamma$. For example,

$$\texttt{CRITBINOM}(100, .1, .8) = 12$$

and one verifies that

$$\texttt{BINOMDIST}(12, 100, .1, TRUE) = .80182$$

while

$$\texttt{BINOMDIST}(11, 100, .1, TRUE) = .70303\ .$$

The answer to the Poisson problem using Excel is not so direct but can be finessed by using `CRITBINOM`. This is possible since the Poisson distribution with mean λ is a very good approximation to the binomial distribution with parameters n and p with $\lambda = np$, provided p is very small. For fixed λ this means that we should choose n very large, say $n = 1000$ or $n = 10000$ and $p = \lambda/n$. Since this is an approximation it may not yield the exact solution k but one can then check the actual probability using `POISSON` in Excel.

As an example, suppose we have $\lambda = 4$ and $\gamma = .8$. Then

$$\texttt{CRITBINOM}(n, \lambda/n, \gamma) = \texttt{CRITBINOM}(1000, 4/1000, .8) = 6\ .$$

We check the actual probability using the cumulative distribution function `POISSON` in Excel, namely:

$$\texttt{POISSON}(6, 4, TRUE) = .889326$$

while

$$\texttt{POISSON}(5, 4, TRUE) = .78513\ .$$

Thus we were successful with the initial value given by `CRITBINOM`.

[7] by Dale E. Robinson

References

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