Pairwise vs. Three-way Independence

This is a very classic example, reported in any book on Probability:

Example 1. We throw two dice. Let A be the event "the sum of the points is 7", B the event "die #1 came up 3", and C the event "die #2 came up 4". Now, $P[A] = P[B] = P[C] = \frac{1}{6}$. Also,

$$P[A \cap B] = P[A \cap C] = P[B \cap C] = \frac{1}{36}$$

so that all events are pairwise independent. However,

$$P[A \cap B \cap C] = P[B \cap C] = \frac{1}{36}$$

while

$$P[A]P[B]P[C] = \frac{1}{216}$$

so they are not independent as a triplet.

First, note that, indeed, $P[A \cap B] = P[B \cap C] = \frac{1}{36}$, since the fact that A *and* B occurred is the same as the fact that B and C occurred.

Example 2. Another example is the case of Ω consisting of four equally likely points, a_1, a_2, a_3, a_4. Let $A = \{a_1, a_2\}$, $B = \{a_2, a_3\}$, $C = \{a_3 a_1\}$. The three are not independent, but they are pairwise.

However, it is also true that, as long as we consider only specific events (that is, we don't take into consideration their complements, or, more generally, other members of their algebra), that mutual (3-way) independence *does not imply pairwise independence*!

Here is a somewhat trivial example:

Example 3. Let $P[A] = p$, $P[B] = q$, $P[A \cap B] \neq pq$, $P[C] = 0$ then, trivially,

$$P[A \cap B \cap C] \leqslant P[C] = 0, \text{ and } P[A]P[B]P[C] = 0$$

but A and B are not pairwise independent.

A less trivial example is the following:

Example 4. Consider the toss of two distinct dice. The sample space is partitioned into equally likely events of the form (i, j), where i and j are the points on the first, respectively second die. Obviously, $P[(i, j)] = \frac{1}{36}$. Now, consider the three events

$$A_1 = ''i = 1, 2, \text{ or } 3'' \quad A_2 = ''i = 3, 4, \text{ or } 5'' \quad A_3 = i + j = 9$$

We have

$$A_1 \cap A_2 = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$$
$$A_1 \cap A_3 = \{(3,6)\}$$
$$A_2 \cap A_3 = \{(3,6),(4,5),(5,4)\}$$
$$A_1 \cap A_2 \cap A_3 = \{(3,6)\}$$

We have the following probabilities:

$$P[A_1] = P[A_2] = \frac{1}{2}, P[A_3] = \frac{1}{9}$$
$$P[A_1 \cap A_2 \cap A_3] = \frac{1}{36} = P[A_1]P[A_2]P[A_3]$$

but

$$P[A_1 \cap A_2] = \frac{1}{6} \neq \frac{1}{4}$$
$$P[A_1 \cap A_3] = \frac{1}{36} \neq \frac{1}{18}$$
$$P[A_2 \cap A_3] = \frac{1}{12} \neq \frac{1}{18}$$

Note, referring to Example 2, that $P[C^c] = 1$, so that $P[C^c \cap A \cap B] = P[A \cap B] \neq 1 \cdot p \cdot q$, so that considering the complement of one of the sets makes the new triplet dependent. Similarly, referring to Example 3, $P[A_3^c] = \frac{8}{9}$, and

$$A_1 \cap A_2 \cap A_3^c = \{(3,1),(3,2),(3,3),(3,4),(3,5)\}$$

which has probability $\frac{5}{36} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{8}{9} = \frac{2}{9}$. Note that this fact does not apply to pairs of events:

Fact: If A is independent of B, then so are, pairwise, A^c and B, A and B^c, and A^c and B^c. That's because, for example, $P[A \cap B^c] = P[A \cap (\Omega \backslash B)]$ and $P[A \cap (\Omega \backslash B)] + P[A \cap B] = P[A]$, so $P[A \cap B^c] + P[A]P[B] = P[A]$, hence $P[A \cap B^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]$. Similarly for the other cases.

This points to a better definition of independence of multiple events:

Theorem: *Suppose events A, B, C satisfy the conditions*

$$P[X \cap Y \cap Z] = P[X]P[Y]P[Z]$$

where X, Y, Z are, respectively, A, or A^c, B, or B^c, and C, or C^c. Then they are also pairwise independent. The result extends to any finite collection of events, in an obvious way.

Proof: We can write $P[A \cap B] = P[(A \cap B \cap C) \cup (A \cap B \cap C^c)] = P[A]P[B]P[C] + P[A]P[B]P[C^c]$, because the two parts are disjoint. This is equal to $P[A]P[B](P[C] + P[C^c]) = P[A]P[B]$. All other cases are treated in the same way.

Remark: Checking all intersections of the sets and their complement can be seen as checking independence of all couples built from the *minimal algebra generated by each of the events*, which, for an event A, is the collection $\{A, A^c, \Omega, \emptyset\}$. Of course, trivially, Ω, and $\emptyset$ are independent of any event.

While some scholars have looked at the taxonomy of events that are $k - $ independent, but not $h - $ independent for $h < k$, this is not a very exciting subject, since, in considering independence and, more generally, conditional probabilities, it is much more significant to look at all events in the algebras the events belong to naturally - at the very least the ones generated by each event and its complement.