Predicting the Vortex Viscosity of a Ferrofluid

By

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Abstract

The vortex viscosity of a ferrofluid is calculated using a two-dimensional approach using Comsol Multiphysics to perform the numerical calculations. This approach is a simplification of one used Feng, et al., where the ratio of vortex viscosity and the shear viscosity of plain fluid is a function of the volume fraction of the ferrofluid. One simulation was performed with a moderate number of particles in the domain, and the result was a vortex viscosity of .0179, 2200% smaller than the corresponding value by Feng, et al.

Introduction

A ferrofluid is a suspension containing particles that are subject to magnetic forces. A magnetic force applied to a ferrofluid causes both translational and rotational movement of the particles creating movement in the fluid. This magnetic force creates motion in an otherwise quiescent fluid. The vortex viscosity of the ferrofluid is the stress on the fluid caused by the particle motion, which in turn, is caused by an external magnetic field. A paper presented by Feng et al. suggests a method to determine the vortex viscosity of a fluid such as a ferrofluid. This method uses a boundary element method (BEM) to calculate the shear viscosity and the vortex viscosity. The simulations are based an a three-dimensional cylindrical domain with 160 spherical particles, and the results were given as μ_v/μ_0 versus the volume fraction of the fluid, ϕ. The purpose of this paper is to investigate another numerical method to determine the vortex viscosity of a ferrofluid, using Comsol Multiphysics. The method proposed is to use a two-dimensional square domain in Comsol representing the suspension, with circles that representing infinitely long cylinders placed inside the domain. These cylinders will be rotated in the domain, imparting motion to the suspension.

Theory

The vortex viscosity is determined by the following relationship.

$$\mu_v = \frac{\phi \overline{G}}{4\pi a^2 (\Omega - \gamma/2)} \qquad (1)$$

$$\overline{G} = \frac{1}{N}\sum_{i=1}^{N} G^i \qquad (2)$$

Where $\overline{G}$ is the average external force on each particle. ϕ is the volume fraction, “a” is the radius of each particle, Ω is the angular velocity of each particle and γ is the effective shear rate on the suspension. This relationship is derived from the three conservation equations for the flow of an incompressible fluid: the conservation of mass, of linear momentum, and of angular momentum. However, in the equation for the conservation of linear momentum, the stress tensor is separated into its symmetric and antisymmetric sections, and the antisymmetric portion is expressed in terms of $\underline{\underline{\tau}}_X$, which is also a term in the conservation of angular momentum equation. These three equations are solved simultaneously to express $\underline{v}$ in terms of $\mathbf{G}$, $\mathbf{\Omega}$, and the vortex viscosity. [1] For the purpose of these simulations, the fluid is an incompressible Newtonian fluid and density is set to zero to neglect inertial terms.

Setting up the Simulation

As mentioned above, a square domain is drawn in Comsol Multiphysics. Twenty-five circles (representing infinitely long cylinders) are drawn inside of the domain and subtracted from the domain when the square and circles are merged into one domain. A picture of the domain is show in Figure 1.

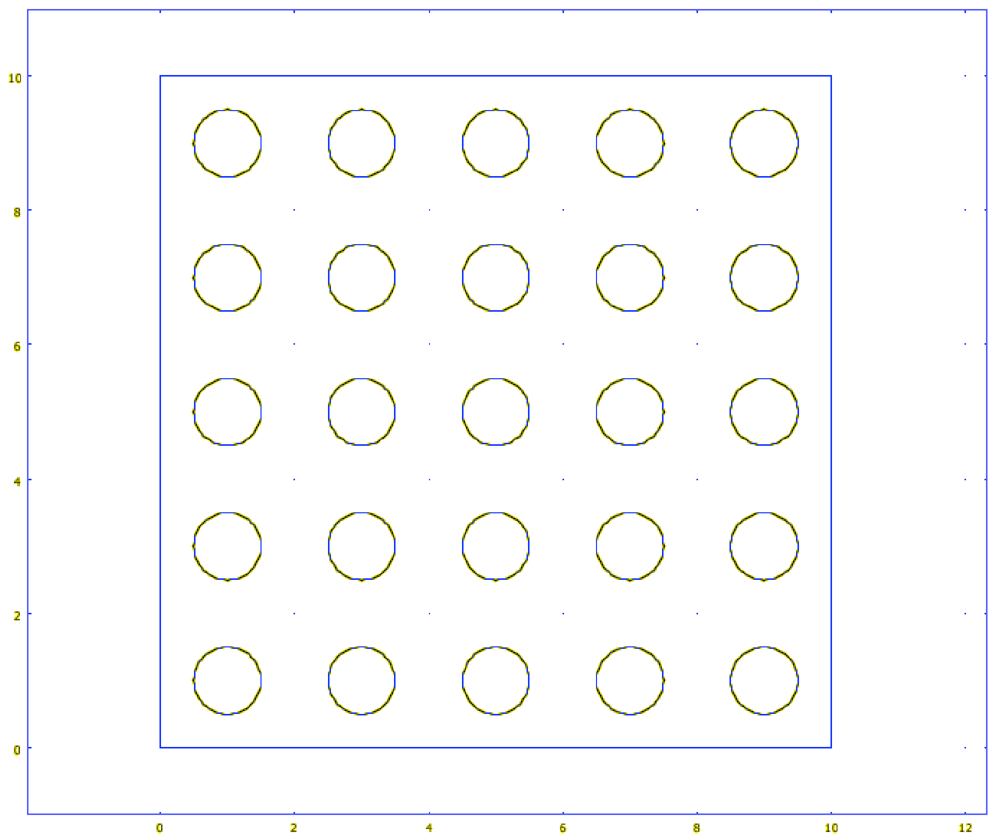

Figure 1. The square domain (in Comsol), 10x10 in size.

The parameters for the domain are: density is zero and the fluid (without the particles) viscosity is one. The boundary conditions for the domain represent Couette flow—the top boundary is slip/symmetry, the bottom boundary is no slip, the right and left boundaries are periodic and set for pressure equals zero.

The cylinders are set to rotate by filling in appropriate equations for the $\underline{u}$ and $\underline{v}$ velocities. These equations derived the following way:

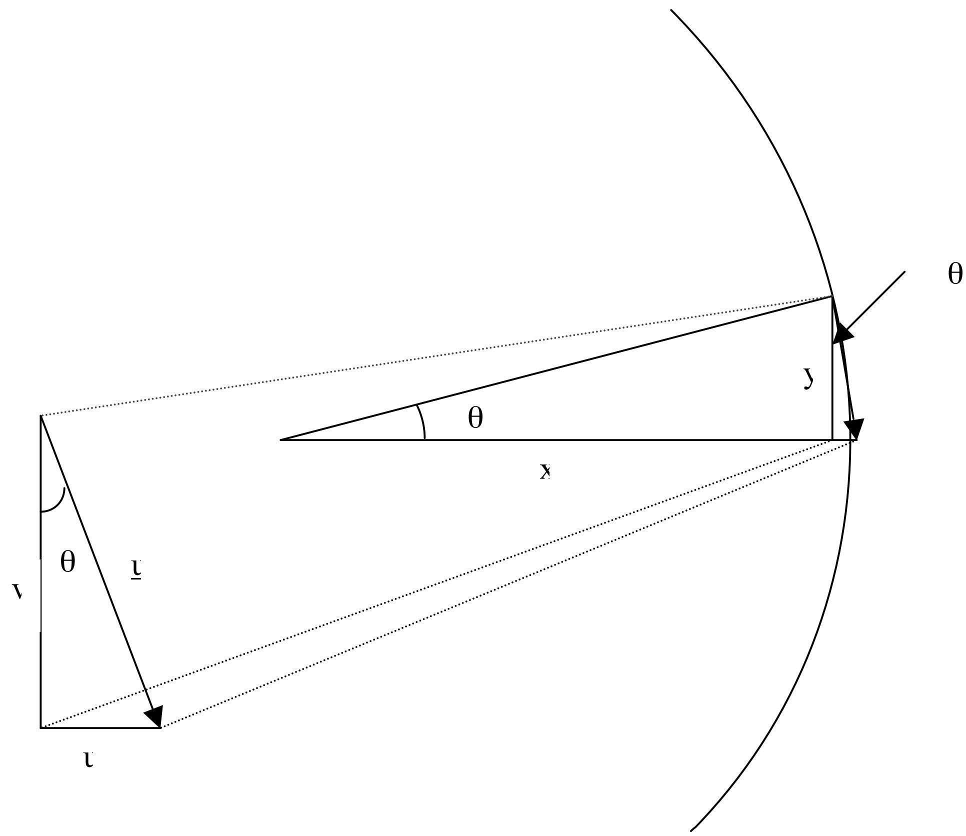

Figure 2. A graphical representation of the relationship between u,v and x,y in the cylinders created in Comsol. This representation is used to derive the u and v components for the velocity u for Comsol.

The center of the cylinder is at (0,0). The following four equations represent simple trigonometric identities applied to Fig. 2

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad (3)$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad (4)$$

$$\sin\theta = \frac{u}{\sqrt{u^2 + v^2}} \qquad (5)$$

$$\cos\theta = \frac{v}{\sqrt{u^2 + v^2}} \qquad (6)$$

Let

$$\sqrt{u^2 + v^2} = m \qquad (7)$$

then substitute “m” into the denominator for Eq. (5) so

$$u = m\sin\theta \tag{8}$$

in (x,y) coordinates, substitute

$$r = \sqrt{x^2 + y^2} \tag{9}$$

into Eq. (3) and the result is

$$\sin\theta = \frac{y}{r} \tag{10}$$

which, when substituted into Eq. (8) is

$$u = m\frac{y}{r} \tag{11}$$

Repeat the procedure with equations (4) and (6), and in order to create a clockwise direction, the v velocity is negative. The result is

$$v = -m\frac{x}{r} \tag{12}$$

However, in Eqs (11) and (12) the x and y need to refer to the global position. Therefore, the angular velocity of each circle was represented by the set of equations:

$$u = \frac{m(y - k_i)}{r} \tag{13}$$

$$v = \frac{m(x - h_i)}{r} \tag{14}$$

where (h_i,k_i) is the center of each circle in the x, y frame. The variable “m” refers to the magnitude of the linear velocity of each cylinder.

With all of the necessary parameters and inputs, the simulation is solved.

In order to determine the value of $\overline{G}$, the value of G_i for each cylinder must be calculated. G_i is the torque on each cylinder. The torque on each particle is calculated by the tangential force applied to the particle cross the radius of the particle. To calculate the torque on each cylinder, the tangential force at each point around the cylinder must be calculated. In order to calculate this by Comsol, the following equations must be entered into the subdomain. The following derivation determines the force in the θ direction.

The force on the cylinder can be written as

$$\underline{\underline{\tau}} \bullet \underline{n} = \underline{\underline{\tau}} \bullet \underline{e}_r \qquad (15)$$

The force in the θ direction is expressed as

$$\underline{F}_\theta = \underline{e}_\theta \bullet \underline{\underline{\tau}} \bullet \underline{e}_r \qquad (16)$$

where

$$\underline{e}_\theta = -\sin\theta \underline{e}_x + \cos\theta \underline{e}_y \qquad (17)$$

$$\underline{e}_r = \cos\theta \underline{e}_x + \sin\theta \underline{e}_y \qquad (18)$$

$$\underline{\underline{\tau}} = \underline{e}_x \underline{e}_x \tau_{xx} + \underline{e}_x \underline{e}_y \tau_{xy} + \underline{e}_y \underline{e}_x \tau_{xy} + \underline{e}_y \underline{e}_y \tau_{yy} \qquad (19)$$

Substitute Eqs. (17), (18), and (19) into (16) and the resulting equation after simplification is

$$\underline{F}_\theta = \left[-\sin\theta \underline{e}_x + \cos\theta \underline{e}_y\right] \bullet \underline{\underline{\tau}} \bullet \left[\cos\theta \underline{e}_x + \sin\theta \underline{e}_y\right] \qquad (20)$$

Assume $\tau_{sy}=\tau_{yx}$, perform the operations as shown, and Eq. (20) simplifies to

$$F_\theta = \sin\theta\cos\theta\left(\tau_{yy} - \tau_{xx}\right) + \tau_{xy}\left(2\cos^2\theta - 1\right) \qquad (21)$$

The supporting expressions to enter into the subdomain expressions are

$$\tau_{xy} = -\mu\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \qquad (22)$$

$$\tau_{yy} = -2\mu\frac{\partial v}{\partial y} \qquad (23)$$

$$\tau_{xx} = -2\mu\frac{\partial u}{\partial x} \qquad (24)$$

$$\sin\theta = \frac{x - h_i}{\sqrt{\left(x - h_i\right)^2 + \left(y - k_i\right)^2}} \qquad (25)$$

$$\cos\theta = \frac{y - k_i}{\sqrt{\left(x - h_i\right)^2 + \left(y - k_i\right)^2}} \qquad (26)$$

where (h_i,k_i) represents the (x,y) coordinates for the center of each cylinder. Finally, the force around each cylinder is calculated as follows.

$$\int_0^{2\pi} \underline{\tau}_i = \int_0^{2\pi} \underline{F}_i r_i = G_i \qquad (27)$$

Two more details necessary to calculate μ_v is the effective shear rate, γ, and the angular velocity, Ω. The shear rate is calculated by integrating the x-velocity over the x-axis and dividing it by the depth of the domain. The angular velocity is given by the relationship,

$$\Omega = r\underline{v} \qquad (28)$$

where r is the radius of the cylinder, and $\underline{v}$ is the linear velocity of the upper surface.

Results

The simulation is solved for 127,160 degrees of freedom. The value for "m" is 1, and the radius of each cylinder is .5. The resulting velocity field is shown below in Figure 3.

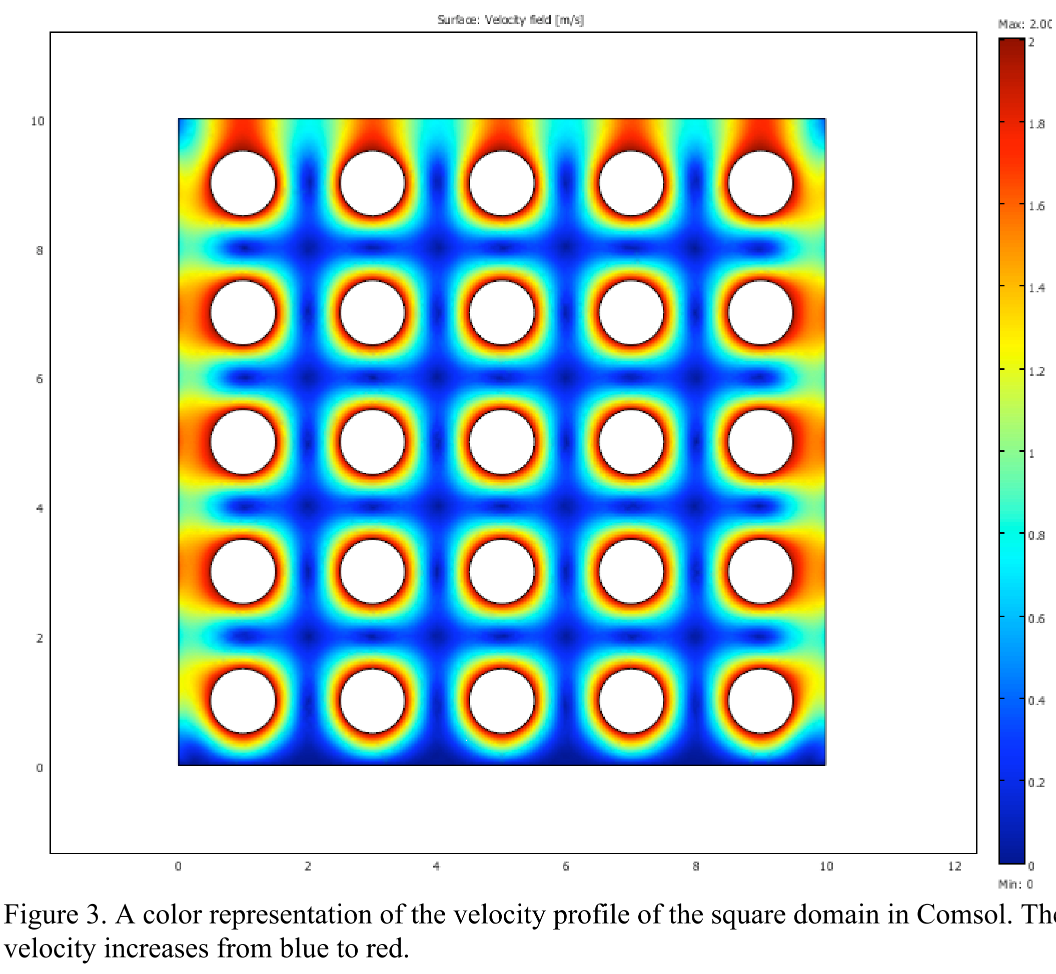

Figure 3. A color representation of the velocity profile of the square domain in Comsol. The velocity increases from blue to red.

It is clear that the rotational motion of the cylinders has created a velocity along the x-axis. A graph of the velocity along the x-axis is shown below.

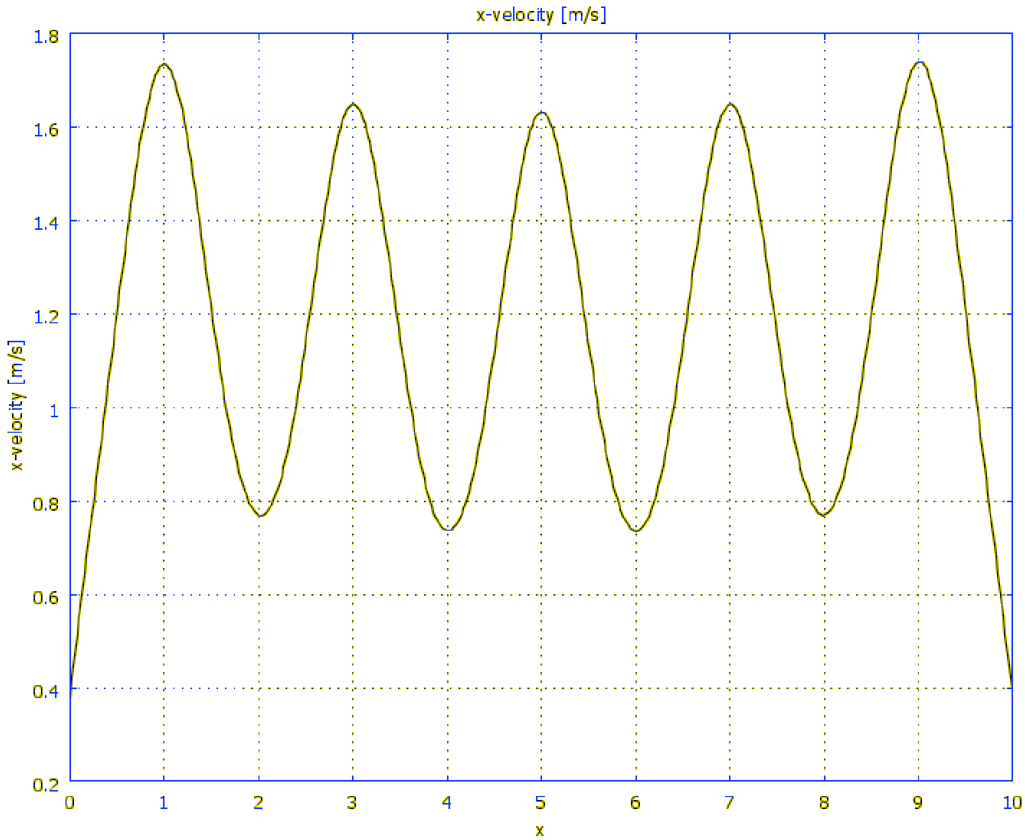

Figure 4. This graph represents the x-velocity along the x-axis of the square domain. Note: although the graph shows units, all calculations are performed as if there were no units.

When the x-velocity is integrated along the x-axis, the value is 11.77, and when divided by the depth of the domain, the effective shear rate is 1.17.

Each cylinder is numbered, with one at the bottom left, counting right until the end of the row, and then starting the new row right above number one. Number twenty-five is at the top right corner. The following two tables summarize the data needed for the calculation of the vortex viscosity.

Table 1. The calculated torque on each cylinder in the domain.

Cylinder Number	**Torque**	**Cylinder Number**	**Torque**
1	.1264	14	.5335
2	.1454	15	.6327
3	.1422	16	.6234
4	.1449	17	.5325
5	.1277	18	.5388
6	.6217	19	.5449
7	.5337	20	.6224
8	.5486	21	.1355
9	.5395	22	.1882
10	.6254	23	.1922
11	.6340	24	.1967
12	.5458	25	.1291
13	.5408	**Average**	**.4058**

Table 2. Summary of results for the quantities needed to calculate the vortex viscosity

Quantity	**Value**
γ	1.17
Ω	2
Average G	.4058
ϕ	.196

When these values are plugged into Eq. (1), the result is **.0179.** With a shear viscosity of one, the ratio μ_v/μ_0 is also .0179 at a volume fraction of .196. The vortex viscosity determined by Feng, et al. at the same volume fraction is about .4. This shows an error with respect to Feng, et al is 2200%, which indicates that there is a severe error in the method proposed in this paper.

The large error in this method may come from several different sources. One source may be that the derivation for the force in the θ direction is imperfect. This is most likely source when ignoring the crudeness of the method itself. This is because all of the other quantities are simple and basic to calculate. In addition, a change in the value of the average G would directly influence the value of the vortex viscosity, while a change in the effective shear rate would only indirectly affect the vortex viscosity. However, it is possible that using a two-dimensional method is inappropriate and too crude of a method. In addition, only 25 particles were used in the simulation and 160 particles were used in the method that Feng, et al. used. It is possible that with more particles and different volume fractions, this type of simulation might yield results that are closer to the literature results.

Conclusion

This two-dimensional numerical method with Comsol Multiphysics yields results that are inconsistent with the results obtained by Feng, et al. Although several sources of error exist, with only one data point, it is difficult to define those sources. However, because little data was obtained, future research with different volume fractions and number of particles may yield more promising results.

Reference

Feng, S., Graham, A.L., Abbott, J.R. & Brenner, H. 2006. Antisymmetric stresses in suspensions: vortex viscosity and energy dissipation. *J. Fluid Mech* **563**, 97-122.

Acknowledgements

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