Flow of Water in Partially-Saturated Soils

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Introduction

The purpose of this research is to create a model for the flow of water in partially-saturated soils in one dimension and two dimensions without the pressure head term, and one dimension with the pressure head included. By creating the model one can predict the flow of fluid through different mediums therefore making educated decisions.

Background Information

In order to approach this problem, the Darcy's Law was applied, which is used to describe the flow of groundwater through porous material, such as soil. Darcy's Law is defined below.

$$q = -\mathrm{K}\nabla h \qquad \text{(Eq.1)}$$

K is hydraulic conductivity, the rate of water flowing through porous medium, and h is potential energy of groundwater. Darcy's Law shows similarity with Fourier's Law, defined below.

$$q = -\kappa\nabla T \qquad \text{(Eq. 2)}$$

k is thermal conductivity and T is temperature. Both Eq. 1 and 2 carry a medium property and a driving force.

Methods

- **Partially Saturated Soil in One and Two Dimensions**

Calculations

For all three cases, Darcy's Law was used which were rearranged in terms of variables needed. For one and two dimensions, the same equations were used. Eq. 3 and 4 are simplified in terms of saturation and permeability (the ability of the material to transmit fluid).

$$q_w = -\frac{\kappa_w}{\mu_w}\left(\frac{\partial p_w}{\partial x} - \rho_w g\right) \qquad \text{(Eq. 3)}$$

k = permeability, μ = viscosity of water, ρ = density of water and g = gravity. The subscript w represents water.

$$-\frac{dS}{dp_c}\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(k_r\frac{\partial p}{\partial x}\right) - \frac{\partial k_r}{\partial x} \qquad \text{(Eq. 4)}$$

S = saturation, p = pressure, subscript r is relative and c is capillary. The subscript w was dropped. For simplicity, the problem was solved without the gravity function. The following equation is shown without the gravity term:

$$-\frac{dS}{dp_c}\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(k_r \frac{\partial p}{\partial x}\right) \qquad \text{(Eq. 5)}$$

The following equations are to be used with Eq. 5:

$$k_r = \frac{1}{1+\left(\frac{p_c L}{B}\right)^{\lambda}} \; , \; \frac{S - S_r}{1 - S_r} = \frac{1}{1+\left(\frac{p_c L}{A}\right)^{\eta}} \qquad \text{(Eq. 6, 7)}$$

S_r, A, B, η, and λ are depended on the type of soil. Table 1 shows the values for a typical soil.

Table 1. Parameters for Typical Soil

S_r	0.32
A	231.0
B	146.0
η	3.65
λ	6.65

The following equations are rearranging and deriving Eq. 7 to insert into Eq. 5:

$$S - S_r = \frac{1 - S_r}{1+\left(\frac{p_c L}{A}\right)^{\eta}} \qquad \text{(Eq. 8)}$$

$$S = \frac{1 - S_r}{1+\left(\frac{p_c L}{A}\right)^{\eta}} + S_r \qquad \text{(Eq. 9)}$$

$$S = (1 - S_r)\left[1+\left(\frac{p_c L}{A}\right)^{\eta}\right]^{-1} + S_r \qquad \text{(Eq. 10)}$$

$$\frac{dS}{dp_c} = \frac{-(1 - S_r)\eta\left(\frac{p_c L}{A}\right)^{\eta-1}\frac{L}{A}}{\left[1+\left(\frac{p_c L}{A}\right)^{\eta}\right]^2} \qquad \text{(Eq. 11)}$$

(Note added in proof: the calculations did not have L/A in Eq. (11). Since this is not far from one this changes the time scale only slightly. For the boundary conditions for one dimension, we assume at x = 0 p = 0, but because the model did not converged it was changed to p = -0.001cm. At x = L $\frac{\partial p}{\partial x} = 0$, and L = 100cm. At initial conditions p (x,0) = -100cm, -200cm and -300cm. The same boundary conditions and initial conditions were used for two dimensions, except the pressure flux was zero for the surrounding soil excluding the surface where water enters.

Schematics

The following two figures are shown for one dimension and two dimensions. For one dimension, water enters from the surface of the soil then penetrates through but only in the y direction. In the two dimension situation, however, water moves in both x and y direction.

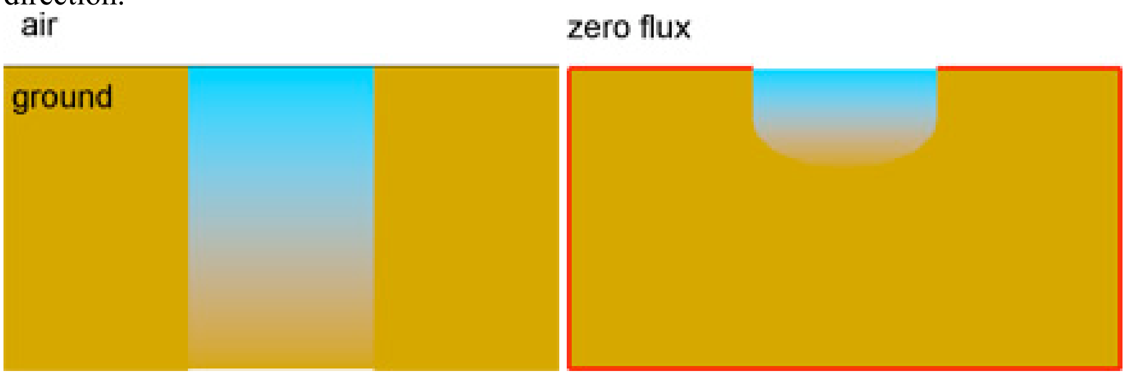

Figure 1. Schematic of water flow in soil in one and two dimensions

- **Partially Saturated Soil in One Dimension With Pressure Head**

Calculations

This time the pressure head is not neglected, thus the following equation is modified from Darcy's Law to show appropriate variables,

$$C^* \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial x} K \text{, where } C^* = \frac{\partial \theta}{\partial h} \qquad \text{(Eq. 12).}$$

Here θ = moisture content, t = time, K = hydraulic conductivity and h = pressure head. The following equations are used to describe the hydraulic properties of different soils:

$$\theta = \theta_r + \frac{(\theta_s - \theta_r)}{\left[1 + (\alpha |h|)^n\right]^m} \text{, where } m = 1 - \frac{1}{n} \qquad \text{(Eq. 13)}$$

$$K = K_s \Theta^{1/2}\left[1-\left(1-\Theta^{1/m}\right)^m\right]^2, \text{ where } \Theta = \frac{\theta-\theta_r}{\theta_s-\theta_r} \qquad \text{(Eq. 14)}$$

And α, n, m = parameters of soil. Table 2 shows the parameters used for various soils.

Table 2. Parameters for soil properties

Soil No.	θ_r (cm^3/cm^3)	θ_s (cm^3/cm^3)	α (cm^{-1})	n (-)	K_s (cm/day)	S_s (cm^{-1})
1. (Clay Loam)	.20	.54	.008	1.8	25.	4.10^{-7}
2. (Dense Layer)	.25	.40	.009	3.	10.	5.10^{-8}
3. (Loamy Sand)	.17	.47	.010	2.	75.	1.10^{-7}
4.	.1611	.4611	.01036	2.178	132.8	1.10^{-7}
5.	.15	.45	.0108	2.4	205.	1.10^{-7}
6.	.14	.44	.0112	2.6	270.	1.10^{-7}
7.	.1311	.4311	.01156	2.778	327.8	1.10^{-7}
8.	.1244	.4244	.01182	2.911	371.1	1.10^{-7}
9. (Sand)	.12	.42	.012	3.0	400.	1.10^{-7}

The red lines represent the four types for soil used in the model. Please see Figure 2 for additional information. The following equations are deriving equations to use Eq. 12.

Deriving moisture content equation:

$$\theta = \theta_r + (\theta_s - \theta_r)\left[1+\left(\alpha|h|\right)^n\right]^{-m} \qquad \text{(Eq. 15)}$$

$$\frac{\partial\theta}{\partial h} = \frac{-m(\theta_s-\theta_r)}{\left[1+\left(\alpha|h|\right)^n\right]^{m+1}}\left(n\left(\alpha|h|\right)^{n-1}\right)\alpha\frac{d\,|h|}{dh}, \text{ where } \frac{d\,|h|}{dh} = -1 \qquad \text{(Eq. 16)}$$

Deriving hydraulic conductivity equation:
From Eq. 12:

$$\frac{\partial K}{\partial x} = \frac{\partial K}{\partial \Theta}\frac{\partial \Theta}{\partial h}\frac{\partial h}{\partial x} \qquad \text{(Eq. 17)}$$

$$\frac{\partial K}{\partial \Theta} = \frac{1}{2}K_s\Theta^{-1/2}\left[1-\left(1-\Theta^{1/m}\right)^m\right]^2 + 2K_s\Theta^{1/2}\left[1-\left(1-\Theta^{1/m}\right)^m\right]\left[-m\left(1-\Theta^{1/m}\right)^{m-1}\right]\left(\frac{-1}{m}\Theta^{1/m-1}\right)$$
(Eq. 18)

For the boundary conditions, at x = 0 $-K\frac{\partial h}{\partial x}+K=25$ and at x = L $\frac{\partial h}{\partial x}=0$, where L = 170 cm.

Schematic

Figure 2 is the schematic of the soil profile. As seen in the figure the soil is 170cm deep, with nine types of soil. For simplicity four soils were used as highlighted in Table 2.

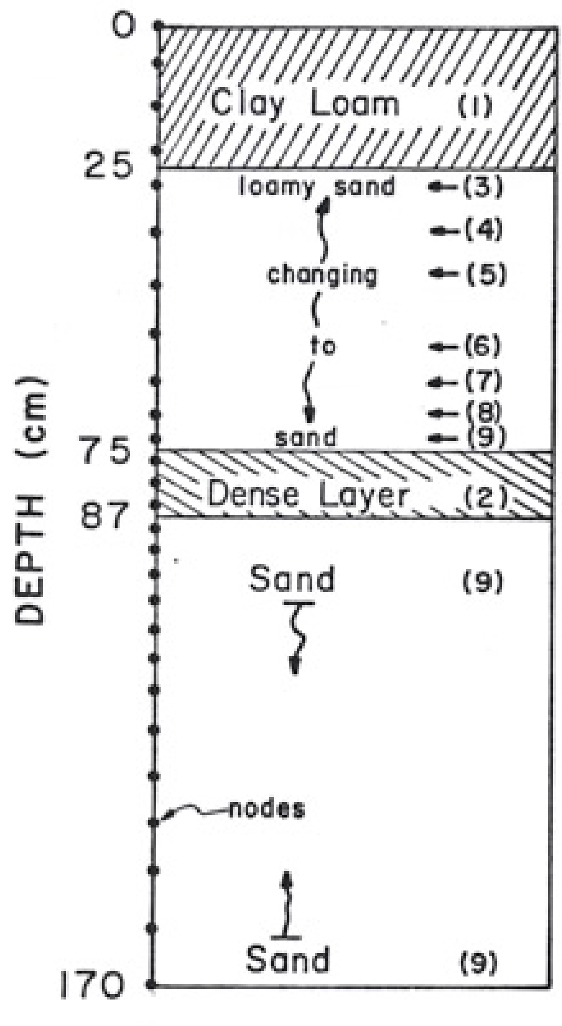

Figure 2. Profile of soil profile

Results and Discussions

- **Partially Saturated Soil in One Dimension**

Figures 3a and 3b are developed using Eq. 3 to 11 and Comsol Multiphysics. Figure 3a is at initial pressure of -200cm while Figure 3b is at -300cm. The x-axis represents the normalized soil depth and the y-axis is the normalized axis with the ratio of pressure over the initial pressure.

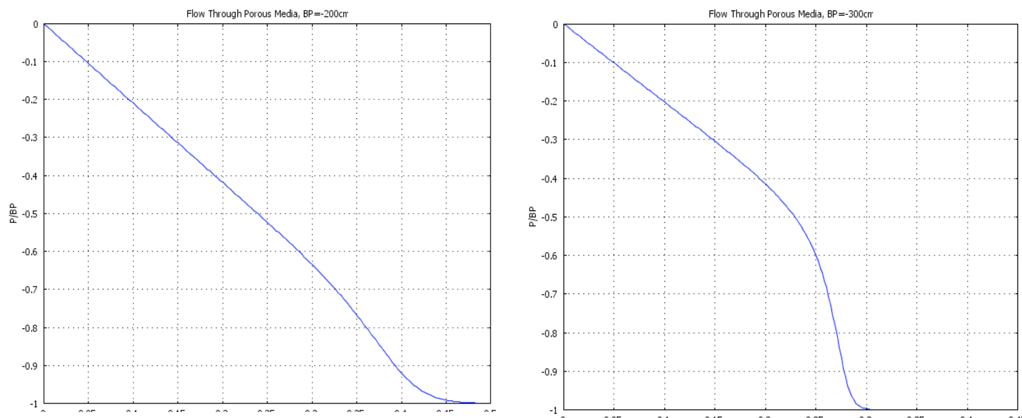

Figure 3. (a) Solution to flow through porous media, initial pressure = -200cm. (b) Solution to flow through porous media, initial pressure = -300cm

One way to make sure the values are correct was to compare the solutions of Figures 3a and 3b with Figure 4, which the results were from the published work of Professor Finlayson [1].

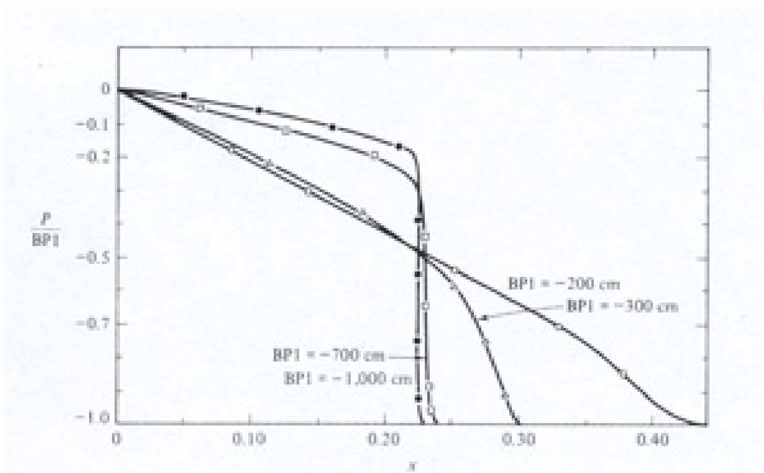

Figure 4. Solution to flow through porous media for various initial pressures

For Comsol, the line was x’ = x/L = 1, and mesh consisted of 15 elements, with number of degrees of freedom solved for 481. The solution time was 1.678s. The following three figures are from Comsol, and each window shows where the values were imputed. As seen in Figure 5, f = Eq. 11 and g = Eq. 6.

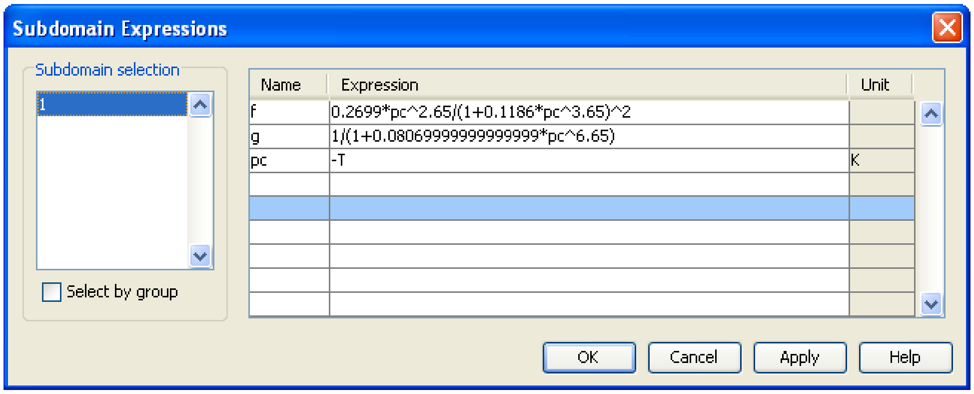

Figure 5. Defining equations in terms of variables for one dimension

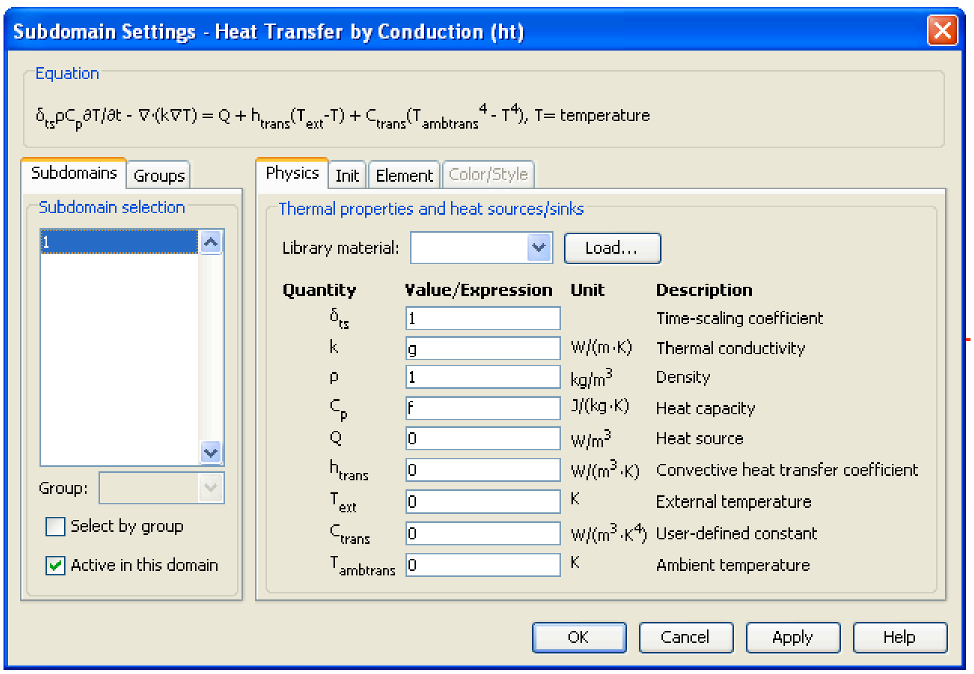

Figure 6. Using Subdomain Expressions to insert into overall equation for one dimension

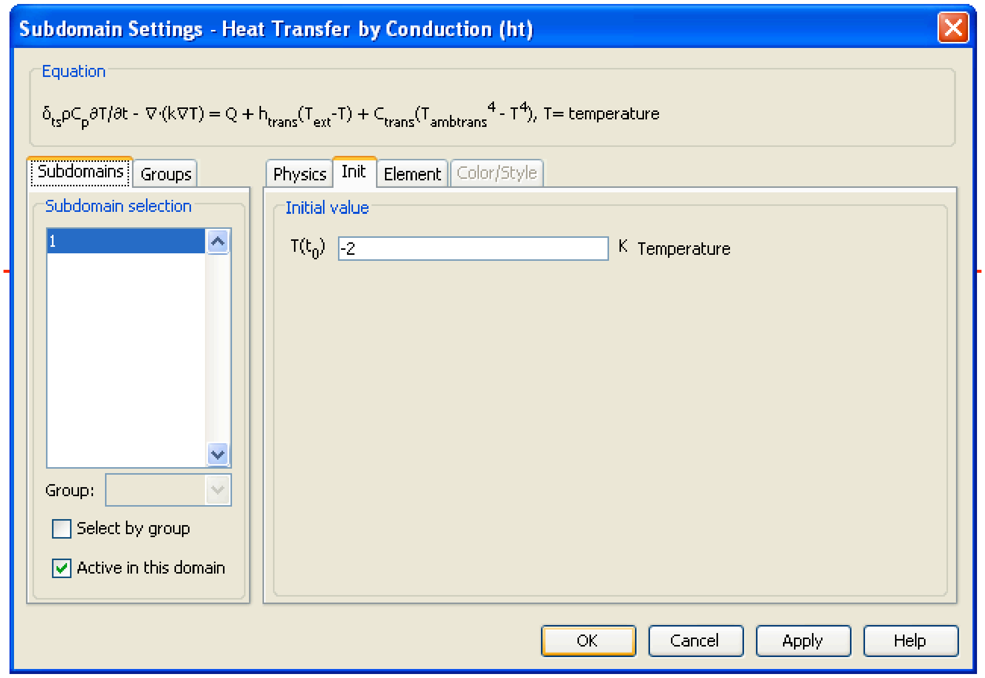

Figure 7. Initial pressure selection; shown with initial pressure = -200cm

- **Partially Saturated Soil in Two Dimensions**

The following figure shown is for a two dimensional case without the gravitational term. As expected, the initial pressure is -0.1cm at the entrance and there is a radial increase of pressure as soil depth increases.

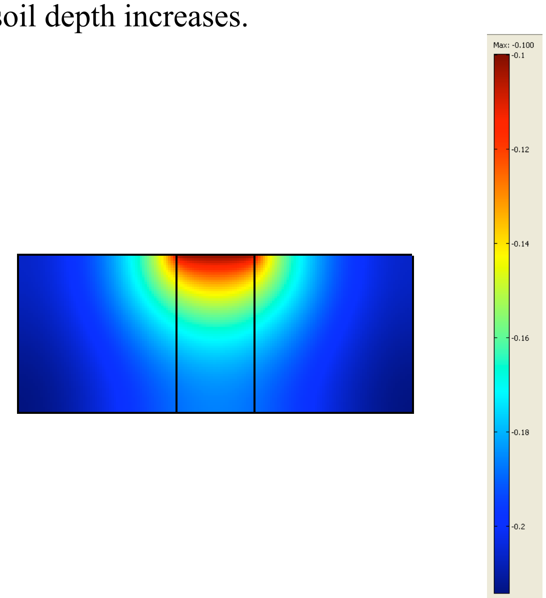

Figure 8. Pressure change for a two dimensional case

Because same equations were used as the previous case, Figures 5,6 and 7 apply for this case as well. The dimension of the box is normalized, with 0.4 by 1. The first and third box has a width of 0.4, and the second box has a width of 0.2. Mesh consisted of 12240 elements, number of degrees of freedom solved for was 24697, and the solution time was 62.89 s. To check the answers, the calculated values were compared to the calculated values of Comsol. Please see the Sample Calculations section.

- **Partially Saturated Soil in One Dimension With Pressure Head**

Figure 9 represents the pressure distribution calculated using Comsol. The x-axis represents the soil depth and the y-axis is the pressure head. The different line colors represent time, the first green line being at time = 0.1 day and the last blue line at time = 1 day. Notice at time = 0.1 water only reaches a soil depth of 30 cm. But as time progresses, the colored lines begin to reach further down the depth, as well as developing an expected pressure profile. This is because the model is for four different types of soil, each with different permeability. As the water moves from one type of soil to the next, the different properties create resistance or ease, depending on soil thus causing different change of pressure.

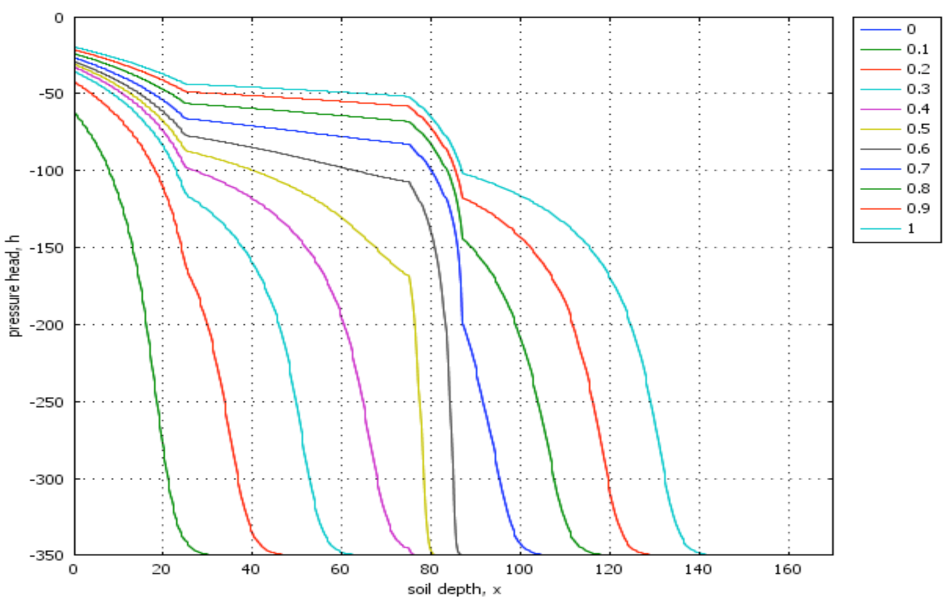

Figure 9. Distribution of pressure change calculated from Comsol

Figure 10 was obtained from the published work of van Genuchten [2], which has the same x and y axis representation as Figure 9. The multiple lines also represent water movement over time. Although Figure 9 does not replicate Figure 10 exactly, it can be concluded that the model works as predicted because it shows a similar trend. The minor differences could be caused from using only four types of soil instead of all nine.

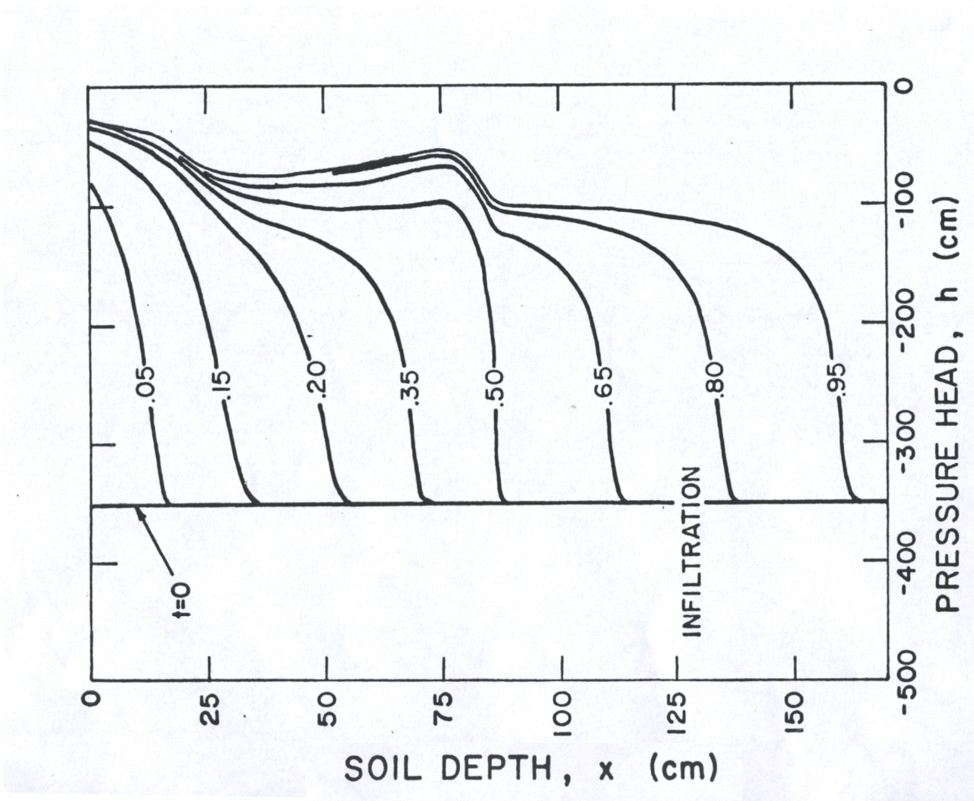

Figure 10. Distribution of pressure change

As shown in Figure 2, four lines were drawn in Comsol: one from 0 to 25cm, 25cm to 75cm, 75cm to 87cm, and 87 to 170cm. These four lines represent the different soils used and expressions were inserted as shown in Figure 11. Not shown are the constants used, shown in expressions f and g as tr, ts, a, n, and k. Each variable represents as shown in Table 2. Mesh consisted of 17 elements, number of degrees of freedom solved for was 35, and the solution time was 23.75 s.

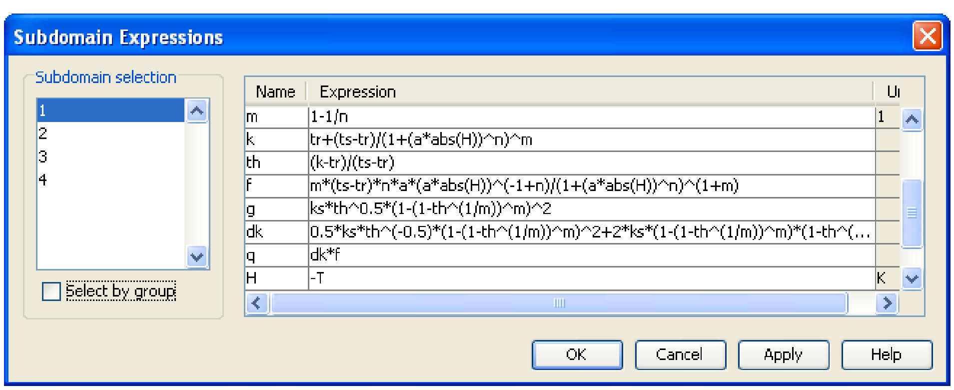

Figure 11. Defining equations in terms of variables for one dimension with pressure head

From Figure 12, f = 16, g = Eq. 14, and q = Eq. 17.

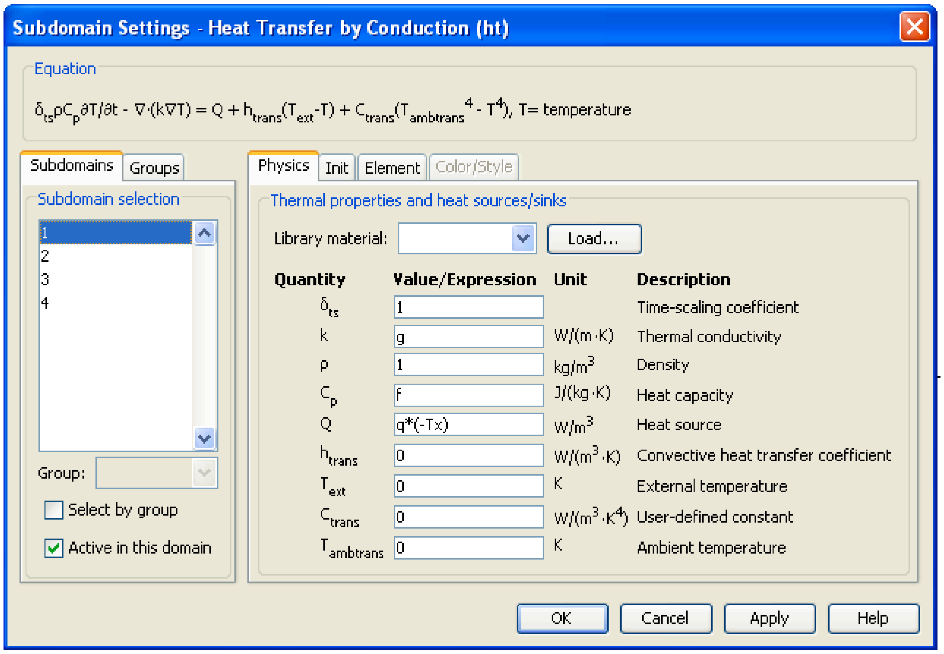

Figure 12. Using Subdomain Expressions to insert into overall equation for one dimension with pressure head

Conclusions

By comparing the model results with literature values and checking the values with the output values of Comsol, it is concluded that the three models work correctly. These models can be used to predict and analyze the movement of water or other fluids though different mediums.

Recommendations

Because not all three models use all the given values, another good check to see that the models work is to graph Figures 3a and 3b with more initial pressure and compare them to Figure 4. Also include all the nine soil variables to acquire accurate model as shown in Figure 10.

References

[1] Finlayson, Bruce A. Nonlinear Analysis in Chemical Engineering. New York: McGraw-Hill, 1980, Seattle: Ravenna Park, 2003.

[2] Finlayson, Bruce A. Numerical Methods for Problems with Moving Fronts. Seattle: Ravenna Park Publishing, Inc. 1992.

[3] van Genuchten, M. Th.. Mass Transport in Saturated-Unsaturated Media: One-Dimensional Solutions. August 1978.

Sample Calculations

- **Partially Saturated Soil in One Dimension**

Value: 1.999506 [K], Expression: pc, Position: (0.5)
Value: 0.273607, Expression: f, Position: (0.5)
Value: 0.109997, Expression: g, Position: (0.5)

f = Eq. 11, g = Eq. 6.

$$\frac{dS}{dp_c} = \frac{-(1-S_r)\eta\left(\frac{p_c L}{A}\right)^{\eta-1}}{\left[1+\left(\frac{p_c L}{A}\right)^{\eta}\right]^2} \qquad \text{(Eq. 11)}$$

Where $S_r = 0.32$, A = 231.0, B = 146.0, $\eta = 3.65$, $\lambda = 6.65$.

$$\frac{dS}{dp_c} = \frac{-(1-0.32)\times 3.65\times\left(\frac{1.999\times 100}{231.0}\right)^{3.65-1}}{\left[1+\left(\frac{1.999\times 100}{231.0}\right)^{3.65}\right]^2} = 0.27$$

$$k_r = \frac{1}{1+\left(\frac{p_c L}{B}\right)^{\lambda}} \qquad \text{(Eq. 6)}$$

$$k_r = \frac{1}{1+\left(\frac{1.999\times 100}{146.0}\right)^{6.65}} = 0.11$$

- **Partially Saturated Soil in Two Dimensions**

For first box:

Value: 0.185301 [K], Expression: pc, Position: (0.3,0)
Value: 0.00134, Expression: f, Position: (0.3,0)
Value: 0.999991, Expression: g, Position: (0.3,0)

$$\frac{dS}{dp_c} = \frac{-(1-S_r)\eta\left(\frac{p_c L}{A}\right)^{\eta-1}}{\left[1+\left(\frac{p_c L}{A}\right)^{\eta}\right]^2} \qquad \text{(Eq. 11)}$$

Where $S_r = 0.32$, $A = 231.0$, $B = 146.0$, $\eta = 3.65$, $\lambda = 6.65$.

$$\frac{dS}{dp_c} = \frac{-(1-0.32)\times 3.65\times\left(\frac{0.185\times 100}{231.0}\right)^{3.65-1}}{\left[1+\left(\frac{0.185\times 100}{231.0}\right)^{3.65}\right]^2} = 0.0013$$

$$k_r = \frac{1}{1+\left(\frac{p_c L}{B}\right)^{\lambda}} \qquad \text{(Eq. 6)}$$

$$k_r = \frac{1}{1+\left(\frac{0.185\times 100}{146.0}\right)^{6.65}} = 0.98$$

- **Partially Saturated Soil in One Dimension With Pressure Head**

For clay loam:

Value: 27.669365 [K], Expression: H, Position: (10)
Value: 0.444444 [1], Expression: m, Position: (10)
Value: 0.530444, Expression: k, Position: (10)
Value: 0.971894, Expression: th, Position: (10)
Value: 5.93597e-4, Expression: f, Position: (10)
Value: 12.393909, Expression: g, Position: (10)
Value: 164.289822, Expression: dk, Position: (10)

m = from Eq. 13, k = Eq. 13, th = from Eq. 14, f = Eq. 16, g = Eq. 14, dk = Eq. 18

$$m = 1 - \frac{1}{n} \qquad \text{(from Eq. 13)}$$

$$\theta = \theta_r + \frac{(\theta_s - \theta_r)}{\left[1+(\alpha|h|)^n\right]^m}, \text{ where } m = 1 - \frac{1}{n} \qquad \text{(Eq. 13)}$$

$$\frac{\partial \theta}{\partial h} = \frac{-m(\theta_s - \theta_r)}{\left[1 + (\alpha|h|)^n\right]^{m+1}} \left(n(\alpha|h|)^{n-1}\right)\alpha \frac{d\,|\,h\,|}{dh}, \text{ where } \frac{d\,|\,h\,|}{dh} = -1 \qquad \text{(Eq. 16)}$$

$$K = K_s \Theta^{1/2}\left[1 - \left(1 - \Theta^{1/m}\right)^m\right]^2, \text{ where } \Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \qquad \text{(Eq. 14)}$$

$$\frac{\partial K}{\partial \theta} = \frac{\partial K}{\partial \Theta} = \frac{1}{2} K_s \Theta^{-1/2}\left[1 - \left(1 - \Theta^{1/m}\right)^m\right]^2 + 2K_s \Theta^{1/2}\left[1 - \left(1 - \Theta^{1/m}\right)^m\right]\left[-m\left(1 - \Theta^{1/m}\right)^{m-1}\right]\left(\frac{-1}{m}\Theta^{1/m-1}\right)$$
(Eq. 18)

Where $\theta_r = 0.2$, $\theta_s = 0.54$, $\alpha = 0.008$, n = 1.8, $K_s = 25$.

$$m = 1 - \frac{1}{1.8} = 0.444$$

$$\theta = 0.2 + \frac{(0.54 - 0.2)}{\left[1 + (0.008 \times 27.67)^{1.8}\right]^{0.44}} = 0.53$$

$$\Theta = \frac{0.53 - 0.2}{0.54 - 0.2} = 0.972$$

$$\frac{\partial \theta}{\partial h} = \frac{0.44(0.54 - 0.2)}{\left[1 + (0.008 \times 27.67)^{1.8}\right]^{1.44}} \left(1.8 \times (0.008 \times 27.67)^{0.8}\right)0.008 = 5.88\text{e-}4$$

$$K = 25 \times 0.972^{1/2}\left[1 - \left(1 - 0.972^{1/0.44}\right)^{0.44}\right]^2 = 12.24$$

$$\frac{\partial K}{\partial \theta} = \frac{1}{2} \times 25 \times 0.972^{-1/2}\left[1 - \left(1 - 0.972^{1/0.44}\right)^{0.44}\right]^2 + 2 \times 25 \times 0.972^{1/2}\left[1 - \left(1 - 0.972^{1/0.44}\right)^{0.44}\right]$$
$$\left[-0.44\left(1 - 0.972^{1/0.44}\right)^{0.44-1}\right]\left(\frac{-1}{0.44} \times 0.972^{1/0.44-1}\right) = 164.3$$