Amath 546/Econ 589 Homework 2

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Question 1. Let R denote the simple daily return on an asset and assume that $R \sim N(0.01, 0.10)$. For the initial wealth $V_0 = 100$, the profit and loss are random variables defined as $\Pi = V_0 R$ and $L = -\Pi$. Let $\alpha \in (0.1)$ denote the confidence level for the daily VaR.

1. Derive the normal distributions for Π and L.
2. Give mathematical expressions for VaR_α based on the normal distributions for R, Π and L.
3. Using the expressions from the previous question, compute VaR_α for $\alpha = 0.95$ and 0.99. The values based on R, Π and L should all be equivalent.

Question 2. Let $\mathbf{R} = (R_1, \ldots, R_n)'$ denote an $n \times 1$ vector of asset returns with $E[\mathbf{R}] = \boldsymbol{\mu}$ and $\text{var}(\mathbf{R}) = \boldsymbol{\Sigma}$. Let $\mathbf{w} = (w_1, \ldots, w_n)'$ denote an $n \times 1$ vector of portfolio weights satisfying $\sum_{i=1}^{n} w_i = 1$. Let $R_p = \mathbf{w}'\mathbf{R}$ denote the portfolio return. Three standard portfolio risk measures are: (1) $\sigma_p(\mathbf{w}) = \sqrt{var(R_p)}$; (2) $VaR_\alpha(\mathbf{w})$ defined as $\Pr(R_p \leq VaR_\alpha(\mathbf{w})) = \alpha$; (3) $ES_\alpha(\mathbf{w}) = E[R_p | R_p \leq VaR_\alpha(\mathbf{w}))$.

1. Show that $\sigma_p(\mathbf{w})$ is a linearly homogenous function of the portfolio weights $\mathbf{w}$.
2. Suppose $R_p \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Give analytic expressions for $VaR_\alpha(\mathbf{w})$ and $ES_\alpha(\mathbf{w})$.
3. Using the above results, show that $VaR_\alpha(\mathbf{w})$ and $ES_\alpha(\mathbf{w})$ are linearly homogenous functions of the portfolio weights $\mathbf{w}$.
4. Give an analytic expression for the $n \times 1$ vector of asset marginal contributions to $\sigma_p(\mathbf{w})$. That is, compute $MCR^\sigma = \partial \sigma_p(\mathbf{w}) / \partial \mathbf{w}$.
5. Show that the asset specific percent contribution to risk

$$PCR_i^\sigma = \frac{w_i MCR_i^\sigma}{\sigma_p(\mathbf{w})}, \ i = 1, \ldots, n$$

can be expressed as

$$PCR_i^{\sigma} = w_i \beta_i$$

where

$$\beta_i = \frac{cov(R_i, R_p)}{var(R_p)}.$$

Question 3. Let R denote the simple daily return on an asset and assume that $R \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown and must be estimated from an observed sample of size T. A natural estimate for daily VaR_{α} is

$$\widehat{VaR}_{\alpha} = -V_0 \hat{q}_{1-\alpha}^R \tag{1}$$

where

$$\hat{q}_{1-\alpha}^R = \hat{\mu} + \hat{\sigma} \times q_{1-\alpha}^Z \tag{2}$$

$\hat{\mu}$ is the sample mean, $\hat{\sigma}$ is the sample standard deviation and and $q_{1-\alpha}^Z$ the $1-\alpha$ lower quantile of $Z \sim N(0,1)$. The Central Limit Theorem gives the result

$$\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \sigma \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{T} & 0 \\ 0 & \frac{\sigma^2}{2T} \end{pmatrix} \right) \tag{3}$$

which implies that, for large enough T, $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{T}\right)$, $\hat{\sigma} \sim N\left(\sigma, \frac{\sigma^2}{2T}\right)$ and that $\hat{\mu}$ and $\hat{\sigma}$ are independent.

1. Use (1) - (3) to derive mathematical expressions for $var(\widehat{VaR}_{\alpha})$ and $SE(\widehat{VaR}_{\alpha}) = \sqrt{var(\widehat{VaR}_{\alpha})}$.

2. Assuming $\sigma = 0.10$ and $V_0 = 100$, plot $SE(\widehat{VaR}_{\alpha})$ for a grid of 25 α values between $\alpha = 0.90$ and $\alpha = 0.995$ for $T = 25, 50$ and 100. How well is VaR_{α} estimated for α values close to 1?

Question 4. In this question you will use the PerformanceAnalytics package to estimate daily historical, normal and modified (Cornish-Fisher) VaR for Microsoft and the S&P 500 index. First, download daily adjusted closing prices on Microsoft (ticker MSFT) and the S&P 500 (ticker ^GSPC) over the period 2000-01-03 to 2012-04-03. Compute simple daily returns from both sets of prices. For automatically downloading data in R, you can use the `getSymbols()` function from the quantmod package or the `get.hist.quote()` function from the tseries package. For calculating returns you can use the `CalculateReturns()` function from the PerformanceAnalytics package.

1. Plot the daily returns on MSFT and the S&P 500. Note any of the stylized facts we discussed in class. Using the PerformanceAnalytics function `chart.Histogram()` plot the histograms with a normal curve overlaid, and

using the PerformanceAnalytics `chart.QQPlot()` plot the normal QQ-plots. Does the normal distribution look appropriate for these two assets? Use the PerformanceAnalytics function `table.Stats()` to compute descriptive statistics for the returns on the two assets. Note the sample values of skewness and excess kurtosis.

2. Using the PerformanceAnalytics function `VaR()`, estimate daily 95% and 99% VaR for the two assets based on the empirical distribution (i.e., historical VaR), the normal distribution and the Cornish-Fisher distribution (i.e., modified VaR). The `VaR()` function calculates VaR based on the distribution of returns and gives VaR_α as the lower $1-\alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% VaR values for the three methods. Note: be sure to read the online help for the function `VaR()`.

3. Using the PerformanceAnalytics function `ES()`, estimate daily 95% and 99% ES for the two assets based on the empirical distribution (i.e., historical ES), the normal distribution and the Cornish-Fisher distribution (i.e., modified ES). The `ES()` function calculates ES based on the distribution of returns and gives ES_α as the mean return less than the lower $1-\alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% ES values for the three methods. Note: be sure to read the online help for the function `ES()`.

Question 5. In this question you will use the PerformanceAnalytics functions StdDev, VaR and ES to estimate risk budgets based volatility, VaR and ES for an equally weighted portfolio of Microsoft and the S&P 500 index.

1. Using the PerformanceAnalytics function `StdDev()`, decompose the volatility of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Which assets contributes most to the volatility of the portfolio?

2. Using the PerformanceAnalytics function `VaR()`, decompose the 95% Gaussian, historical and modified VaR of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% VaR of the portfolio? Are the results similar to the decomposition based on the volatility?

3. Using the PerformanceAnalytics function `ES()`, decompose the 95% Gaussian, historical and modified ES of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% ES of the portfolio? Are the results similar to the decomposition based on the VaR?