Trend-Cycle Decompositions

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1 Introduction

A convenient way of representing an economic time series y_t is through the so-called *trend-cycle decomposition*

$$y_t = TD_t + Z_t \tag{1}$$

where TD_t represents the deterministic trend and Z_t represents the stochastic, and possibly cyclic, noise component. For simplicity, the deterministic trend takes the form of a simple time trend

$$TD_t = \kappa + \delta t \tag{2}$$

and the noise component has a finite order ARIMA(p,d,q) representation

$$\phi(L)Z_t = \theta(L)\varepsilon_t,\ \varepsilon_t \sim WN(0, \sigma^2)$$

where $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$and $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$. It is assumed that the polynomial $\phi(z) = 0$ has at most one root on the complex unit circle and $\theta(z) = 0$ has all roots outside the unit circle.

1.1 Trend Stationary and Difference Processes

The series y_t is called *trend stationary* if the roots of $\phi(z) = 0$ are outside the unit circle and is called *difference stationary* if $\phi(z) = 0$ has one root on the unit circle and the others outside the unit circle. If y_t is trend stationary then $\phi(L)$ is invertible and Z_t has the stationary or Wold representation

$$\begin{aligned} Z_t &= \phi(L)^{-1}\theta(L)\varepsilon_t \\ &= \psi(L)\varepsilon_t \end{aligned}$$

where $\psi(L) = \phi(L)^{-1}\theta(L) = \sum_{k=0}^{\infty} \psi_k L^k$ with $\psi_0 = 1$ and $\psi(1) \neq 0$. Alternatively, if y_t is difference stationary then $\phi(L)$ can be factored as $\phi(L) = (1 - L)\phi^*(L)$ where

$\phi^*(z) = 0$ has all $p-1$ roots outside the unit circle. In this case, ΔZ_t has the stationary ARMA$(p-1,q)$ representation

$$\begin{aligned} \Delta Z_t &= \phi^*(L)^{-1}\theta(L)\varepsilon_t \\ &= \psi^*(L)\varepsilon_t \end{aligned} \tag{3}$$

where $\psi^*(L) = \phi^*(L)^{-1}\theta(L) = \sum_{k=0}^{\infty}\psi_k^* L^k$ with $\psi_0^* = 1$ and $\psi^*(1) \neq 0$.

Example 1 *Difference stationary AR(2)*

Let $\phi(L)Z_t = \varepsilon_t$ with $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$. Assume that $\phi(z) = 0$ has one root equal to unity and the other root real valued with absolute value less than 1. Factor $\phi(L)$ so that

$$\phi(L) = (1 - \phi^* L)(1 - L) = \phi^*(L)(1 - L)$$

where $\phi^*(L) = 1 - \phi^* L$ with $|\phi^*| < 1$. Then

$$\phi(L)Z_t = (1 - \phi^* L)(1 - L)Z_t = (1 - \phi^* L)\Delta Z_t$$

so that ΔZ_t follows an AR(1) process.

1.2 I(1) and I(0) Processes

If the noise series Z_t is difference stationary then we say that Z_t is integrated of order 1 and we write $Z_t \sim I(1)$. To see why, note that from (3) above $\Delta Z_t = \psi^*(L)\varepsilon_t = u_t$ where u_t is stationary. It follows that $Z_t = Z_{t-1} + u_t$ and by recursive substitution starting at time $t = 0$ we have

$$Z_t = Z_0 + \sum_{k=1}^{t} u_k$$

so that Z_t can be represented as the (integrated) sum of t stationary innovations $\{u_k\}_{k=1}^t$. Moreover, since u_t is stationary we say that u_t is integrated of order zero, and write $u_t \sim I(0)$, to signify that u_t cannot be written as the sum of stationary innovations. It should be clear that an $I(1)$ series can be converted to an $I(0)$ series by first differencing.

If ΔZ_t is an ARMA(p,q) process then Z_t is called an ARIMA(p,1,q) process. The term ARIMA refers to an autoregressive integrated moving average process.

1.3 Impulse Response Functions from I(1) Processes

Consider an $I(1)$ process with Wold representation $\Delta y_t = \psi^*(L)\varepsilon_t$. Since $\Delta y_t = y_t - y_{t-1}$ the level y_t may be represented as

$$y_t = y_{t-1} + \Delta y_t$$

Similarly, the level at time $t+h$ may be represented as

$$y_{t+h} = y_{t-1} + \Delta y_t + \Delta y_{t+1} + \cdots + \Delta y_{t+h}$$

The impulse response on the level of y_{t+h} of a shock to ε_t is

$$\begin{aligned} \frac{\partial y_{t+h}}{\partial \varepsilon_t} &= \frac{\partial \Delta y_t}{\partial \varepsilon_t} + \frac{\partial \Delta y_{t+1}}{\partial \varepsilon_t} + \cdots + \frac{\partial \Delta y_{t+h}}{\partial \varepsilon_t} \\ &= 1 + \psi_1^* + \cdots + \psi_h^* \end{aligned} \tag{4}$$

From (4), the long-run impact of a shock to the level of y_t is given by

$$\lim_{h\to\infty} \frac{\partial y_{t+h}}{\partial \varepsilon_t} = \sum_{j=1}^{\infty} \psi_j^* = \psi^*(1).$$

Hence, $\psi^*(1)$ measures the permanent effect of a shock, ε_t, to the level of y_t.
Remarks:

1. Since $\frac{\partial y_t}{\partial \varepsilon_t} = 1$ it follows that $\psi^*(1)$ can also be interpreted as the long-run effect of a shock relative to the immediate effect of a shock.

2. $\psi^*(1)$ is a natural measure of the importance of a permanent shock. If $\psi^*(1) = 1$ then the long-run effect of a shock is equal to the immediate effect; if $\psi^*(1) > 1$ the long-run effect is greater than the immediate effect and if $\psi^*(1) < 1$ the long-run effect is less than the immediate effect.

3. If $\psi^*(1) = 0$ then $Z_t \sim I(0)$. To see this suppose $Z_t \sim I(0)$ and has the Wold representation $Z_t = \psi(L)\varepsilon_t$ with $\psi(1) \neq 0$. Then $\Delta Z_t = (1-L)Z_t = (1-L)\psi(L)\varepsilon_t = \psi^*(L)\varepsilon_t$ where $\psi^*(L) = (1-L)\psi(L)$. It follows that $\psi^*(1) = (1-1)\psi(1) = 0$.

1.4 Forecasting from an I(1) Process

Forecasting from an I(1) process follows directly from writing y_{t+h} as

$$y_{t+h} = y_t + \Delta y_{t+1} + \Delta y_{t+2} + \cdots + \Delta y_{t+h}$$

Then

$$\begin{aligned} y_{t+h|t} &= y_t + \Delta y_{t+1|t} + \Delta y_{t+2|t} + \cdots + \Delta y_{t+h|t} \\ &= y_t + \sum_{s=1}^{h} \Delta y_{t+s|t} \end{aligned}$$

Notice that forecasting an I(1) process proceeds from the most recent observation.

Example 2 *Forecasting from an AR(1) model for* Δy_t

Let Δy_t follow an AR(1) process

$$\Delta y_t - \mu = \phi(\Delta y_{t-1} - \mu) + \varepsilon_t,\ \ \varepsilon_t \sim WN(0, \sigma^2)$$

where $|\phi| < 1$. Using the chain-rule of forecasting, the h-step ahead forecast of Δy_{t+h}based on information at time t is

$$\Delta y_{t+h|t} = \mu + \phi^h(\Delta y_t - \mu) \tag{5}$$

Using (5), the h-step ahead forecast of y_{t+h} is

$$\begin{aligned} y_{t+h|t} &= y_t + \sum_{s=1}^{h} [\mu + \phi^s(\Delta y_t - \mu)] \\ &= y_t + h\mu + (\Delta y_t - \mu) \sum_{s=1}^{h} \phi^h \end{aligned} \tag{6}$$

2 The Trend-Cycle Decomposition with Stochastic Trends

Assume that $Z_t \sim I(1)$. Then it is possible to decompose Z_t into a stochastic (random walk) trend and a stationary, $I(0)$, "cyclical" component:

$$Z_t = TS_t + C_t \tag{7}$$

The stochastic trend, TS_t, captures shocks that have a permanent effect on the level of y_t and the stationary component, C_t, captures shocks that only have a temporary effect on the level of y_t. Given the decomposition of Z_t, the representation for y_t becomes

$$y_t = TD_t + TS_t + C_t \tag{8}$$

where $TD_t + TS_t$ measures the overall or total trend and C_t represents the deviations about the trend.

The decomposition of Z_t in (7) is not unique. In fact, there are an infinite number of such combinations depending on how TS_t and C_t are defined. Two decompositions have been popular in the empirical literature: the Beveridge-Nelson (BN) decomposition; and the orthogonal unobserved components (UC0) decomposition. Both decompositions define TS_t as a pure random walk. They primarily differ in how they model the serial correlation in ΔZ_t. The BN decomposition uses an unrestricted ARMA(p, q) model for ΔZ_t, and the UC0 model uses a restricted ARMA(p,q) for ΔZ_t.

2.1 The Beveridge-Nelson Decomposition

Beveridge and Nelson (1980) proposed a definition of the permanent component of an $I(1)$ time series y_t with drift μ as the limiting forecast as horizon goes to infinity, adjusted for the mean rate of growth over the forecast horizon,

$$TD_t + BN_t = \lim_{h \to \infty} y_{t+h|t} - \delta h \tag{9}$$

The stochastic part of the permanent component (11), BN_t, is referred to as the *BN trend*. The implied cycle at time t is then

$$C_t = y_t - TD_t - BN_t$$

Beveridge and Nelson showed that if Δy_t has a Wold representation

$$\Delta y_t = \delta + \psi^*(L)\varepsilon_t \tag{10}$$

then BN_t follows a pure random walk without drift

$$\begin{aligned} BN_t &= BN_{t-1} + \psi^*(1)\varepsilon_t \\ &= BN_0 + \psi^*(1)\sum_{j=1}^{t} \varepsilon_t \end{aligned} \tag{11}$$

The derivation of the BN trend (11) relies on the following algebraic result.

Proposition 3 *Let* $\psi(L) = \sum_{k=0}^{\infty} \psi_k L^k$ *with* $\psi_0 = 1$. *Then*

$$\begin{aligned} \psi(L) &= \psi(1) + (1-L)\widetilde{\psi}(L), \\ \psi(1) &= \sum_{k=0}^{\infty} \psi_k, \\ \widetilde{\psi}(L) &= \sum_{j=0}^{\infty} \widetilde{\psi}_j L^j,\ \widetilde{\psi}_j = -\sum_{k=j+1}^{\infty} \psi_k. \end{aligned} \tag{12}$$

In addition, if $\sum_{k=0}^{\infty} k|\psi_k| < \infty$ (1-summability) then $\sum_{k=0}^{\infty} |\widetilde{\psi}_k| < \infty$. 1-summability is satisfied by all covariance stationary ARMA(p,q) processes. For an algebraic proof, see Hamilton (1993) pages 534 and 535.

Now, consider the Wold representation for Δy_t given in (10). Then by recursive substitution

$$y_t = y_0 + \delta t + \psi^*(L)\sum_{j=1}^{t} \varepsilon_j$$

Applying (12) to $\psi^*(L)$

$$\begin{aligned} y_t &= y_0 + \delta t + \left(\psi^*(1) + (1-L)\widetilde{\psi}^*(L)\right)\sum_{j=1}^{t}\varepsilon_j \\ &= y_0 + \delta t + \psi^*(1)\sum_{j=1}^{t}\varepsilon_j + \widetilde{\varepsilon}_t - \widetilde{\varepsilon}_0 \\ &= TD_t + TS_t + C_t \end{aligned}$$

where $\widetilde{\varepsilon}_t = \widetilde{\psi}^*(L)\varepsilon_t$, $TD_t = y_0 + \delta t$, $TS_t = Z_0 + \psi^*(1)\sum_{j=1}^{t}\varepsilon_j$ and $C_t = \widetilde{\varepsilon}_t - \widetilde{\varepsilon}_0$. To show that $\psi^*(1)\sum_{j=1}^{t}\varepsilon_j$ is the BN trend defined by (9), consider the series at time $t+h$

$$y_{t+h} = y_0 + \delta(t+h) + \psi^*(1)\sum_{j=1}^{t+h}\varepsilon_j + \widetilde{\varepsilon}_{t+h}$$

The forecast of y_{t+h} at time t is

$$y_{t+h|t} = y_0 + \delta(t+h) + \psi^*(1)\sum_{j=1}^{t}\varepsilon_j + \widetilde{\varepsilon}_{t+h|t}$$

The limiting forecast as horizon goes to infinity, adjusted for mean growth, is

$$\begin{aligned} \lim_{h\to\infty} y_{t+h|t} - \delta h &= y_0 + \delta t + \psi^*(1)\sum_{j=1}^{t}\varepsilon_j + \lim_{h\to\infty}\widetilde{\varepsilon}_{t+h|t} \\ &= y_0 + \delta t + \psi^*(1)\sum_{j=1}^{t}\varepsilon_j \\ &= TD_t + BN_t \end{aligned}$$

as $\lim_{h\to\infty}\widetilde{\varepsilon}_{t+h|t} = 0$ since $\widetilde{\varepsilon}_{t+h}$ is a mean-zero stationary process. We have just proved that $\psi^*(1)\sum_{j=1}^{t}\varepsilon_j$ is the BN trend, and that the BN trend follows a pure random walk.

Example 4 *BN decomposition from MA(1) process for* Δy_t

BN decomposition of U.S. real GDP (Stock and Watson (1987)). Let $y_t = \ln(\text{rgdp}_t)$. Using postwar quarterly data from 1947:II-1985:IV, Stock and Watson fit the following MA(1) model to the growth rate of real gdp:

$$\Delta y_t = 0.008 + \varepsilon_t + 0.3\varepsilon_{t-1},\ \varepsilon_t \sim iid(0,\sigma^2),\ \widehat{\sigma} = 0.0106$$

For the MA(1) model, the Wold representation for Δy_t has the simple form $\Delta y_t = \delta + \psi^*(L)\varepsilon_t$ where $\psi^*(L) = 1 + \psi_1^* L$ and $\psi_1^* = 0.03$. Straightforward calculations give

$$\begin{aligned}
\psi^*(1) &= 1 + \psi_1^* = 1.03 \\
\widetilde{\psi}_0^* &= -\sum_{j=1}^{\infty} \psi_j^* = -\psi_1^* = -0.03 \\
\widetilde{\psi}_j^* &= -\sum_{j=k+1}^{\infty} \psi_j^* = 0, \;\; j = 1, 2 \ldots
\end{aligned}$$

so that the trend-cycle decomposition of y_t using the BN decomposition becomes

$$\begin{aligned}
y_t &= (y_0 + \delta t) + \psi^*(1) \sum_{j=1}^{t} \varepsilon_j + \widetilde{\varepsilon}_t \\
&= y_0 + 0.008t + 1.3 \sum_{j=1}^{t} \varepsilon_j - 0.3\varepsilon_t
\end{aligned}$$

so that

$$\begin{aligned}
TD_t &= y_0 + 0.008t \\
BN_t &= 1.3 \sum_{j=1}^{t} \varepsilon_j \\
C_t &= -0.3\varepsilon_t
\end{aligned}$$

Note that $\frac{\partial y_t}{\partial \varepsilon_t} = 1$ and $\frac{\partial y_{t+s}}{\partial \varepsilon_t} = 1.3$ for $s > 0$.

2.1.1 Computing the BN Decomposition from an Estimated ARMA for Δy_t

The naive computation of the BN decomposition requires the following steps

1. Estimation of ARMA(p,q) model for Δy_t
2. Estimation of $\psi^*(1)$ from estimated ARMA(p,q) model for Δy_t
3. Estimation of $\sum_{j=1}^{t} \varepsilon_j$ using residuals from estimated ARMA(p,q) model for Δy_t

Example 5 *BN decomposition from ARMA(2,2) model for* Δy_t

Morley, Nelson and Zivot (2003) fit the following ARMA(2,2) model to the growth rate of postwar quarterly real GDP over the period 1947:I - 1998:II

$$\begin{aligned}
\Delta y_t &= 0.816 + 1.342\Delta y_{t-1} - 0.706\Delta y_{t-2} + \hat{\varepsilon}_t - 1.054\hat{\varepsilon}_{t-1} + 0.519\hat{\varepsilon}_{t-2} \\
\phi(L) &= 1 - 1.342L + 0.706L^2, \;\; \theta(L) = 1 - 1.054L + 0.519L^2
\end{aligned}$$

To compute an estimate of $\psi^*(1)$ from the ARMA(p,q) model, solve for the wold representation

$$\begin{aligned}\phi(L)\Delta y_t &= \theta(L)\varepsilon_t \\ &\Rightarrow \Delta y_t = \phi(L)^{-1}\theta(L)\varepsilon_t = \psi^*(L)\varepsilon_t\end{aligned}$$

where $\psi^*(L) = \phi(L)^{-1}\theta(L)$. Therefore,

$$\psi^*(1) = \phi(1)^{-1}\theta(1) \tag{13}$$

Using (13), the estimate of $\psi^*(1)$ from the ARMA(2,2) model is

$$\psi^*(1) = \frac{1 - 1.054 + 0.519}{1 - 1.342 + 0.706} = 1.276$$

The estimate of the permanent component is then

$$TD_t + BN_t = y_0 + 0.816t + 1.276\sum_{j=1}^{t}\hat{\varepsilon}_j$$

The previous example shows that the computation of the BN decomposition from an estimated ARMA(p,q) model is straightforward but somewhat tedious. The following example shows that the computation of the BN decomposition from an AR(1) model is simple and elegant.

Example 6 *BN decomposition from AR(1) process for* Δy_t

From the $h-$step ahead forecast for y_t given in (6), it is easy to compute the BN trend for y_t:

$$\begin{aligned}TD_t + BN_t &= \lim_{h\to\infty}\left(y_{t+h|t} - h\delta\right) \\ &= y_t + (\Delta y_t - \delta)\lim_{h\to\infty}\sum_{s=1}^{h}\phi^h \\ &= y_t + \frac{\phi}{1-\phi}(\Delta y_t - \delta)\end{aligned}$$

The cycle component is then

$$\begin{aligned}C_t &= y_t - TD_t - BN_t \\ &= \frac{\phi}{1-\phi}(\Delta y_t - \mu)\end{aligned}$$

Notice that the cycle inherits the behavior of the AR(1) model for Δy_t.

Morley (2002) shows how the BN decomposition for an AR(1) model for Δy_t may be extended to any model for Δy_t that can be represented in state space form. In particular, suppose $\Delta y_t - \mu$ is a linear combination of the elements of the $m \times 1$ state vector $\boldsymbol{\alpha}_t$:

$$\Delta y_t - \mu = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_m \end{array} \right] \boldsymbol{\alpha}_t$$

where z_i $(i = 1, \ldots, m)$ is the weight of the ith element of $\boldsymbol{\alpha}_t$ in determining $\Delta y_t - \mu$. Suppose further that

$$\boldsymbol{\alpha}_t = \mathbf{T}\boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t,\ \ \boldsymbol{\eta}_t \sim iid\ N(\mathbf{0}, \mathbf{Q}),$$

such that all of the eigenvalues of $\mathbf{T}$ have modulus less than unity, and $\mathbf{T}$ is invertible. Then, Morley shows that

$$\begin{aligned} TD_t + BN_t &= \mathbf{y}_t + \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_m \end{array} \right] \mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1} \mathbf{a}_{t|t} \\ C_t &= y_t - TD_t - BN_t = - \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_m \end{array} \right] \mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1} \mathbf{a}_{t|t} \end{aligned} \tag{14}$$

where $\mathbf{a}_{t|t}$ denotes the filtered estimate of $\boldsymbol{\alpha}_t$ from the Kalman filter recursions.

2.2 The Orthogonal Unobserved Components Model

The basic idea behind the unobserved components (UC) model is to give structural equations for the components on the trend-cycle decomposition (1). For example, Watson (1986) considers UC-ARIMA models of the form

$$\begin{aligned} y_t &= \mu_t + C_t \\ \mu_t &= \alpha + \mu_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \\ \phi(L) C_t &= \theta(L)\eta_t,\ \ \eta_t \sim iid(0, \sigma_\eta^2) \end{aligned} \tag{15}$$

where $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$. Note that the trend component, μ_t, is a random walk with drift and the cyclical component, C_t, is an ARMA(p,q) process. As it stands, however, the parameters of the UC model are not identified without further restrictions. Restrictions commonly used in practice to identify all of the parameters are: (1) the roots of $\phi(z) = 0$ are outside the unit circle; (2) $\theta(L) = 1$, and (3) $cov(\varepsilon_t, \eta_t) = 0$. These restrictions identify C_t as a transitory autoregressive "cyclical" component, and μ_t as the permanent trend component. The restriction $cov(\varepsilon_t, \eta_t) = 0$ states that shocks to C_t and μ_t are uncorrelated. As shown in Morley, Nelson and Zivot (2003), for certain models the assumption that $cov(\varepsilon_t, \eta_t) = 0$ turns out to be an over-identifying restriction.

Example 7 *Clark's (1986) JPE Model*

Clark considered the UC-ARIMA(2,0) model

$$\begin{aligned} y_t &= \mu_t + C_t \\ \mu_t &= \alpha + \mu_{t-1} + \varepsilon_t,\ \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \\ C_t &= \phi_1 C_{t-1} + \phi_2 C_{t-2} + \eta_t, \eta_t \sim iid(0, \sigma_\eta^2) \\ cov(\varepsilon_t, \eta_t) &= 0 \end{aligned} \tag{16}$$

The model may be put in state space form in several way. For example, one may define the transition equation for the state vector $\boldsymbol{\alpha}_t = (\mu_t, C_t, C_{t-1})'$ as

$$\begin{pmatrix} \mu_t \\ C_t \\ C_{t-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ C_{t-1} \\ C_{t-2} \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix}$$

and the measurement as

$$y_t = (1, 1, 0) \begin{pmatrix} \mu_t \\ C_t \\ C_{t-1} \end{pmatrix}$$

$$\mathbf{Q}_t = \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix}$$

Notice that the state vector contains the $I(1)$ component $\alpha_{1,t} = \mu_t$ and the covariance stationary component $\boldsymbol{\alpha}_{2,t}' = (C_t, C_{t-1})'$. As a result, the distribution of the initial state vector $\boldsymbol{\alpha}_0 = (\mu_0, C_0, C_{-1})'$ cannot be determined in the usual way. This is because, $a_{1,0}$ does not have a simple stationary unconditional distribution. Since $var(\mu_t) = \mu_0 + \sigma_\varepsilon^2 \cdot t$, the usual approach is to assume

$$\begin{aligned} a_{1,0} &= 0 \\ var(\alpha_{1,0}) &= \kappa \cdot 10^6 \end{aligned}$$

The distribution for $\boldsymbol{\alpha}_{2,0}$ may be based on its unconditional stationary distribution. Therefore, the initial state distribution is characterized by

$$\begin{aligned} \mathbf{a}_0 &= \mathbf{0} \\ \mathbf{P}_0 &= \begin{pmatrix} \kappa \cdot 10^6 & 0 \\ 0 & \mathbf{P}_2 \end{pmatrix} \end{aligned}$$

where $\mathbf{P}_2$ is the unconditional variance of $\boldsymbol{\alpha}_{2,0}$.

An alternative state space representation that only treats the unobserved cycle as a state variable is based on the transformed model

$$\begin{aligned} \Delta y_t &= \Delta \mu_t + \Delta C_t \\ &= \alpha + \Delta C_t + \varepsilon_t \end{aligned}$$

Defining the state variables as $\boldsymbol{\alpha}_t = (C_t, C_{t-1})'$, the transition equation is

$$\begin{pmatrix} C_t \\ C_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_{t-1} \\ C_{t-2} \end{pmatrix} + \begin{pmatrix} \eta_t \\ 0 \end{pmatrix}$$

The corresponding measurement equation is

$$\Delta y_t = \begin{pmatrix} 1 & -1 \end{pmatrix} \boldsymbol{\alpha}_t + \varepsilon_t$$

Since $\boldsymbol{\alpha}_t$ is covariance stationary, the distribution of the initial state vector $\boldsymbol{\alpha}_0$ may be determined in the usual way.

Proposition 8 *Any UC-ARIMA model of the form (15) with a specified correlation between ε_t and η_t is observationally equivalent to an ARMA model for Δy_t with non-linear restrictions on the parameters. The restricted ARMA model for Δy_t is called the* reduced form *of the UC-ARIMA model.*

Proposition 9 *(Lippi and Reichlin (1992) JME) $\psi^*(1)$ computed from the reduced form ARMA model for Δy_t based on a UC-ARIMA model for y_t is always less than 1.*

Example 10 *Random walk plus noise model*

Consider the random walk plus noise model

$$\begin{aligned} y_t &= \mu_t + \eta_t,\ \eta_t \sim iid(0, \sigma_\eta^2) \\ \mu_t &= \mu_{t-1} + \varepsilon_t\ \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \end{aligned}$$

where η_t and ε_t are independent. Define the signal-to-noise ratio $q = \frac{\sigma_\varepsilon^2}{\sigma_\eta^2}$. The reduced form ARMA model for Δy_t is then

$$\Delta y_t = \Delta \mu_t + \Delta \eta_t = \varepsilon_t + \eta_t - \eta_{t-1}.$$

It is straightforward to show that Δy_t follows an ARMA(0,1) process. To see this, consider the autocovariances of Δy_t

$$\begin{aligned} \gamma_0^* &= var(\Delta y_t) = var(\varepsilon_t + \eta_t - \eta_{t-1}) = \sigma_\varepsilon^2 + 2\sigma_\eta^2 \\ &= \sigma_\eta^2 (q+2) \\ \gamma_1^* &= cov(\Delta y_t, \Delta y_{t-1}) = E[(\varepsilon_t + \eta_t - \eta_{t-1})(\varepsilon_{t-1} + \eta_{t-1} - \eta_{t-2})] \\ &= -\sigma_\eta^2 \\ \gamma_j^* &= 0,\ j > 1 \end{aligned}$$

Clearly, the autocovariances for Δy_t are the same as the autocovariances for an ARMA(0,1) model, and so Δy_t has the representation

$$\Delta y_t = \varsigma_t + \theta \varsigma_{t-1},\ \varsigma_t \sim iid(0, \sigma_\varsigma^2)$$

where the ARMA(0,1) parameters θ and σ_ς^2 are nonlinearly related to the UC-ARIMA parameters σ_η^2 and σ_ε^2, and the error term ς_t encompasses the structural shocks η_t and ε_t.

Notice that the reduced form ARMA(0,1) model has two parameters θ and σ_ς^2, and the structural random walk plus noise model also has two parameters σ_ε^2 and σ_η^2. Since the reduced form and structural models have the same number of parameters the order condition for identification is satisfied. If ε_t and η_t were allowed to be correlated then there would be three structural parameters (two variances and a covariance) and only two reduced form parameters, and the order condition for identification would not be satisfied. Hence, setting $cov(\varepsilon_t, \eta_t) = 0$ is an identifying restriction in this model.

The mapping from UC-ARIMA parameters to the reduced form parameters is determined as follows. The first order autocorrelation for the reduced form ARMA(0,1) is

$$\rho_1^* = \frac{\theta}{1+\theta^2}$$

and for the UC-ARIMA it is

$$\rho_1 = \frac{\gamma_1^*}{\gamma_0^*} = \frac{-1}{q+2}$$

Setting $\rho_1^* = \rho_1$ and solving for θ gives

$$\theta = \frac{-(q+2) \pm \sqrt{(q+2)^2 - 4}}{2}$$

The invertible solution is

$$\theta = \frac{-(q+2) + \sqrt{q^2+4q}}{2}, \ \theta < 0.$$

Notice that the MA coefficient θ for the reduced form ARMA(0,1) model is restricted to be negative. If $q = 0$, then $\theta = -1$. Similarly, matching variances for the two models gives

$$\sigma_\varsigma^2 = \frac{\sigma_\eta^2}{-\theta}.$$

Finally, note that since $\theta < 0$ it follows that $\psi^*(1) = 1 + \theta < 1$.

3 Cochrane's Variance Ratio Statistic

Cochrane (1988) considered the question of the relative importance of permanent shocks to temporary shocks in the analysis of U.S. real GDP. Specifically, he was interested in determining the fraction of the quarterly variation in log real GDP that is attributable to permanent shocks. To answer this question, Cochrane proposed the use of variance ratio statistics computed as the normalized ratio of the estimated variance of the k^{th} difference of GDP to the variance of the first difference. to be more specific, let y_t denote the log-level of real GDP, $\mu = E[\Delta y_t]$ and define the variance of the k^{th} difference as

$$V_k = \frac{1}{k} var(y_{t+k} - y_t - k\mu)$$

Cochrane's variance ratio statistic is then defined as

$$R_k = \frac{V_k}{V_1}.$$

Example 11 *Pure random walk model*

Consider the pure random walk

$$y_t = y_{t-1} + \varepsilon_t,\ \varepsilon_t \sim iid(0, \sigma^2_\varepsilon).$$

Then $y_t = y_0 + \sum_{j=1}^{t} \varepsilon_j$, $y_{t+k} = y_0 + \sum_{j=1}^{t+k} \varepsilon_t$ and $y_{t+k} - y_t = \sum_{j=t+1}^{t+k} \varepsilon_j$ so that $V_k = V_1 = \sigma^2_\varepsilon$ and $R_k = 1$ for all values of k.

Example 12 *White noise process*

Let $y_t = \varepsilon_t \sim iid(0, \sigma^2_\varepsilon)$ be a white noise process. Then $var(y_{t+k} - y_t) = var(y_{t+1} - y_t) = 2\sigma^2_e$ and so $R_k = \frac{1}{k}$. Notice that $\lim_{k\to\infty} R_k = 0$.

The examples above indicate that the variance ratio statistic computed from a difference stationary process is non-zero for any value of k whereas it converges to zero as k gets large for a trend stationary process.

The variance ratio statistic can be rewritten in several different ways. First, note that

$$\begin{aligned} V_k &= \frac{1}{k} E\left[((\Delta y_{t+1} - \mu) + \cdots + (\Delta y_{t+k} - \mu))^2\right] \\ &= \gamma^*_0 + 2\sum_{j=1}^{k-1} \frac{k-j}{k} \gamma^*_j,\ \gamma^*_j = cov(\Delta y_t, \Delta y_{t-j}) \end{aligned}$$

which is a weighted average of autocovariances of Δy_t. Furthermore, it can be shown that

$$\begin{aligned} \lim_{k\to\infty} V_k &= \sum_{j=-\infty}^{\infty} \gamma^*_j = \gamma^*_0 + 2\sum_{j=1}^{\infty} \gamma^*_j \equiv V, \\ \lim_{k\to\infty} R_k &= \frac{V}{\gamma^*_0} \equiv R. \end{aligned}$$

To interpret the limiting form of the variance ratio statistic we require the following results.

Proposition 13 *Let* $\Delta y_t = \psi^*(L)\varepsilon_t, \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$ *and* $\sum_{j=0}^{\infty} j|\psi_j^*| < \infty$*. Then as* $T \to \infty$

$$\sqrt{T}\Delta y_t \xrightarrow{d} N(0, \sigma_\varepsilon^3 \psi^*(1)^2)$$

where $\sigma_\varepsilon^3 \psi^*(1)^2$ *is the asymptotic or* long-run variance *of* $\sqrt{T}\Delta y_t$*.*

Proposition 14 $\sigma_\varepsilon^3 \psi^*(1)^2 = var(\Delta TS_t)$ *where* TS_t *is computed from the BN decomposition of* y_t*.*

From the BN decomposition, $TS_t = \psi^*(1) \sum_{j=1}^{t} \varepsilon_j$ so that $\Delta TS_t = \psi^*(1)\varepsilon_t$ and hence $var(\Delta TS_t) = \sigma_\varepsilon^3 \psi^*(1)^2$.

Proposition 15 $\sigma_\varepsilon^3 \psi^*(1)^2 = \sum_{j=-\infty}^{\infty} \gamma_j^*$

Combining the results in Propositions, the limiting form of the variance ratio statistic, R, may be re-expressed as

$$\begin{aligned} R &= \frac{\sigma_\varepsilon^3 \psi^*(1)^2}{\gamma_0^*} \\ &= \frac{var(\Delta TS_t)}{var(\Delta y_t)} \end{aligned}$$

The expression above shows that R may be interpreted as the ratio of the variance of the stochastic trend or permanent shock to the variance of the total change in y_t.
Remarks

1. The variance ratio statistics are usually computed for various values of k and reported in graphical form.

2. If $R = 0$ the series y_t is trend stationary or $I(0)$; if $R > 0$ then y_t is difference stationary or $I(1)$; if $R < 1$ then y_t is called trend reverting and if $R > 1$ it is called trend averting.

3. R can be computed nonparametrically using the sample autocovariances of Δy_t. Alternatively, parametric estimates of R can be computed from the either the estimated BN decomposition of Δy_t or the UC-ARIMA model for y_t.

4. Estimates of R based on the BN decomposition are always greater than estimates based on UC-ARIMA models because for UC-ARIMA models $\psi^*(1)$ is always less than 1.

5. Campbell and Mankiw (1986) derive the following relationship between R and $\psi^*(1)$. Define

$$\rho^2 = 1 - \frac{\sigma_\varepsilon^2}{\sigma_{\Delta y}^2} = 1 - \frac{\sigma_\varepsilon^2}{\gamma_0^*}$$

which is a measure of the predictability of Δy_t. Then it is easy to show that

$$\psi^*(1) = \sqrt{\frac{R}{1-\rho^2}}$$

and so $\sqrt{R}$ is a lower bound for $\psi^*(1)$.

6. There are problems with nonparametric estimates of R. First, R cannot be computed exactly with a finite amount of data. As a practical matter, one must choose a value of k. Choosing k too small may obscure the presence of trend reversion that is picked up in higher order values of ρ_j^*. Alternatively, choosing k too large may produce excess or spurious trend reversion. The reason for this result is that as $k \to T$, $R_k \to 0$. In other words, R_k is a downward biased estimate of R.

4 References

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