Structural VAR Modeling for I(1) Data that is Not Cointegrated

Assume $\mathbf{y}_t = (y_{1t}, y_{2t})'$ be $I(1)$ and not cointegrated. That is, y_{1t} and y_{2t} are both $I(1)$ and there is no linear combination of y_{1t} and y_{2t} that is $I(0)$.

In this case, $\Delta\mathbf{y}_t = (\Delta y_{1t}, \Delta y_{2t})'$ is $I(0)$ and is assumed to have the SVAR representation

$$
\begin{gathered}
\mathbf{B}\Delta\mathbf{y}_t = \gamma_0 + \mathbf{\Gamma}_1\Delta\mathbf{y}_t + \varepsilon_t \\
\mathbf{B}(L)\Delta\mathbf{y}_t = \gamma_0 + \varepsilon_t \\
\varepsilon_t \sim \text{iid } (0, \mathbf{D}) \\
\mathbf{D} \text{ is diagonal}
\end{gathered}
$$

The reduced form VAR for $\Delta \mathbf{y}_t$ is

$$\begin{aligned} \Delta \mathbf{y}_t &= \mathbf{a}_0 + \mathbf{A}_1 \Delta \mathbf{y}_{t-1} + \mathbf{u}_t \\ \mathbf{A}(L)\Delta \mathbf{y}_t &= \mathbf{a}_0 + \mathbf{u}_t \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\alpha}_0 &= \mathbf{B}^{-1}\boldsymbol{\gamma}_0, \mathbf{A}_1 = \mathbf{B}^{-1}\boldsymbol{\Gamma}_1, \mathbf{u}_t = \mathbf{B}^{-1}\boldsymbol{\varepsilon}_t, \\ E[\mathbf{u}_t \mathbf{u}_t'] &= \mathbf{B}^{-1}\mathbf{D}\mathbf{B}^{-1\prime},\ \mathbf{A}(L) = \mathbf{I}_2 - \mathbf{A}_1 L \end{aligned}$$

The Wold MA representation is

$$\begin{aligned} \Delta \mathbf{y}_t &= \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\mathbf{u}_t, \\ \boldsymbol{\mu} &= \mathbf{A}(1)^{-1}\mathbf{a}_0,\ \boldsymbol{\Psi}(L) = \mathbf{A}(L)^{-1} \end{aligned}$$

and the SMA representation is

$$\begin{aligned} \Delta \mathbf{y}_t &= \boldsymbol{\mu} + \boldsymbol{\Theta}(L)\boldsymbol{\varepsilon}_t, \\ \boldsymbol{\Theta}(L) &= \boldsymbol{\Psi}(L)\mathbf{B}^{-1} \end{aligned}$$

Impulse Response Functions

Consider the SMA representation at time $t+s$

$$\begin{bmatrix} \Delta y_{1t+s} \\ \Delta y_{2t+s} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \theta_{11}^{(0)} & \theta_{12}^{(0)} \\ \theta_{21}^{(0)} & \theta_{22}^{(0)} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t+s} \\ \varepsilon_{2t+s} \end{bmatrix} + \cdots + \begin{bmatrix} \theta_{11}^{(s)} & \theta_{12}^{(s)} \\ \theta_{21}^{(s)} & \theta_{22}^{(s)} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \cdots .$$

The structural dynamic multipliers are

$$\begin{aligned} \frac{\partial \Delta y_{1t+s}}{\partial \varepsilon_{1t}} &= \theta_{11}^{(s)}, \frac{\partial \Delta y_{1t+s}}{\partial \varepsilon_{2t}} = \theta_{12}^{(s)} \\ \frac{\partial \Delta y_{2t+s}}{\partial \varepsilon_{1t}} &= \theta_{21}^{(s)}, \frac{\partial \Delta y_{2t+s}}{\partial \varepsilon_{2t}} = \theta_{22}^{(s)} \end{aligned}$$

which give the impact of the structural shocks on the *first difference* of y at horizon $t+s$.

Using the fact that

$$y_{it+s} = y_{it-1} + \Delta y_{it} + \Delta y_{it+1} + \cdots + \Delta y_{it+s},\ i = 1,2$$

the impacts of the structural shocks on the level of y are

$$\begin{aligned} \frac{\partial y_{it+s}}{\partial \varepsilon_{jt}} &= \frac{\partial \Delta y_{it}}{\partial \varepsilon_{jt}} + \frac{\partial \Delta y_{it+1}}{\partial \varepsilon_{jt}} + \cdots + \frac{\partial \Delta y_{it+s}}{\partial \varepsilon_{jt}} \\ &= \theta_{ij}^{(0)} + \theta_{ij}^{(1)} + \cdots + \theta_{ij}^{(s)} \\ &= \sum_{k=0}^{s} \theta_{ij}^{(k)},\ i,j = 1,2. \end{aligned}$$

The *long-run impact* of a shock to ε_j on the level of y_i is then

$$\lim_{s \to \infty} \frac{\partial y_{it+s}}{\partial \varepsilon_{jt}} = \theta_{ij}(1),\ i,j = 1,2.$$

For stationary y this long-run impact is always zero but for nonstationary y this impact may or may not be zero for some combination of i and j.

Beveridge-Nelson Decomposition

Using the Wold representation for $\Delta \mathbf{y}_t$, the *multivariate BN* decomposition of $\mathbf{y}_t$ is

$$\begin{aligned}
\mathbf{y}_t &= \mathbf{y}_0 + \boldsymbol{\mu} \cdot t + \boldsymbol{\Psi}(1) \sum_{k=1}^{t} \mathbf{u}_k + \tilde{\mathbf{u}}_t - \tilde{\mathbf{u}}_0, \\
\tilde{\mathbf{u}}_t &= \tilde{\boldsymbol{\Psi}}(L)\mathbf{u}_t \\
\boldsymbol{\Psi}(1) &= \sum_{k=0}^{\infty} \boldsymbol{\Psi}_k = (\mathbf{I}_2 - \mathbf{A}_1)^{-1} \\
\tilde{\boldsymbol{\Psi}}(L) &= \sum_{k=0}^{\infty} \tilde{\boldsymbol{\Psi}}_k L^k, \tilde{\boldsymbol{\Psi}}_k = -\sum_{j=k+1}^{\infty} \boldsymbol{\Psi}_j
\end{aligned}$$

The BN decomposition gives the multivariate stochastic trends in $\mathbf{y}_t$ in terms of the *reduced form error terms* $\mathbf{u}_t$

$$\begin{aligned}
\mathbf{TS}_t &= \boldsymbol{\Psi}(1) \sum_{k=1}^{t} \mathbf{u}_k \\
&= \begin{pmatrix} \psi_{11}(1) & \psi_{12}(1) \\ \psi_{21}(1) & \psi_{22}(1) \end{pmatrix} \begin{pmatrix} \sum_{k=1}^{t} u_{1k} \\ \sum_{k=1}^{t} u_{2k} \end{pmatrix}
\end{aligned}$$

Using

$$\mathbf{u}_t = \mathbf{B}^{-1}\boldsymbol{\varepsilon}_t,\ \mathbf{\Theta}(1) = \mathbf{\Psi}(1)\mathbf{B}^{-1}$$

the multivariate stochastic trends in $\mathbf{y}_t$ may also be represented in terms of the *structural errors* ε_t

$$\mathbf{TS}_t = \mathbf{\Theta}(1)\sum_{k=1}^{t}\boldsymbol{\varepsilon}_k$$

$$= \begin{pmatrix} \theta_{11}(1) & \theta_{12}(1) \\ \theta_{21}(1) & \theta_{22}(1) \end{pmatrix} \begin{pmatrix} \sum_{k=1}^{t}\varepsilon_{1k} \\ \sum_{k=1}^{t}\varepsilon_{2k} \end{pmatrix}$$

Remarks:

1. $\mathbf{TS}_t$ is invariant to the use of $\mathbf{u}_t$ or ε_t

2.The bivariate BN decomposition uses different information set than univariate BN decomposition: they may differ substantially! This is an open area of research.

Testing Long-run Neutrality

King and Watson (1997) survey the use of bivariate SVAR models to test some simple long-run neutrality propositions in macroeconomics. The key feature of long-run neutrality propositions is that changes in nominal variables have no effect on real economic variables in the long-run. Some examples of long-run neutrality propositions are:

1. A permanent change in the nominal money stock has no long-run effect on the level of real output

2. A permanent change in the rate of inflation has no long-run effect on unemployment (a vertical Phillips curve)

3. A permanent change in the rate of inflation has no long-run effect on real interest rates (the long-run Fisher relationship).

KW show that testing long-run neutrality within a SVAR framework requires the data to be $I(1)$. They characterize long-run neutrality of money using the SMA representation for $\Delta\mathbf{y}_t$ written as

$$\begin{aligned}\text{output:} \quad & \Delta y_t = \mu_y + \theta_{yy}(L)\varepsilon_{yt} + \theta_{ym}(L)\varepsilon_{mt} \\ \text{money} \ : \quad & \Delta m_t = \mu_m + \theta_{my}(L)\varepsilon_{yt} + \theta_{mm}(L)\varepsilon_{mt}\end{aligned}$$

where ε_{yt} represents exogenous shocks to output that are uncorrelated with exogenous shocks to nominal money ε_{mt}.

Long-run neutrality of money involves the answer to the question:

- Does an unexpected and exogenous permanent change in the level of money (m) lead to a permanent change in the level of output (y)?

If the answer is no, then money is long-run neutral towards output.

In terms of the SMA representation, ε_{mt} represents exogenous unexpected changes in money.

$$\begin{aligned} \theta_{mm}(1)\varepsilon_{mt} &= \text{permanent effect of } \varepsilon_{mt} \text{ on the } m \\ \theta_{ym}(1)\varepsilon_{mt} &= \text{permanent effect of } \varepsilon_{mt} \text{ on the } y \end{aligned}$$

With the data in logs, the long-run elasticity of output with respect to permanent changes in money is

$$\gamma_{ym} = \frac{\theta_{ym}(1)}{\theta_{mm}(1)}$$

Result: money is neutral in the long-run when

$$\theta_{ym}(1) = 0 \text{ or } \gamma_{ym} = 0$$

That is, money is neutral in the long-run when the exogenous shocks that permanently alter money, ε_{mt}, have no permanent effect on output.

The restriction that money is long-run neutral for output imposes the restriction that the long-run impact matrix $\mathbf{\Theta}(1)$ is lower triangular. The lower triangularity of $\mathbf{\Theta}(1)$ implies that the multivariate stochastic trend for y_t has the form

$$\begin{bmatrix} TS_{yt} \\ TS_{mt} \end{bmatrix} = \begin{bmatrix} \theta_{yy}(1) & 0 \\ \theta_{my}(1) & \theta_{mm}(1) \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{t} \varepsilon_{yk} \\ \sum_{k=1}^{t} \varepsilon_{mk} \end{bmatrix}.$$

Hence, the stochastic trend in $y_t, TS_{yt},$ only involves shocks to ε_y.

SVAR(1) Model

$$
\begin{aligned}
\Delta y_t &= c_y + \lambda_{ym}\Delta m_t \\
&\quad +\gamma_{yy}^1 \Delta y_{t-1} + \gamma_{ym}^1 \Delta m_{t-1} + \varepsilon_{yt} \\
\Delta m_t &= c_m + \lambda_{my}\Delta y_t \\
&\quad +\gamma_{my} \Delta y_{t-1} + \gamma_{mm}^1 \Delta m_{t-1} + \varepsilon_{mt} \\
cov(\varepsilon_{yt}, \varepsilon_{mt}) &= 0 \\
\mathbf{B} &= \begin{pmatrix} 1 & -\lambda_{ym} \\ -\lambda_{my} & 1 \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{ym} &= \text{impact elasticity of } y \text{ wrt } m \\
\lambda_{my} &= \text{impact elasticity of } m \text{ wrt } y
\end{aligned}
$$

To test the long-run neutrality proposition, the SVAR model for $\Delta \mathbf{y}_t$ must be identified and estimated and then the long-run impact coefficients $\theta_{ym}(1)$ and $\theta_{mm}(1)$ can be estimated from the derived SMA model. From the previous discussion of identification, at least one restriction on the parameters of SVAR is need for identification. KW consider the following identifying assumptions:

- the impact elasticity of y with respect to m, λ_{ym}, is known,
- the impact elasticity of m with respect to y , λ_{my}, is known,
- the long-run elasticity of y with respect to m, γ_{ym}, is known,
- the long-run elasticity of m with respect to y, γ_{my}, is known.

Estimating the SVAR assuming λ_{ym} or λ_{my} is known

Suppose λ_{ym} is known. Given that λ_{ym} is known the SVAR(1) may be rewritten as

$$\Delta y_t + \lambda_{ym}\Delta y_{2t} = c_y + \gamma_{yy}^1 \Delta y_{t-1} + \gamma_{ym}^1 \Delta m_{t-1} + \varepsilon_{yt}$$
$$\Delta y_{2t} = \gamma_{20} + \lambda_{my}\Delta y_t + \gamma_{my}\Delta y_{t-1} + \gamma_{mm}^1 \Delta m_{t-1} + \varepsilon_{mt}$$

- The first equation may be estimated by OLS since only lagged values of Δy and Δm are on the right-hand-side.

- However, the second equation cannot be estimated by OLS because Δy_t will be correlated with ε_{mt} unless $\lambda_{my} = 0$.

- Need an instrument for Δy_t in the second equation

Result: If $\lambda_{ym} \neq 0$, the second equation may be estimated by IV/GMM using the residual from the estimated first equation, $\hat{\varepsilon}_{yt}$, together with Δy_{t-1} and Δm_{t-1} as instruments.

The residual ε_{yt} is a valid instrument because

$$p \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \hat{\varepsilon}_{yt} \Delta y_t \neq 0$$

since $E[\varepsilon_{yt} \Delta y_t] \neq 0$ and

$$p \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \hat{\varepsilon}_{yt} \varepsilon_{mt} = 0$$

since $E[\varepsilon_{yt} \varepsilon_{mt}] = 0$.

Remark: Hausman, Newey and Taylor (1987) show that this procedure is asymptotically equivalent to maximum likelihood estimation assuming normal errors. However, the OLS standard errors for the parameters in the second equation must be adjusted because ε_{yt} is used instead of ε_{yt}. See the appendix of King and Watson for details.

GMM estimation of SVAR (preferred method)

- Ignoring the variances and with λ_{ym} known, the SVAR has 7 parameters: $c_y, \gamma_{yy}^1, \gamma_{ym}^1, c_m, \lambda_{my}, \gamma_{my}^1, \gamma_{mm}^1$

- There are 7 population moment conditions

$$\begin{aligned} E[\varepsilon_{yt}] &= E[\varepsilon_{mt}] = E[\varepsilon_{yt}\varepsilon_{mt}] = 0 \\ E[\Delta y_{t-1}\varepsilon_{yt}] &= E[\Delta y_{t-1}\varepsilon_{mt}] = 0 \\ E[\Delta m_{t-1}\varepsilon_{yt}] &= E[\Delta m_{t-1}\varepsilon_{mt}] = 0 \end{aligned}$$

- SVAR with λ_{ym} known is exactly identified, and GMM estimation may proceed using the identity weight matrix.

- Do not have to adjust standard errors for 2nd equation estimates

Estimating long-run elasticities and extracting structural shocks

Given the estimates $\hat{\mathbf{B}}, \hat{\gamma}_0$ and $\hat{\mathbf{\Gamma}}$ of the SVAR parameters obtained using the above IV procedure, the SMA representation may be obtained directly. The process is

- Solve for the reduced form VAR

$$\begin{aligned}\mathbf{\Delta y}_t &= \hat{\mathbf{B}}^{-1}\hat{\mathbf{c}}_0 + \hat{\mathbf{B}}^{-1}\hat{\mathbf{\Gamma}}\mathbf{\Delta y}_{t-1} + \hat{\mathbf{B}}^{-1}\hat{\boldsymbol{\varepsilon}}_t \\ &= \hat{\mathbf{a}}_0 + \hat{\mathbf{A}}_1\mathbf{Y}_{t-1} + \hat{\mathbf{B}}^{-1}\hat{\boldsymbol{\varepsilon}}_t\end{aligned}$$

  Define $\mathbf{A}(L) = \mathbf{I} - \hat{\mathbf{A}}_1 L$ so that $\hat{\mathbf{A}}(1) = \mathbf{I} - \hat{\mathbf{A}}_1$.

- Solve for the SMA representation by inverting the reduced form VAR

$$\mathbf{\Delta y}_t = \hat{\boldsymbol{\mu}} + \hat{\mathbf{\Theta}}_0\hat{\boldsymbol{\varepsilon}}_t + \hat{\mathbf{\Theta}}_1\hat{\boldsymbol{\varepsilon}}_{t-1} + \cdots$$

$$\hat{\boldsymbol{\mu}} = \hat{\mathbf{A}}(1)^{-1}\hat{\mathbf{a}}_0, \hat{\mathbf{\Theta}}_0 = \hat{\mathbf{B}}^{-1}$$

$$\hat{\mathbf{\Theta}}_k = \hat{\mathbf{A}}_1^k\hat{\mathbf{B}}^{-1} = (\hat{\mathbf{B}}^{-1}\hat{\mathbf{\Gamma}})^k\hat{\mathbf{B}}^{-1}$$

- Solve for the long-run elasticity

$$\hat{\gamma}_{ym} = \frac{\hat{\theta}_{ym}(1)}{\hat{\theta}_{mm}(1)}$$

$$\hat{\mathbf{\Theta}}(1) = \hat{\mathbf{A}}(1)^{-1}\hat{\mathbf{B}}^{-1} = \begin{bmatrix} \hat{\theta}_{yy}(1) & \hat{\theta}_{ym}(1) \\ \hat{\theta}_{my}(1) & \hat{\theta}_{mm}(1) \end{bmatrix}$$

and compute standard errors using delta method

- The estimated structural errors $\hat{\boldsymbol{\varepsilon}}_t = \mathbf{\Delta y}_t - \hat{\boldsymbol{\gamma}}_0 - \hat{\mathbf{\Gamma}}_1 \mathbf{\Delta y}_{t-1}$ may be used to compute the BN decompostion and extract the stochastic trends

$$\begin{bmatrix} \widehat{TS}_{yt} \\ \widehat{TS}_{mt} \end{bmatrix} = \begin{bmatrix} \theta_{yy}(1) & \theta_{ym}(1) \\ \theta_{my}(1) & \theta_{mm}(1) \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{t} \hat{\varepsilon}_{yk} \\ \sum_{k=1}^{t} \hat{\varepsilon}_{mk} \end{bmatrix}$$

One may also use the reduced form errors $\hat{\mathbf{u}}_t = \mathbf{\Delta y}_t - \hat{\mathbf{a}}_0 - \hat{\mathbf{A}}_1 \mathbf{Y}_{t-1}$

$$\begin{bmatrix} \widehat{TS}_{yt} \\ \widehat{TS}_{mt} \end{bmatrix} = \begin{bmatrix} \hat{\psi}_{yy}(1) & \hat{\psi}_{ym}(1) \\ \hat{\psi}_{my}(1) & \hat{\psi}_{mm}(1) \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{t} \hat{u}_{yk} \\ \sum_{k=1}^{t} \hat{u}_{mk} \end{bmatrix}$$

where $\hat{\mathbf{\Psi}}(1) = \hat{\mathbf{A}}(1)^{-1}$.

Summary of King and Watson Results

- Use quarterly data from 1949:I - 1990:4

- Reduced form VAR is estimated with 6 lags of all variables

- Long-run money neutrality is not rejected at 5% level for values of λ_{my} (initial impact of money to ouput) < 1.40

- Long-run money neutrality is not rejected at 5% level for values of λ_{ym} (initial impact of output to money) > -4.61

Structural VARs with Combinations of I(1) and I(0) Data

Consider two observed series y_{1t} and y_{2t} such that y_{1t} is $I(1)$ and y_{2t} is $I(0)$. For example, in the analysis in Blanchard and Quah (1989)

$$\begin{aligned} y_1 &= \text{log of real GDP} \\ y_2 &= \text{unemployment rate.} \end{aligned}$$

Define

$$\begin{aligned} \mathbf{y}_t &= (\Delta y_{1t}, y_{2t})' \\ &\Rightarrow \mathbf{y}_t \sim I(0) \end{aligned}$$

Suppose $\mathbf{y}_t$ has the structural representations

$$\begin{aligned} \mathbf{B}\mathbf{y}_t &= \boldsymbol{\gamma}_0 + \boldsymbol{\Gamma}_1 \mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t \\ \mathbf{y}_t &= \boldsymbol{\mu} + \boldsymbol{\Theta}(L)\boldsymbol{\varepsilon}_t \\ \boldsymbol{\Theta}(L) &= \boldsymbol{\Psi}(L)\mathbf{B}^{-1} \\ E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] &= \mathbf{D} = \text{diagonal} \end{aligned}$$

with reduced form representations

$$\begin{aligned}
\mathbf{y}_t &= \mathbf{a}_0 + \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{u}_t \\
&= \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\mathbf{u}_t \\
\boldsymbol{\Psi}(L) &= (\mathbf{I}_2 - \mathbf{A}_1 L)^{-1} \\
E[\mathbf{u}_t\mathbf{u}_t'] &= \boldsymbol{\Omega} = \mathbf{B}^{-1}\mathbf{D}\mathbf{B}^{-1\prime}
\end{aligned}$$

Note: BQ loosely interpret ε_{1t} as a permanent (supply) shock since it is the innovation to the $I(1)$ real output series y_t and interpret ε_{2t} as a transitory (demand) shock since it is the innovation to the $I(0)$ unemployment series.

The structural IRFs are given by

$$\begin{aligned}
\frac{\partial \Delta y_{1t+s}}{\partial \varepsilon_{1t}} &= \theta_{11}^{(s)}, \frac{\partial \Delta y_{1t+s}}{\partial \varepsilon_{2t}} = \theta_{12}^{(s)} \\
\frac{\partial y_{2t+s}}{\partial \varepsilon_{1t}} &= \theta_{21}^{(s)}, \frac{\partial y_{2t+s}}{\partial \varepsilon_{2t}} = \theta_{22}^{(s)}
\end{aligned}$$

Since y_t (output) is $I(1)$, the long-run impacts on the *level* of y_1 of shocks to ε_1 and ε_2 are

$$\begin{aligned}\lim_{s\to\infty}\frac{\partial y_{1t+s}}{\partial \varepsilon_{1t}} &= \theta_{yy}(1)=\sum_{s=0}^{\infty}\theta_{yy},\\ \lim_{s\to\infty}\frac{\partial y_{1t+s}}{\partial \varepsilon_{2t}} &= \theta_{12}(1)=\sum_{s=0}^{\infty}\theta_{12}^{(s)}.\end{aligned}$$

Since y_{2t} (unemployment) is $I(0)$, the long-run impacts on the *level* of y_2 of shocks to ε_1 and ε_2 are zero:

$$\lim_{s\to\infty}\frac{\partial y_{2t+s}}{\partial \varepsilon_{jt}}=\lim_{s\to\infty}\theta_{2j}^{(s)}=0.$$

For y_2,

$$\begin{aligned}\theta_{21}(1) &= \sum_{s=0}^{\infty}\theta_{21}^{(s)}\\ \theta_{22}(1) &= \sum_{s=0}^{\infty}\theta_{22}^{(s)}\end{aligned}$$

represent the *cumulative impact* of shocks to ε_1 and ε_2 on the level of y_2.

Identifying the SVAR Using Long-Run Restrictions

BQ achieve identification of the SVAR/SMA by assuming

- Transitory (demand) shocks (shocks to ε_2) have no long-run impact on the level of output or unemployment.

- They allow permanent (supply) shocks (shocks to ε_1) to have a long-run impact on the level of output but not on the level of unemployment.

The restriction that shocks to ε_2 have no long-run impact on the level of y_1 implies that

$$\theta_{12}(1) = \sum_{s=0}^{\infty} \theta_{12}^{(s)} = 0.$$

The restriction that shocks to ε_1 and ε_2 have no long-run effect on the level of y_2 is

$$\lim_{s\to\infty} \frac{\partial y_{2t+s}}{\partial \varepsilon_{jt}} = \lim_{s\to\infty} \theta_{2j}^{(s)} = 0.$$

which follows automatically since $y_2 \sim I(0)$.

BQ assumptions imply that the long-run impact matrix $\mathbf{\Theta}(1)$ is lower triangular

$$\mathbf{\Theta}(1) = \left[\begin{array}{cc} \theta_{11}(1) & 0 \\ \theta_{21}(1) & \theta_{22}(1) \end{array} \right]$$

Claim: The lower triangularity of $\mathbf{\Theta}(1)$ can be used to indentify $\mathbf{B}$.

To see why, consider the *long-run covariance matrix* of $\mathbf{y}_t$ defined from the Wold representation

$$\begin{aligned}\mathbf{\Lambda} &= \mathbf{\Psi}(1)\mathbf{\Omega}\mathbf{\Psi}(1)' \\ &= (\mathbf{I}_2 - \mathbf{A}_1)^{-1}\mathbf{\Omega}(\mathbf{I}_2 - \mathbf{A}_1)^{-1\prime}.\end{aligned}$$

Since

$$\begin{aligned}\mathbf{\Omega} &= \mathbf{B}^{-1}\mathbf{D}\mathbf{B}^{-1\prime} \\ \mathbf{\Theta}(1) &= \mathbf{\Psi}(1)\mathbf{B}^{-1} = (\mathbf{I}_2 - \mathbf{A}_1)^{-1}\mathbf{B}^{-1}\end{aligned}$$

$\mathbf{\Lambda}$ may be re-expressed as

$$\begin{aligned}\mathbf{\Lambda} &= (\mathbf{I}_2 - \mathbf{A}_1)^{-1}\mathbf{B}^{-1}\mathbf{D}\mathbf{B}^{-1\prime}(\mathbf{I}_2 - \mathbf{A}_1)^{-1\prime} \\ &= \mathbf{\Theta}(1)\mathbf{D}\mathbf{\Theta}(1)'\end{aligned}$$

In order to identify $\mathbf{B}$, BQ make the additional assumption

$$\mathbf{D} = \mathbf{I}_2$$

so that the structural shocks ε_{1t} and ε_{2t} have unit variances. Then

$$\mathbf{\Lambda} = \mathbf{\Theta}(1)\mathbf{\Theta}(1)'.$$

Since $\boldsymbol{\Theta}(1)$ is lower triangular, Λ can be obtained *uniquely* using the Choleski factorization; that is, $\boldsymbol{\Theta}(1)$ can be computed as the lower triangular Choleski factor of $\boldsymbol{\Lambda}$.The Choleski factorization of Λ is

$$\begin{aligned} \boldsymbol{\Lambda} &= \mathbf{P}\mathbf{P}' = \boldsymbol{\Theta}(1)\boldsymbol{\Theta}(1)' \\ &\Rightarrow \boldsymbol{\Theta}(1) = \mathbf{P} \end{aligned}$$

Given that $\boldsymbol{\Theta}(1) = \mathbf{P}$ can be computed directly from Λ, $\mathbf{B}$ can then be computed using

$$\begin{aligned} \mathbf{P} &= \boldsymbol{\Theta}(1) = \boldsymbol{\Psi}(1)\mathbf{B}^{-1} = (\mathbf{I}_2 - \mathbf{A}_1)^{-1}\mathbf{B}^{-1} \\ &\Rightarrow \mathbf{B} = [(\mathbf{I}_2 - \mathbf{A}_1)\mathbf{P}]^{-1}. \end{aligned}$$

and the SVAR model is exactly identified!

Estimating the SVAR in the Presence of Long-Run Restrictions

The estimation of $\mathbf{B}$ and $\Theta(L)$ using the BQ identification scheme can be accomplished in two steps.

- Estimate the reduced form VAR by OLS equation by equation:

$$\begin{aligned}\mathbf{y}_t &= \widehat{\mathbf{a}}_0 + \widehat{\mathbf{A}}_1 \mathbf{y}_{t-1} + \widehat{\mathbf{u}}_t \\ \widehat{\boldsymbol{\Omega}} &= \frac{1}{T}\sum_{t=1}^{T} \widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t'\end{aligned}$$

- Compute a parametric estimate of the long-run covariance matrix:

$$\hat{\boldsymbol{\Lambda}} = (\mathbf{I}_2 - \hat{\mathbf{A}}_1)^{-1} \hat{\boldsymbol{\Omega}} (\mathbf{I}_2 - \hat{\mathbf{A}}_1)^{-1\prime}$$

- Compute the Choleski factorization of $\hat{\boldsymbol{\Lambda}}$:

$$\hat{\boldsymbol{\Lambda}} = \widehat{\mathbf{P}} \widehat{\mathbf{P}}'$$

- Define the estimate of $\mathbf{\Theta}(1)$ as the lower triangular Choleski factor of $\hat{\mathbf{\Lambda}}$:

$$\hat{\mathbf{\Theta}}(1) = \hat{\mathbf{P}}$$

- Estimate $\mathbf{B}$ using

$$\hat{\mathbf{B}} = \left[(\mathbf{I}_2 - \hat{\mathbf{A}}_1)\hat{\mathbf{\Theta}}(1)\right]^{-1}$$

- Estimate $\mathbf{\Theta}_k$ using

$$\begin{aligned} \hat{\mathbf{\Theta}}_k &= \hat{\mathbf{\Psi}}_k \hat{\mathbf{B}}^{-1} \\ &= \hat{\mathbf{A}}_1^k \hat{\mathbf{B}}^{-1}. \end{aligned}$$

From the estimated $\mathbf{\Theta}_k$ matrices the structural IRF and FEVD may then be computed. Also, estimates of the structural shocks $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ may be extracted using $\hat{\boldsymbol{\varepsilon}}_t = \hat{\mathbf{B}}\hat{\mathbf{u}}_t$.