Example: stylized consumption function (Campbell and Mankiw (1990)

$$\begin{aligned}\Delta c_t &= \delta_0 + \delta_1 \Delta y_t + \delta_2 r_t + \varepsilon_t,\ t = 1, \ldots, T \\ &= \boldsymbol{\delta}' \mathbf{z}_t + \varepsilon_t \\ L &= 3\end{aligned}$$

where

$c_t =$ the log of real per capita consumption (excluding durables),

$y_t =$ the log of real disposable income, and

$r_t =$ the ex post real interest rate (T-bill rate - inflation rate).

Note: See Zivot and Wang (2005), Chapter 21 for S-PLUS code to replicate this example.

Assumptions

$\{\Delta c_t, \Delta y_t, r_t\}$ are stationary and ergodic

$\{\varepsilon_t, I_t\}$ is a stationary and ergodic martingale difference sequence (MDS) where $I_t = \{\Delta c_s, \Delta y_s, r_s\}_{s=1}^{t}$ denotes the observed information set at time t.

Endogeneity and Instruments

The variables Δy_t and r_t are likely to be contemporaneously correlated with ε_t

Because $\{\varepsilon_t, I_t\}$ is a stationary and ergodic MDS, $E[\varepsilon_t | I_{t-1}] = 0$ which implies that any variable in I_{t-1} is a potential instrument.

For any variable $x_{t-1} \subset I_{t-1}$, $\{x_{t-1}\varepsilon_t\}$ is an uncorrelated sequence.

Data: Annual data over the period 1960 to 1995 taken from Wooldridge (2002)

Example: Testing the Permanent Income Hypothesis

The pure permanent income hypothesis (PIH) due to Hall (1978) states that c_t is a martingale so that $\Delta c_t = \varepsilon_t$ is a MDS.

Hence, the PIH implies the linear restrictions

$$H_0 : \delta_1 = \delta_2 = 0$$

which are of the form $\mathbf{R}\boldsymbol{\delta} = \mathbf{r}$ with

$$\begin{aligned} \mathbf{R} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \text{rank}(\mathbf{R}) &= 2 \end{aligned}$$

If there are temporary income consumers, then $\delta_1 > 0$.

$\Delta c_t = \delta_1 + \delta_2 \Delta y_t + \delta_3 r_t + \varepsilon_t$ $\mathbf{x}_t = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'$, $E[\mathbf{x}_t \varepsilon_t] = 0,\ E[\mathbf{x}_t \mathbf{x}_t' \varepsilon_t^2] = \mathbf{S}$				
Estimator	δ_1	δ_2	δ_3	J
2-step	.007 (.004)	.627 (.150)	$-$.010 (.098)	1.578 (.209)
Iterated	.008 (.004)	.591 (.144)	$-$.032 (.095)	1.855 (.173)
CU	.008 (.003)	.574 (.139)	$-$.054 (.095)	1.747 (.186)
1-step $\mathbf{W} = \mathbf{I}_4$	.003 (.005)	.801 (.223)	$-$.024 (.116)	– –
2SLS	.008 (.003)	.586 (.133)	$-$.027 (.175)	2.018 (.155)

Table 1: GMM estimates of the consumption function parameters.

GMM Wald Statistic

This is based on any of the unrestricted GMM estimates (efficient or inefficient). Using the iterated GMM estimate, the Wald statistics is

$$\text{Wald} = n(\mathbf{R}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}) - \mathbf{r})' \left[\mathbf{R}\widehat{\text{avar}}(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}))\mathbf{R}'\right]^{-1}$$
$$\times(\mathbf{R}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}) - \mathbf{r}) = 16.99$$

Since $\text{rank}(\mathbf{R}) = 2$, Wald $\sim \chi^2(2)$. The p-value is 0.0002, so we reject the PIH at any reasonable level of significance.

$H_0: \delta_1 = \delta_2 = 0$

$H_0: \delta_1 = \delta_2 = 0$

GMM-LR Statistic

This statistic can only be computed using an efficient GMM estimator (2-step, iterated, CU, 2SLS).

It is based on the difference between a restricted and unrestricted J-statistic.

The unrestricted model is

$$\Delta c_t = \delta_0 + \delta_1 \Delta y_t + \delta_2 r_t + \varepsilon_t, \ t = 1, \ldots, T$$

The J-statistic from the iterated efficient GMM estimation is

$$J(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}), \hat{\mathbf{S}}_{\text{iter}}^{-1}) = 1.855$$

The restricted model imposes $H_0 : \delta_1 = \delta_2 = 0$

$$\Delta c_t = \delta_0 + \varepsilon_t,\ t = 1, \ldots, T$$

To ensure a positive GMM-LR statistic, the restricted model should be estimated using the unrestricted efficient weight matrix $\hat{\mathbf{S}}_{\text{iter}}^{-1}$. (Some software, eg. Eviews 6, cannot do this).

The J-statistic from the restricted efficient GMM estimation is

$$J(\tilde{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}), \hat{\mathbf{S}}_{\text{iter}}^{-1}) = 18.8505$$

The GMM-LR statistic is then

$$\begin{aligned} \text{LR}_{\text{GMM}} &= J(\tilde{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}), \hat{\mathbf{S}}_{\text{iter}}^{-1}) - J(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{iter}}^{-1}), \hat{\mathbf{S}}_{\text{iter}}^{-1}) \\ &= 18.8505 - 1.855 = 16.99 \end{aligned}$$

Note: $\text{LR}_{\text{GMM}} = \textbf{Wald}$ since $H_0 : \delta_1 = \delta_2 = 0$ is a linear hypothesis.

Example: Testing Endogeneity of r_t in consumption function

$$\begin{aligned}\Delta c_t &= \delta_0 + \delta_1 \Delta y_t + \delta_2 r_t + \varepsilon_t,\ t = 1, \ldots, T \\ \mathbf{x}_t &= (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'\end{aligned}$$

Consider testing

$$\begin{aligned}H_0 &: E[r_t \varepsilon_t] = 0 \text{ } (r_t \text{ is exogenous}) \\ H_1 &: E[r_t \varepsilon_t] \neq 0 \text{ } (r_t \text{ is endogenous})\end{aligned}$$

Under H_0 the full set of instruments is

$$\mathbf{x}_t = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1}, r_t)'$$

and under H_1 the valid instruments are

$$\mathbf{x}_{1t} = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'$$

Therefore

$$\begin{aligned} x_{2t} &= r_t = \text{suspect instrument} \\ K &= 5,\ K_1 = 4, K - K_1 = K_2 = 1 \end{aligned}$$

Unrestricted GMM based on $\mathbf{x}_t = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1}, r_t)'$ gives

$$J(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{Full}}^{-1}), \hat{\mathbf{S}}_{\text{Full}}^{-1}) = 1.9528$$

Restricted GMM based on $\mathbf{x}_{1t} = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'$ and $\hat{\mathbf{S}}_{11,\text{Full}}^{-1}$ gives

$$J(\tilde{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{11,\text{Full}}^{-1}), \hat{\mathbf{S}}_{11,\text{Full}}^{-1}) = 1.9346$$

The C-statistic is therefore

$$\begin{aligned} C &= J(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{Full}}^{-1}), \hat{\mathbf{S}}_{\text{Full}}^{-1}) - J(\tilde{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{11,\text{Full}}^{-1}), \hat{\mathbf{S}}_{11,\text{Full}}^{-1}) \\ &= 0.0182 \end{aligned}$$

The p-value based on the $\chi^2(1)$ distribution is 0.892 so we do not reject the null that r_t is exogenous in the consumption function.

Example: Testing Exogeneity of Δc_{t-1} in consumption function

$$\begin{aligned}\Delta c_t &= \delta_0 + \delta_1 \Delta y_t + \delta_2 r_t + \varepsilon_t,\ t = 1, \ldots, T \\ \mathbf{x}_t &= (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'\end{aligned}$$

Here

$$\begin{aligned}H_0 &: E[\Delta c_{t-1}\varepsilon_t] = 0 \text{ } (\Delta c_{t-1} \text{ is exogenous}) \\ H_1 &: E[\Delta c_{t-1}\varepsilon_t] \neq 0 \text{ } (\Delta c_{t-1} \text{ is endogenous})\end{aligned}$$

Under H_0 the valid instruments are

$$\mathbf{x}_t = (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'$$

and under H_1 the valid instruments are

$$\mathbf{x}_{1t} = (1, \Delta y_{t-1}, r_{t-1})', K_1 = 3$$

Remark: Under H_1, δ_0 is just identified $(K_1 = L)$ so Restricted GMM based on $\mathbf{x}_{1t} = (1, \Delta y_{t-1}, r_{t-1})'$ and $\hat{\mathbf{S}}_{11,\text{Full}}^{-1}$ gives

$$J(\tilde{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{11,\text{Full}}^{-1}), \hat{\mathbf{S}}_{11,\text{Full}}^{-1}) = 0$$

Therefore, the C stat is

$$C = J(\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{\text{Full}}^{-1}), \hat{\mathbf{S}}_{\text{Full}}^{-1}) = 1.855$$

which is identical to the $J-$statistic for the unrestricted model.

Remark:

We will get exactly the same result if we test

$$\begin{aligned} H_0 &: E[\Delta y_{t-1}\varepsilon_t] = 0 \\ H_1 &: E[\Delta y_{t-1}\varepsilon_t] \neq 0 \end{aligned}$$

or if we test

$$\begin{aligned} H_0 &: E[r_{t-1}\varepsilon_t] = 0 \\ H_1 &: E[r_{t-1}\varepsilon_t] \neq 0 \end{aligned}$$

$\text{rank}(\Sigma_{xz}) = L$ $\quad S_{xz} \xrightarrow{p} \Sigma_{xz}$

H_0: Σ_{xz} has rank L based on examining rank of S_{xz}

Example: Testing instrument relevance in the consumption function

$$
\begin{aligned}
\Delta c_t &= \delta_0 + \delta_1 \Delta y_t + \delta_2 r_t + \varepsilon_t, \; t = 1, \ldots, T \\
\mathbf{x}_t &= (1, \Delta c_{t-1}, \Delta y_{t-1}, r_{t-1})'
\end{aligned}
$$

There are 2 endogenous variables so we have 2 reduced form equations

$$
\begin{aligned}
\Delta y_t &= \pi_{10} + \pi_{11} \Delta c_{t-1} + \pi_{12} \Delta y_{t-1} + \pi_{13} r_{t-1} + v_{1t} \\
r_t &= \pi_{20} + \pi_{21} \Delta c_{t-1} + \pi_{22} \Delta y_{t-1} + \pi_{23} r_{t-1} + v_{2t}
\end{aligned}
$$

Instruments are completely irrelevant if

$$
\begin{aligned}
\pi_{11} &= \pi_{12} = \pi_{13} = 0 \\
\pi_{21} &= \pi_{22} = \pi_{23} = 0
\end{aligned}
$$

Least squares estimation of the first stage for Δy_t gives

```
Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept)  0.0067  0.0055     1.2323  0.2271
  tslag(GC)  1.2345  0.3955     3.1214  0.0039
  tslag(GY) -0.5226  0.2781    -1.8787  0.0697
  tslag(R3)  0.0847  0.1395     0.6069  0.5483
```

Least squares estimation of the first stage for r_t gives

```
Coefficients:
               Value Std. Error t value Pr(>|t|)
(Intercept)  0.0083  0.0044     1.8987  0.0669
  tslag(GC)  0.1645  0.3167     0.5192  0.6073
  tslag(GY) -0.4290  0.2228    -1.9259  0.0633
  tslag(R3)  0.8496  0.1117     7.6049  0.0000
```