Topics

- Course introduction
- Large sample theory
  - Introduction to asymptotic analysis
  - Laws of large numbers
  - Consistency of estimators

Motivating Example

\( X_1, \ldots, X_n \) iid \( E(X_i) = \mu, \text{var}(X_i) = \sigma^2 \)

A natural estimator of \( \mu \) is

\[ \hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \]

Properties

\[ E[\bar{X}] = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu \]

\[ \text{var}(\bar{X}) = \text{var}(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(X_i) \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{n \sigma^2}{n^2} = \frac{\sigma^2}{n} \]
As $n \to \infty$ the pdf of $\bar{x}$ collapses at $\mu$.

Notation:

\[ \text{Var}(\bar{x}) = \frac{\sigma^2}{n}, \quad \sqrt{\text{Var}(\bar{x})} = \text{SE}(\bar{x}) = \frac{\sigma}{\sqrt{n}} \]

Asymptotic Normality

Finite sample mean of $\bar{x} = \frac{1}{n} \sum x_i$ is not known, but the mean of $\bar{x}$ is not known. \( f(\bar{x}) \) is unknown.
However, we know that

\[ E[\bar{x}] = \mu \]
\[ \text{var}(\bar{x}) = \frac{\sigma^2}{n} \]

Create the standardized r.v.

\[ z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{\bar{x} - \mu}{\sigma} \]
\[ E[z] = 0 \quad \text{var}(z) = 1 \]

So the CLT says for large enough \( n \)

\[ z = \sqrt{n} \left( \frac{\bar{x} - \mu}{\sigma} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\sigma} \right) \]

has a distn that is well approximated

by a standard normal distribution.
Notation

$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

"is asymptotically distributed as"

$N(0, 1)$ is the approximate distribution for

$\sqrt{n} \left( \frac{\bar{x} - \mu}{\sigma} \right)$
What about $\bar{x}$?

$$\Gamma_n\left(\bar{x} - \mu \left| \frac{\sigma}{\sqrt{n}}\right.\right) = z \sim N(0,1)$$

\[\Rightarrow \quad \bar{x} - \mu = \frac{\sigma}{\sqrt{n}} \cdot z\]

\[\Rightarrow \quad \bar{x} = \mu + \frac{\sigma}{\sqrt{n}} \cdot z\]

$$\mathbb{E}[\bar{x}] = \mu$$

$$\text{Var}(\bar{x}) = \frac{\sigma^2}{n}$$

$$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$$

Notation:

$$\text{var}(\bar{x}) = \frac{\sigma^2}{n} = \text{variance of the asymptotic normal distribution.}$$

$$\text{var}(\Gamma_n(\bar{x} - \mu)) = \sigma^2$$
\[ \text{A natural estimate of } \sigma^2 \text{ is the sample variance} \]
\[ \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]

Inference: 95\% confidence interval

\[ \bar{x} \pm 1.96 \times \text{ASE} (\bar{x}) \]

\[ \bar{x} \pm 1.96 \times \text{ASE} (\hat{\sigma}) \]

\[ \bar{x} \pm 1.96 \times \frac{\hat{\sigma}}{\sqrt{n}} \]
$H_0: \mu = \mu_0 \text{ vs. } H_1: \mu \neq \mu_0$

$X \sim N(\mu, \frac{\sigma^2}{n})$

$t_{\mu=\mu_0} = \frac{\bar{X} - \mu_0}{\sigma \sqrt{\frac{1}{n}}} = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$

Under $H_0: \mu = \mu_0$

$t_{\mu=\mu_0} \overset{D}{\sim} N(0,1)$

Reject $H_0: \mu = \mu_0 \text{ vs. } H_1: \mu \neq \mu_0$ if $|t_{\mu=\mu_0}| \geq 1.96$

Asymptotic size of the test is $\alpha_0$, but in finite samples (e.g. $n=25$) the actual size could be different from $\alpha_0$. 