Econ 424
Portfolio Theory with No Short Sales

Eric Zivot

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Efficient Portfolios without Short-Sales

Short Sale

- Borrow asset from broker and sell now
- To close short position, buy back asset and return to broker
- Profit if asset price drops after short sale
- If asset i is sold short then

$$x_i < 0$$

where $x_i =$ share of wealth in asset i

No Short Sale Restrictions

- Exchanges (e.g. NYSE, NASDAQ) may prevent short sales in some assets
- Some institutions (e.g. pension funds) are prevented from short-selling assets
- Certain accounts do not allow short sales (e.g. retirement accounts)
- Short selling often requires substantial credit qualifications

Markowitz Algorithm with No Short Sales Restrictions

$$
\begin{aligned}
\min_{\mathbf{x}} \sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \quad \text{s.t.} \\
\mu_{p,x} &= \mathbf{x}'\mu = \mu_p^0 \\
\mathbf{x}'\mathbf{1} &= 1 \\
x_i &\geq 0 \ (i = 1, \ldots, n)
\end{aligned}
$$

Remark: With inequality constraints, the Lagrange multiplier method no longer works because it imposes an equality in the constraint.

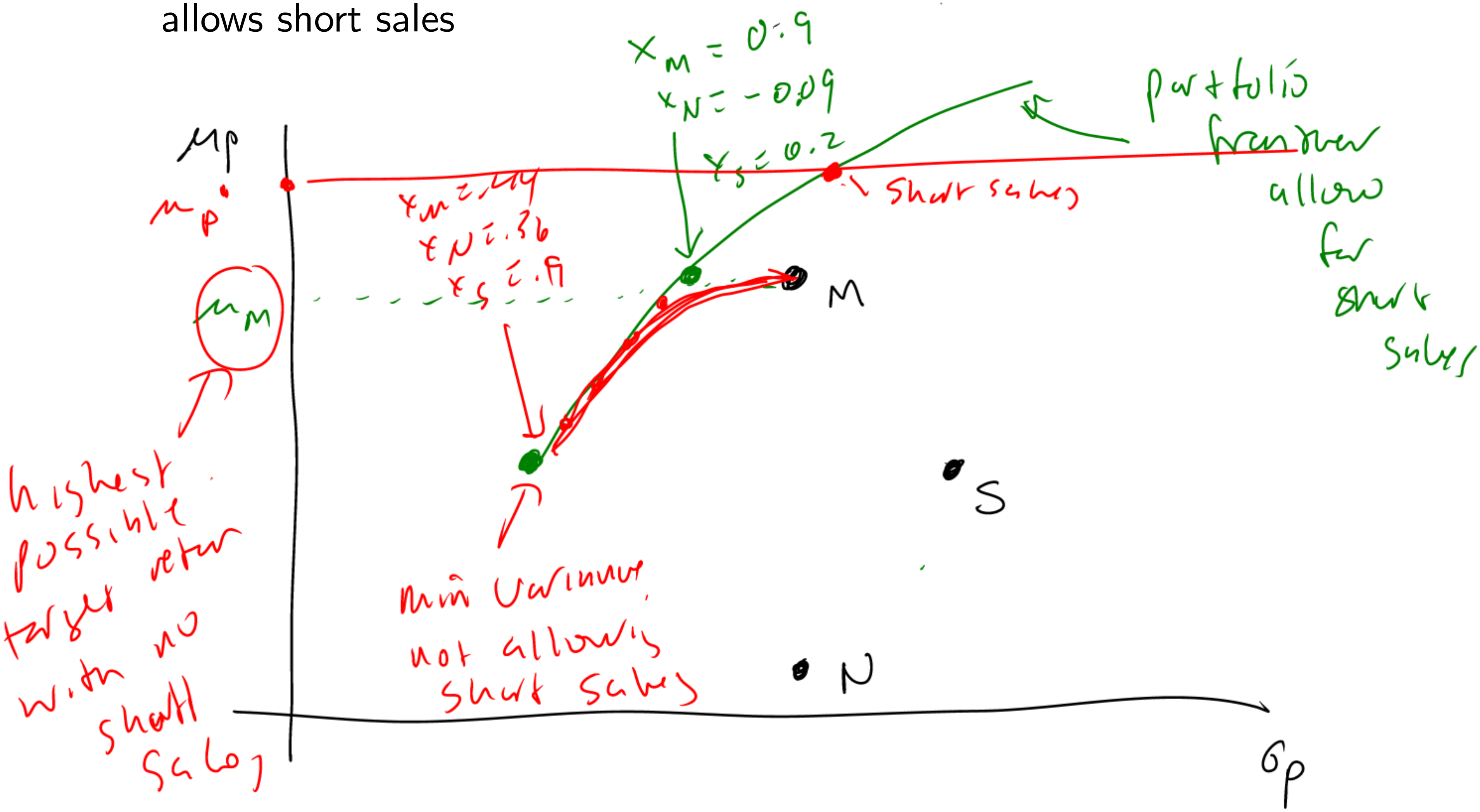
Remarks:

- Problem must be solved numerically, e.g. using the Solver in Excel or the function `solve.QP()` in the R package quadprog.

- Portfolio Frontier can no longer be constructed from any two efficient portfolios (cannot guarantee positive weights). It has to be computed by brute force for each portfolio with target expected return above the global minimum variance expected return.

- There may not be a feasible solution; i.e., there may not exist a no-short sale portfolio that reaches the target return μ_p^0.

- No short sale portfolio frontier must lie “inside” the portfolio frontier that allows short sales



When $\mu_p^{target} = \mu_{ms,2}$ then the No short sales constraint becomes binding b/c we know the efficient portfolio that allows shorts has $x_N = -0.09$

R functions for Minimum Variance Portfolios with No Short Sales Restrictions

IntroCompFin R

- My `portfolio_noshorts.r` functions can restrict short-sales for all assets (set optional argument `shorts=FALSE`)

- R package quadprog function `solve.QP()`

- R package tseries function `portfolio.optim()` (wrapper for `solve.QP`)

- Guy Yollin's R in Finance presentation on R tools for portfolio optimization

  http://www.rinfinance.com/RinFinance2009/presentations/yollin_slides.pdf

- Rmetrics (`www.Rmetrics.org`) package fPortfolio

- PortfolioAnalytics package functions (from the authors of PerformanceAnalytics) available on R-forge (not ready yet for CRAN)

Solving the Markowitz Algorithm with No Short Sales using the R function solve.QP

Ruppert chapter 11 section 6 shows how the portfolio optimization problem with inequality constraints can be set up as a quadratic programming problem that can be solved with the R package `quadprog` function `solve.QP()`.

Quadratic programming problems are of the form

$$\min_{x} \; \frac{1}{2}\mathbf{x}'\mathbf{D}\mathbf{x} - \mathbf{d}'\mathbf{x}$$
$$\mathbf{A}'_{neq}\mathbf{x} \geq \mathbf{b}_{neq} \text{ for } m \text{ inequality constraints}$$
$$\mathbf{A}'_{eq}\mathbf{x} = \mathbf{b}_{eq} \text{ for } l \text{ equality constraints}$$

where $\mathbf{D}$ is an $n \times n$ matrix, $\mathbf{x}$ and $\mathbf{d}$ are $n \times 1$ vectors, $\mathbf{A}'_{neq}$ is an $m \times n$ matrix, $\mathbf{b}_{neq}$ is an $m \times 1$ vector, $\mathbf{A}'_{eq}$ is an $l \times n$ matrix, and $\mathbf{b}_{eq}$ is an $l \times 1$ vector.

Consider the portfolio optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \sigma^2_{p,x} &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\mu = \mu^0_p \\ \mathbf{x}'\mathbf{1} &= 1 \\ x_i &\geq 0 \ (i = 1, \ldots, n) \end{aligned}$$

1. For the objective function $\frac{1}{2}\mathbf{x}'\mathbf{D}\mathbf{x} - \mathbf{d}'\mathbf{x}$, set

$$\mathbf{D} = 2 \cdot \mathbf{\Sigma} \text{ and } \mathbf{d} = (0, \ldots, 0)'$$

Then

$$\frac{1}{2}\mathbf{x}'\mathbf{D}\mathbf{x} - \mathbf{d}'\mathbf{x} = \mathbf{x}'\mathbf{\Sigma}\mathbf{x}$$

2. For the $l = 2$ equality constraints $\mathbf{A}'_{eq}\mathbf{x} = \mathbf{b}_{eq}$, we have

$$\mathbf{x}'\mu = \mu'\mathbf{x} = \mu_p^0,$$
$$\mathbf{x}'\mathbf{1} = \mathbf{1}'\mathbf{x} = 1,$$

and so set

$$\underset{(2\times n)}{\mathbf{A}'_{eq}} = \begin{pmatrix} \mu' \\ \mathbf{1}' \end{pmatrix},$$
$$\underset{(2\times 1)}{\mathbf{b}_{eq}} = \begin{pmatrix} \mu_p^0 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{A}'_{eq}\mathbf{x} = \begin{pmatrix} \mu' \\ \mathbf{1}' \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mu'\mathbf{x} \\ \mathbf{1}'\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mu_p^0 \\ 1 \end{pmatrix}$$

3. For the $m = n$ inequality constraints $\mathbf{A}'_{neq}\mathbf{x} \geq \mathbf{b}_{neq}$ we have

$$x_i \geq 0, \ i = 1, \ldots, n$$
$$\Rightarrow \mathbf{x} \geq \mathbf{0},$$

so set

$$\underset{(n \times n)}{\mathbf{A}'_{neq}} = \mathbf{I}_n,$$
$$\underset{(n \times 1)}{\mathbf{b}_{neq}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{A}'_{neq}\mathbf{x} = \mathbf{I}_n\mathbf{x} = \mathbf{x} \geq \mathbf{0}$$

Remark: The function `solve.QP()` assumes that the inequality constraints and equality constraints are combined into a single $(m+l) \times n$ matrix $\mathbf{A}'$ and a single $(m+l) \times 1$ vector $\mathbf{b}$ of the form

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A}'_{eq} \\ \mathbf{A}'_{neq} \end{bmatrix}, \ \mathbf{b} = \begin{pmatrix} \mathbf{b}_{eq} \\ \mathbf{b}_{neq} \end{pmatrix}$$

For the portfolio problem we have

$$\mathbf{A}' = \begin{pmatrix} \mu' \\ \mathbf{1}' \\ \mathbf{I}_n \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} \mu_p^0 \\ 1 \\ \mathbf{0}_n \end{pmatrix}$$

where $\mathbf{0}_n = (0, \ldots, 0)'$.

Solving for the Global Minimum Variance Portfolio with No Short Sales

The optimization problem is

$$\min_{\mathbf{m}} \sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t.}$$
$$\mathbf{m}'\mathbf{1} = 1, \; m_i \geq 0$$

Here, the restriction matrices are

$$\underset{(1\times n)}{\mathbf{A}'_{eq}} = \mathbf{1}', \;\; \underset{(1\times 1)}{\mathbf{b}_{eq}} = 1$$
$$\underset{(n\times n)}{\mathbf{A}'_{neq}} = \mathbf{I}_n \text{ and } \underset{(n\times 1)}{\mathbf{b}_{neq}} = (0,\ldots,0)'$$

So that

$$\mathbf{A}' = \begin{pmatrix} \mathbf{1}' \\ \mathbf{I}_n \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$$

Finding the Efficient Frontier with No Short Sales

- Compute the global minimum variance portfolio with no-short sales
- Set an initial grid of target expected returns between the expected return on the global minimum variance portfolio with no short sales and the highest single asset expected return
- Solve the Markowitz algorithm with no-short sales for each target expected return in the grid
- Test to see if feasible solutions exist for target expected returns above the highest single asset expected return