Introduction to Computational Finance and Financial Econometrics

Portfolio Theory with Matrix Algebra

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Outline

1. Portfolios with Three Risky Assets
   - Portfolio characteristics using matrix notation
   - Finding the global minimum variance portfolio
   - Finding efficient portfolios
   - Computing the efficient frontier
   - Mutual fund separation theorem again

Example

Example: Three risky assets

Let R_i $(i = A, B, C)$ denote the return on asset i and assume that R_i follows CER model:

$$R_i \sim iid\ N(\mu_i, \sigma_i^2)$$

$$\text{cov}(R_i, R_j) = \sigma_{ij}$$

Portfolio "**x**":

$$x_i = \text{share of wealth in asset } i$$

$$x_A + x_B + x_C = 1$$

Portfolio return:

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$

Example cont.

Stock i	μ_i	σ_i	Pair (i,j)	σ_{ij}
A (Microsoft)	0.0427	0.1000	(A,B)	0.0018
B (Nordstrom)	0.0015	0.1044	(A,C)	0.0011
C (Starbucks)	0.0285	0.1411	(B,C)	0.0026

Three asset example data.

In matrix algebra, we have:

$$\mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} = \begin{pmatrix} (0.1000)^2 & 0.0018 & 0.0011 \\ 0.0018 & (0.1044)^2 & 0.0026 \\ 0.0011 & 0.0026 & (0.1411)^2 \end{pmatrix}$$

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Matrix Algebra Representation

$$\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix},\ \mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix},\ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix},\ \mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}$$

Portfolio weights sum to 1:

$$\mathbf{x}'\mathbf{1} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= x_1 + x_2 + x_3 = 1$$

Portfolio return

$$R_{p,x} = \mathbf{x}'\mathbf{R} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}$$

$$= x_A R_A + x_B R_B + x_C R_C$$

Portfolio expected return:

$$\mu_{p,x} = \mathbf{x}'\mu = \begin{pmatrix} x_A & x_B & x_X \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}$$

$$= x_A \mu_A + x_B \mu_B + x_C \mu_C$$

Computational tools

R formula:

```
t(x.vec)%*%mu.vec

crossprod(x.vec, mu.vec)
```

Excel formula:

```
MMULT(transpose(xvec),muvec)

<ctrl>-<shift>-<enter>
```

Portfolio variance

$$
\begin{aligned}
\sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\
&= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\
&+ 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}
\end{aligned}
$$

Portfolio distribution:

$$R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2)$$

Computational tools

R formulas:

```
t(x.vec)%*%sigma.mat%*%x.vec
```

Excel formulas:

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))

MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)

<ctrl>-<shift>-<enter>
```

Covariance Between 2 Portfolio Returns

2 portfolios:

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix}$$

$$\mathbf{x}'\mathbf{1} = 1, \ \mathbf{y}'\mathbf{1} = 1$$

Portfolio returns:

$$R_{p,x} = \mathbf{x}'\mathbf{R}$$

$$R_{p,y} = \mathbf{y}'\mathbf{R}$$

Covariance:

$$\begin{aligned} \text{cov}(R_{p,x}, R_{p,y}) &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} \\ &= \mathbf{y}'\mathbf{\Sigma}\mathbf{x} \end{aligned}$$

Computational tools

R formula:

```
t(x.vec)%*%sigma.mat%*%y.vec
```

Excel formula:

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))

MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))

<ctrl>-<shift>-<enter>
```

Derivatives of Simple Matrix Functions

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be an $n \times 1$ vectors. Then,

$$\underset{n\times 1}{\frac{\partial}{\partial \mathbf{x}}}\mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{y} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{y} \end{pmatrix} = \mathbf{y}, \qquad \frac{\partial m'1}{\partial m} = 1 \tag{1}$$

$$\underset{n\times 1}{\frac{\partial}{\partial \mathbf{x}}}\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{A}\mathbf{x} \end{pmatrix} = 2\mathbf{A}\mathbf{x}. \qquad \frac{\partial m'\Sigma m}{\partial m} = 2\Sigma m \tag{2}$$

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Computing Global Minimum Variance Portfolio

Problem: Find the portfolio $\mathbf{m} = (m_A, m_B, m_C)'$ that solves:

$$\min_{m_A, m_B, m_C} \sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

1. Analytic solution using matrix algebra
2. Numerical Solution in Excel Using the Solver (see 3firmExample.xls)

Analytic solution using matrix algebra

The Lagrangian is:

$$L(\mathbf{m}, \lambda) = \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m} + \lambda(\mathbf{m}'\mathbf{1} - 1)$$

First order conditions (use matrix derivative results):

$$\underset{(3\times1)}{\mathbf{0}} = \frac{\partial L(\mathbf{m}, \lambda)}{\partial \mathbf{m}} = \frac{\partial \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m}}{\partial \mathbf{m}} + \frac{\partial}{\partial \mathbf{m}}\lambda(\mathbf{m}'\mathbf{1} - 1) = 2 \cdot \boldsymbol{\Sigma}\mathbf{m} + \lambda\mathbf{1}$$

$$\underset{(1\times1)}{0} = \frac{\partial L(\mathbf{m}, \lambda)}{\partial \lambda} = \frac{\partial \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m}}{\partial \lambda} + \frac{\partial}{\partial \lambda}\lambda(\mathbf{m}'\mathbf{1} - 1) = \mathbf{m}'\mathbf{1} - 1$$

Write FOCs in matrix form:

$$\begin{pmatrix} 2\boldsymbol{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \begin{matrix} 3 \times 1 \\ 1 \times 1 \end{matrix}.$$

Analytic solution using matrix algebra cont.

The FOCs are the linear system:

$$\mathbf{A}_m \mathbf{z}_m = \mathbf{b}$$

where,

$$\mathbf{A}_m = \begin{pmatrix} 2\boldsymbol{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \; \mathbf{z}_m = \begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

The solution for $\mathbf{z}_m$ is:

$$\mathbf{z}_m = \mathbf{A}_m^{-1}\mathbf{b}.$$

The first three elements of $\mathbf{z}_m$ are the portfolio weights $\mathbf{m} = (m_A, m_B, m_C)'$ for the global minimum variance portfolio with expected return $\mu_{p,m} = \mathbf{m}'\mu$ and variance $\sigma^2_{p,m} = \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m}$.

Alternative Derivation of Global Minimum Variance Portfolio

The first order conditions from the optimization problem can be expressed in matrix notation as:

$$\underset{(3\times1)}{\mathbf{0}} = \frac{\partial L(\mathbf{m}, \lambda)}{\partial \mathbf{m}} = 2 \cdot \mathbf{\Sigma m} + \lambda \cdot \mathbf{1},$$

$$\underset{(1\times1)}{0} = \frac{\partial L(\mathbf{m}, \lambda)}{\partial \lambda} = \mathbf{m}'\mathbf{1} - 1.$$

Using first equation, solve for $\mathbf{m}$:

$$\mathbf{m} = -\frac{1}{2} \cdot \lambda \mathbf{\Sigma}^{-1}\mathbf{1}.$$

Alternative Derivation of Global Minimum Variance Portfolio cont.

Next, multiply both sides by $\mathbf{1}'$ and use second equation to solve for λ:

$$1 = \mathbf{1}'\mathbf{m} = -\frac{1}{2} \cdot \lambda \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$$

$$\Rightarrow \lambda = -2 \cdot \frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}.$$

Finally, substitute the value for λ in the equation for $\mathbf{m}$:

$$\mathbf{m} = -\frac{1}{2}(-2)\frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\mathbf{\Sigma}^{-1}\mathbf{1}$$

$$= \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}.$$

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Efficient Portfolios of Risky Assets: Markowitz Algorithm

Problem 1: find portfolio $\mathbf{x}$ that has the highest expected return for a given level of risk as measured by portfolio variance.

$$\max_{x_A, x_B, x_C} \mu_{p,x} = \mathbf{x}'\mu \text{ s.t}$$

$$\sigma^2_{p,x} = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} = \sigma^0_p = \text{ target risk}$$

$$\mathbf{x}'\mathbf{1} = 1$$

Efficient Portfolios of Risky Assets: Markowitz Algorithm cont.

Problem 2: find portfolio $\mathbf{x}$ that has the smallest risk, measured by portfolio variance, that achieves a target expected return.

$$\min_{x_A, x_B, x_C} \sigma_{p,x}^2 = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu_p^0 = \text{target return}$$

$$\mathbf{x}'\mathbf{1} = 1$$

Remark: Problem 2 is usually solved in practice by varying the target return between a given range.

Solving for Efficient Portfolios

1. Analytic solution using matrix algebra
2. Numerical solution in Excel using the solver

Analytic solution using matrix algebra

The Lagrangian function associated with Problem 2 is:

$$L(x, \lambda_1, \lambda_2) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \lambda_1(\mathbf{x}'\mu - \mu_{p,0}) + \lambda_2(\mathbf{x}'\mathbf{1} - 1)$$

The FOCs are:

$$\underset{(3\times 1)}{\mathbf{0}} = \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \mathbf{x}} = 2\mathbf{\Sigma}\mathbf{x} + \lambda_1 \mu + \lambda_2 \mathbf{1},$$

$$\underset{(1\times 1)}{0} = \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_1} = \mathbf{x}'\mu - \mu_{p,0},$$

$$\underset{(1\times 1)}{0} = \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_2} = \mathbf{x}'\mathbf{1} - 1.$$

These FOCs consist of five linear equations in five unknowns $(x_A, x_B, x_C, \lambda_1, \lambda_2)$.

Analytic solution using matrix algebra cont.

We can represent the FOCs in matrix notation as:

$$\begin{pmatrix} 2\mathbf{\Sigma} & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

or,

$$\mathbf{A}_x \mathbf{z}_x = \mathbf{b}_0$$

where,

$$\mathbf{A}_x = \begin{pmatrix} 2\mathbf{\Sigma} & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix}, \ \mathbf{z}_x = \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b}_0 = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

Analytic solution using matrix algebra cont.

The solution for $\mathbf{z}_x$ is then:

$$\mathbf{z}_x = \mathbf{A}_x^{-1}\mathbf{b}_0.$$

The first three elements of $\mathbf{z}_x$ are the portfolio weights $\mathbf{x} = (x_A, x_B, x_C)'$ for the efficient portfolio with expected return $\mu_{p,x} = \mu_{p,0}$.

Example

Example: Find efficient portfolios with the same expected return as MSFT and SBUX

For MSFT, we solve:

$$\min_{x_A, x_B, x_C} \sigma_{p,x}^2 = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu_{MSFT} = 0.0427$$

$$\mathbf{x}'\mathbf{1} = 1$$

For SBUX, we solve:

$$\min_{y_A, y_B, y_C} \sigma_{p,x}^2 = \mathbf{y}'\mathbf{\Sigma}\mathbf{y} \text{ s.t.}$$

$$\mu_{p,y} = \mathbf{y}'\mu = \mu_{SBUX} = 0.0285$$

$$\mathbf{y}'\mathbf{1} = 1$$

Example cont.

Using the matrix algebra formulas (see R code in PowerPoint slides) we get:

$$\mathbf{x} = \begin{pmatrix} x_{msft} \\ x_{nord} \\ x_{sbux} \end{pmatrix} = \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix}, \; \mathbf{y} = \begin{pmatrix} y_{msft} \\ y_{nord} \\ y_{sbux} \end{pmatrix} = \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix}$$

Also,

$$\mu_{p,x} = \mathbf{x}'\mu = 0.0427, \; \mu_{p,y} = \mathbf{y}'\mu = 0.0285$$

$$\sigma_{p,x} = (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2} = 0.09166, \; \sigma_{p,y} = (\mathbf{y}'\mathbf{\Sigma}\mathbf{y})^{1/2} = 0.07355$$

$$\sigma_{xy} = \mathbf{x}'\mathbf{\Sigma}\mathbf{y} = 0.005914, \; \rho_{xy} = \sigma_{xy}/(\sigma_{p,x}\sigma_{p,y}) = 0.8772$$

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Computing the Portfolio Frontier

Result: The portfolio frontier can be represented as convex combinations of any two frontier portfolios. Let $\mathbf{x}$ be a frontier portfolio that solves:

$$\min_{\mathbf{x}} \sigma_{p,x}^2 = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu_p^0$$

$$\mathbf{x}'\mathbf{1} = 1$$

Let $\mathbf{y} \neq \mathbf{x}$ be another frontier portfolio that solves:

$$\min_{\mathbf{y}} \sigma_{p,y}^2 = \mathbf{y}'\boldsymbol{\Sigma}\mathbf{y} \text{ s.t.}$$

$$\mu_{p,y} = \mathbf{y}'\mu = \mu_p^1 \neq \mu_p^0$$

$$\mathbf{y}'\mathbf{1} = 1$$

Let α be any constant. Then the portfolio:

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y}$$

portfolio z solves
$z'\Sigma z$ s.t
$z'\mu = \mu_{p,z}$
$z'1 = 1$

is a frontier portfolio. Furthermore,

$$\mu_{p,z} = \mathbf{z}'\mu = \alpha \cdot \mu_{p,x} + (1-\alpha)\mu_{p,y}$$

$$\begin{aligned}\sigma^2_{p,z} &= \mathbf{z}'\boldsymbol{\Sigma}\mathbf{z} \\ &= \alpha^2\sigma^2_{p,x} + (1-\alpha)^2\sigma^2_{p,y} + 2\alpha(1-\alpha)\sigma_{x,y}\end{aligned}$$

$$\sigma_{x,y} = \mathrm{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}$$

Example

Example: 3 asset case

$$
\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y} \\
&= \alpha \cdot \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} + (1-\alpha) \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \\
&= \begin{pmatrix} \alpha x_A + (1-\alpha) y_A \\ \alpha x_B + (1-\alpha) y_B \\ \alpha x_C + (1-\alpha) y_C \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}
\end{aligned}
$$

Example

Example: Compute efficient portfolio as convex combination of efficient portfolio with same mean as MSFT and efficient portfolio with same mean as SBUX.

Let $\mathbf{x}$ denote the efficient portfolio with the same mean as MSFT, $\mathbf{y}$ denote the efficient portfolio with the same mean as SBUX, and let $\alpha = 0.5$. Then,

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y}$$

$$= 0.5 \cdot \begin{pmatrix} 0.82745 \\ -0.09075 \\ 0.26329 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix}$$

$$= \begin{pmatrix} (0.5)(0.82745) \\ (0.5)(-0.09075) \\ (0.5)(0.26329) \end{pmatrix} + \begin{pmatrix} (0.5)(0.5194) \\ (0.5)(0.2732) \\ (0.5)(0.2075) \end{pmatrix} = \begin{pmatrix} 0.6734 \\ 0.0912 \\ 0.2354 \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}.$$

Example cont.

The mean of this portfolio can be computed using:

$$\mu_{p,z} = \mathbf{z}'\mu = (0.6734, 0.0912, 0.2354)' \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix} = 0.0356$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,x} + (1-\alpha)\mu_{p,y} = 0.5(0.0427) + (0.5)(0.0285) = 0.0356$$

The variance can be computed using:

$$\sigma^2_{p,z} = \mathbf{z}'\mathbf{\Sigma}\mathbf{z} = 0.00641$$

$$\begin{aligned}\sigma^2_{p,z} &= \alpha^2\sigma^2_{p,x} + (1-\alpha)^2\sigma^2_{p,y} + 2\alpha(1-\alpha)\sigma_{xy} \\ &= (0.5)^2(0.09166)^2 + (0.5)^2(0.07355)^2 + 2(0.5)(0.5)(0.005914) \\ &= 0.00641\end{aligned}$$

Example

Example: Find efficient portfolio with expected return 0.05 from two efficient portfolios Use,

$$0.05 = \mu_{p,z} = \alpha \cdot \mu_{p,x} + (1-\alpha)\mu_{p,y}$$

to solve for α:

$$\alpha = \frac{0.05 - \mu_{p,y}}{\mu_{p,x} - \mu_{p,y}} = \frac{0.05 - 0.0285}{0.0427 - 0.0285} = 1.514$$

Then, solve for portfolio weights using:

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y}$$

$$= 1.514 \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix} - 0.514 \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} = \begin{pmatrix} 0.9858 \\ -0.2778 \\ 0.2920 \end{pmatrix}$$

Strategy for Plotting Portfolio Frontier

1. Set global minimum variance portfolio = first frontier portfolio

$$\min_{\mathbf{m}} \sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

   and compute $\mu_{p,m} = \mathbf{m}'\mu$

2. Find asset i that has highest expected return. Set target return to $\mu^0 = \max(\mu)$ and solve:

$$\min_{\mathbf{x}} \sigma^2_{p,x} = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu^0_p = \max(\mu)$$

$$\mathbf{x}'\mathbf{1} = 1$$

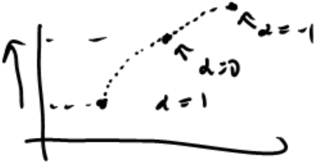
3. Create grid of α values, initially between 1 and -1, and compute

$$\mathbf{z} = \alpha \cdot \mathbf{m} + (1-\alpha)\cdot \mathbf{x}$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,m} + (1-\alpha)\mu_{p,x}$$

$$\sigma^2_{p,z} = \alpha^2\sigma^2_{p,m} + (1-\alpha)^2\sigma^2_{p,x} + 2\alpha(1-\alpha)\sigma_{m,x}$$

$$\sigma_{m,x} = \mathbf{m}'\mathbf{\Sigma}\mathbf{x}$$



$$\alpha = \begin{bmatrix} 1 \\ .9 \\ .8 \\ \vdots \\ 0 \\ -.1 \\ \vdots \\ -1 \end{bmatrix}$$

4. Plot $\mu_{p,z}$ against $\sigma_{p,z}$. Expand or contract the grid of α values if necessary to improve the plot.

Finding the Tangency Portfolio

The tangency portfolio $\mathbf{t}$ is the portfolio of risky assets that maximizes Sharpe's slope:

$$\max_{\mathbf{t}} \text{ Sharpe's ratio } = \frac{\mu_{p,t} - r_f}{\sigma_{p,t}}$$

subject to,

$$\mathbf{t}'\mathbf{1} = 1$$

In matrix notation,

$$\text{Sharpe's ratio } = \frac{\mathbf{t}'\mu - r_f}{\left(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right)^{1/2}}$$

Solving for Efficient Portfolios

1. Analytic solution using matrix algebra
2. Numerical solution in Excel using the solver

Analytic solution using matrix algebra

The Lagrangian for this problem is:

$$L(\mathbf{t}, \lambda) = \left(\mathbf{t}'\mu - r_f\right)\left(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}\right)^{-\frac{1}{2}} + \lambda(\mathbf{t}'\mathbf{1} - 1)$$

Using the chain rule, the first order conditions are:

$$\underset{(3\times 1)}{\mathbf{0}} = \frac{\partial L(\mathbf{t}, \lambda)}{\partial \mathbf{t}} = \mu(\mathbf{t}'\mathbf{\Sigma}\mathbf{t})^{-\frac{1}{2}} - \left(\mathbf{t}'\mu - r_f\right)\left(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}\right)^{-3/2}\mathbf{\Sigma}\mathbf{t} + \lambda\mathbf{1}$$

$$\underset{(1\times 1)}{0} = \frac{\partial L(\mathbf{t}, \lambda)}{\partial \lambda} = \mathbf{t}'\mathbf{1} - 1 = 0$$

After much tedious algebra, it can be shown that the solution for $\mathbf{t}$ is:

$$\mathbf{t} = \frac{\mathbf{\Sigma}^{-1}(\mu - r_f \cdot \mathbf{1})}{\mathbf{1}'\mathbf{\Sigma}^{-1}(\mu - r_f \cdot \mathbf{1})}$$

(Recall, for min variance port

$$m = \frac{\Sigma^{-1} 1}{1'\Sigma^{-1} 1}$$)

Remakrs

- If the risk free rate, r_f, is less than the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio has a positive Sharpe slope
- If the risk free rate, r_f, is equal to the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio is not defined
- If the risk free rate, r_f, is greater than the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio has a negative Sharpe slope

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Mutual Fund Separation Theorem Again

Efficient Portfolios of T-bills and Risky assets are combinations of two portfolios (mutual funds).

- T-bills
- Tangency portfolio

Efficient Portfolios:

$$x_t = \text{ share of wealth in tangency portfolio } \mathbf{t}$$

$$x_f = \text{share of wealth in T-bills}$$

$$x_t + x_f = 1 \Rightarrow \; x_f = 1 - x_t$$

$$\mu_p^e = r_f + x_t(\mu_{p,t} - r_f), \; \mu_{p,t} = \mathbf{t}'\mu$$

$$\sigma_p^e = x_t \sigma_{p,t}, \; \sigma_{p,t} = \left(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}\right)^{1/2}$$

Remarks

The weights x_t and x_f are determined by an investor's risk preferences.

- Risk averse investors hold mostly T-Bills ($x_t \approx 0$)
- Risk tolerant investors hold mostly tangency portfolio ($x_t \approx 1$)
- If Sharpe's slope for the tangency portfolio is negative then the efficient portfolio involve shorting the tangency portfolio

Example

Example: Find efficient portfolio with target risk (SD) equal to 0.02
Solve,

$$0.02 = \sigma_p^e = x_t \sigma_{p,t} = x_t(0.1116)$$

$$\Rightarrow x_t = \frac{0.02}{0.1116} = 0.1792$$

$$x_f = 1 - x_t = 0.8208$$

Also,

$$\mu_p^e = r_f + x_t(\mu_{p,t} - r_f) = 0.005 + (0.1116)\,(0.05189 - 0.005) = 0.0134$$

$$\sigma_p^e = x_t \sigma_{p,t} = (0.1792)(0.1116) = 0.02$$

Example

Example: Find efficient portfolio with target ER equal to 0.07 Solve,

$$0.07 = \mu_p^e = r_f + x_t(\mu_{p,t} - r_f)$$

$$\Rightarrow x_t = \frac{0.07 - r_f}{\mu_{p,t} - r_f} = \frac{0.07 - 0.005}{0.05189 - 0.005} = 1.386$$

Also,

$$\sigma_p^e = x_t \sigma_{p,t} = (1.386)(0.1116) = 0.1547$$

Portfolio Value-at-Risk

Let $\mathbf{x} = (x_1, \ldots, x_n)'$ denote a vector of asset share for a portfolio. Portfolio risk is measured by $\text{var}(R_{p,x}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$. Alternatively, portfolio risk can be measured using Value-at-Risk:

$$\begin{aligned}
\text{VaR}_\alpha &= W_0 q_\alpha^R \\
W_0 &= \text{initial investment} \\
q_\alpha^R &= 100 \cdot \alpha\% \text{ Simple return quantile} \\
\alpha &= \text{ loss probability}
\end{aligned}$$

If returns are normally distributed then:

$$q_{\alpha}^{R} = \mu_{p,x} + \sigma_{p,x} q_{\alpha}^{Z}$$

$$\mu_{p,x} = \mathbf{x}'\mu$$

$$\sigma_{p,x} = \left(\mathbf{x}'\mathbf{\Sigma}\mathbf{x}\right)^{1/2}$$

$$q_{\alpha}^{Z} = 100 \cdot \alpha\% \text{ quantile from } N(0,1)$$

Example

Example: Using VaR to evaluate an efficient portfolio

Invest in 3 risky assets (Microsoft, Starbucks, Nordstrom) and T-bills. Assume $r_f = 0.005$.

1. Determine efficient portfolio that has same expected return as Starbucks
2. Compare $\text{VaR}_{.05}$ for Starbucks and efficient portfolio based on \$100,000 investment

Solution for 1

$$\mu_{\text{SBUX}} = 0.0285$$
$$\mu_p^e = r_f + x_t(\mu_{p,t} - r_f)$$
$$r_f = 0.005$$
$$\mu_{p,t} = \mathbf{t}'\mu = .05186, \sigma_{p,t} = 0.111$$

Solve,

$$0.0285 = 0.005 + x_t(0.05186 - 0.005)$$
$$x_t = \frac{0.0285 - .005}{0.05186 - .005} = 0.501$$
$$x_f = 1 - 0.501 = 0.499$$

Solution for 1 cont.

Note:

$$\mu_p^e = 0.005 + 0.501 \cdot (0.05186 - 0.005) = 0.0285$$

$$\sigma_p^e = x_t \sigma_{p,t} = (0.501)(0.111) = 0.057$$

Solution for 2

$$
\begin{aligned}
q_{.05}^{\text{SBUX}} &= \mu_{\text{SBUX}} + \sigma_{\text{SBUX}} \cdot (-1.645) \\
&= 0.0285 + (0.141) \cdot (-1.645) \\
&= -0.203 \\
q_{.05}^{e} &= \mu_p^e + \sigma_p^e \cdot (-1.645) \\
&= .0285 + (.057) \cdot (-1.645) \\
&- 0.063
\end{aligned}
$$

Solution for 2 cont.

Then,

$$
\begin{aligned}
\mathrm{VaR}_{.05}^{SBUX} &= \$100,000 \cdot q_{.05}^{\mathrm{SBUX}} \\
&= \$100,000 \cdot (-0.203) = -\$20,300 \\
\mathrm{VaR}_{.05}^{e} &= \$100,000 \cdot q_{.05}^{e} \\
&= \$100,000 \cdot (-0.063) = -\$6,300
\end{aligned}
$$

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