Introduction to Computational Finance and Financial Econometrics

Portfolio Theory with Matrix Algebra

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Spring 2015
1 Portfolios with Three Risky Assets
   • Portfolio characteristics using matrix notation
   • Finding the global minimum variance portfolio
   • Finding efficient portfolios
   • Computing the efficient frontier
   • Mutual fund separation theorem again
**Example: Three risky assets**

Let $R_i \ (i = A, B, C)$ denote the return on asset $i$ and assume that $R_i$ follows CER model:

$$R_i \sim iid \ N(\mu_i, \sigma_i^2)$$

$$\text{cov}(R_i, R_j) = \sigma_{ij}$$

**Portfolio “x”:**

$$x_i = \text{share of wealth in asset } i$$

$$x_A + x_B + x_C = 1$$

**Portfolio return:**

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$
Example cont.

<table>
<thead>
<tr>
<th>Stock</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
<th>Pair $(i,j)$</th>
<th>$\sigma_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (Microsoft)</td>
<td>0.0427</td>
<td>0.1000</td>
<td>(A,B)</td>
<td>0.0018</td>
</tr>
<tr>
<td>B (Nordstrom)</td>
<td>0.0015</td>
<td>0.1044</td>
<td>(A,C)</td>
<td>0.0011</td>
</tr>
<tr>
<td>C (Starbucks)</td>
<td>0.0285</td>
<td>0.1411</td>
<td>(B,C)</td>
<td>0.0026</td>
</tr>
</tbody>
</table>

Three asset example data.

In matrix algebra, we have:

$$
\begin{align*}
\mu &= \begin{pmatrix} 
\mu_A \\
\mu_B \\
\mu_C 
\end{pmatrix} = \begin{pmatrix} 0.0427 \\
0.0015 \\
0.0285 
\end{pmatrix} \\
\Sigma &= \begin{pmatrix} 
\sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\
\sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\
\sigma_{AC} & \sigma_{BC} & \sigma_C^2 
\end{pmatrix} = \begin{pmatrix} 
(0.1000)^2 & 0.0018 & 0.0011 \\
0.0018 & (0.1044)^2 & 0.0026 \\
0.0011 & 0.0026 & (0.1411)^2 
\end{pmatrix}
\end{align*}
$$
1 Portfolios with Three Risky Assets

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Matrix Algebra Representation

\[
\mathbf{R} = \begin{pmatrix}
R_A \\
R_B \\
R_C
\end{pmatrix}, \quad \mu = \begin{pmatrix}
\mu_A \\
\mu_B \\
\mu_C
\end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]

\[
\mathbf{x} = \begin{pmatrix}
x_A \\
x_B \\
x_C
\end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix}
\sigma^2_A & \sigma_{AB} & \sigma_{AC} \\
\sigma_{AB} & \sigma^2_B & \sigma_{BC} \\
\sigma_{AC} & \sigma_{BC} & \sigma^2_C
\end{pmatrix}
\]

Portfolio weights sum to 1:

\[
\mathbf{x}'\mathbf{1} = \begin{pmatrix}
x_A & x_B & x_C
\end{pmatrix} \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
= x_1 + x_2 + x_3 = 1
\]
Portfolio return

\[ R_{p,x} = x' R = ( x_A \ x_B \ x_C ) \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix} \]

\[ = x_A R_A + x_B R_B + x_C R_C \]

Portfolio expected return:

\[ \mu_{p,x} = x' \mu = ( x_A \ x_B \ x_X ) \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} \]

\[ = x_A \mu_A + x_B \mu_B + x_C \mu_C \]
Computational tools

R formula:

\[ t(x.\text{vec}) \times \mu.\text{vec} \]
\[ \text{crossprod}(x.\text{vec}, \mu.\text{vec}) \]

Excel formula:

\[ \text{MMULT}(\text{transpose}(x.\text{vec}), \mu.\text{vec}) \]
\[ <\text{ctrl}>-<\text{shift}>-<\text{enter}> \]
Portfolio variance

\[ \sigma_{p,x}^2 = x' \Sigma x \]

\[ = ( x_A \ x_B \ x_C ) \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \]

\[ = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + x_C^2 \sigma_C^2 \]

\[ + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC} \]

Portfolio distribution:

\[ R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2) \]
Computational tools

R formulas:

\[ t(x.\text{vec}) \times \sigma.\text{mat} \times x.\text{vec} \]

Excel formulas:

\[
\text{MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))}
\]

\[
\text{MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)}
\]

\[<\text{ctrl}>-<\text{shift}>-<\text{enter}>\]
Covariance Between 2 Portfolio Returns

2 portfolios:

\[ \mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \]

\[ \mathbf{x}' \mathbf{1} = 1, \quad \mathbf{y}' \mathbf{1} = 1 \]

Portfolio returns:

\[ R_{p,x} = \mathbf{x}' \mathbf{R} \]

\[ R_{p,y} = \mathbf{y}' \mathbf{R} \]

Covariance:

\[ \text{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}' \Sigma \mathbf{y} \]

\[ = \mathbf{y}' \Sigma \mathbf{x} \]
R formula:

\[ t(x.\text{vec})\%*\%sigma.\text{mat}\%*\%y.\text{vec} \]

Excel formula:

\[
\text{MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))} \\
\text{MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))} \\
<\text{ctrl}>-<\text{shift}>-<\text{enter}>\]
Derivatives of Simple Matrix Functions

Let \( A \) be an \( n \times n \) symmetric matrix, and let \( x \) and \( y \) be an \( n \times 1 \) vectors. Then,

\[
\frac{\partial}{\partial x} x' y = \begin{pmatrix}
\frac{\partial}{\partial x_1} x'y \\
\vdots \\
\frac{\partial}{\partial x_n} x'y
\end{pmatrix} = y, \tag{1}
\]

\[
\frac{\partial}{\partial x} x' A x = \begin{pmatrix}
\frac{\partial}{\partial x_1} x' A x \\
\vdots \\
\frac{\partial}{\partial x_n} x' A x
\end{pmatrix} = 2Ax. \tag{2}
\]
Outline

1 Portfolios with Three Risky Assets
   • Portfolio characteristics using matrix notation
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   • Computing the efficient frontier
   • Mutual fund separation theorem again
Problem: Find the portfolio \( \mathbf{m} = (m_A, m_B, m_C)' \) that solves:

\[
\min_{m_A, m_B, m_C} \sigma_{p,m}^2 = \mathbf{m}' \Sigma \mathbf{m} \quad \text{s.t.} \quad \mathbf{m}' \mathbf{1} = 1
\]

1. Analytic solution using matrix algebra
2. Numerical Solution in Excel Using the Solver (see 3firmExample.xls)
Analytic solution using matrix algebra

The Lagrangian is:

\[ L(m, \lambda) = m'\Sigma m + \lambda(m'1 - 1) \]

First order conditions (use matrix derivative results):

\[
\begin{align*}
\mathbf{0}_{(3 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial m} = m'\Sigma m + \frac{\partial}{\partial m} \lambda(m'1 - 1) = 2 \cdot \Sigma m + \lambda \mathbf{1} \\
\mathbf{0}_{(1 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial \lambda} = m'\Sigma m + \frac{\partial}{\partial \lambda} \lambda(m'1 - 1) = m'1 - 1
\end{align*}
\]

Write FOCs in matrix form:

\[
\begin{pmatrix}
2\Sigma & 1 \\
1' & 0
\end{pmatrix}
\begin{pmatrix}
m \\
\lambda
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\quad 3 \times 1
\]

\[
\begin{pmatrix}
m \\
\lambda
\end{pmatrix}
\quad 1 \times 1
\]

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The FOCs are the linear system:

\[ A_m z_m = b \]

where,

\[ A_m = \begin{pmatrix} 2\Sigma & 1 \\ 1' & 0 \end{pmatrix}, \quad z_m = \begin{pmatrix} m \\ \lambda \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The solution for \( z_m \) is:

\[ z_m = A_m^{-1} b. \]

The first three elements of \( z_m \) are the portfolio weights \( m = (m_A, m_B, m_C)' \) for the global minimum variance portfolio with expected return \( \mu_{p,m} = m'\mu \) and variance \( \sigma^2_{p,m} = m'\Sigma m. \)
The first order conditions from the optimization problem can be expressed in matrix notation as:

\[
\begin{align*}
0_{(3 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial m} = 2 \cdot \Sigma m + \lambda \cdot 1, \\
0_{(1 \times 1)} &= \frac{\partial L(m, \lambda)}{\partial \lambda} = m'1 - 1.
\end{align*}
\]

Using first equation, solve for \( m \):

\[
m = -\frac{1}{2} \cdot \lambda \Sigma^{-1} 1.
\]
Next, multiply both sides by $1'$ and use second equation to solve for $\lambda$:

$$1 = 1'm = -\frac{1}{2} \cdot \lambda 1'\Sigma^{-1}1$$

$$\Rightarrow \lambda = -2 \cdot \frac{1}{1'\Sigma^{-1}1}.$$

Finally, substitute the value for $\lambda$ in the equation for $m$:

$$m = -\frac{1}{2} (-2) \frac{1}{1'\Sigma^{-1}1} \Sigma^{-1}1$$

$$= \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}.$$
Portfolios with Three Risky Assets

- Portfolio characteristics using matrix notation
- Finding the global minimum variance portfolio
- Finding efficient portfolios
- Computing the efficient frontier
- Mutual fund separation theorem again
Problem 1: find portfolio $\mathbf{x}$ that has the highest expected return for a given level of risk as measured by portfolio variance.

$$\max_{x_A, x_B, x_C} \mu_{p,x} = \mathbf{x}' \bm{\mu} \quad \text{s.t}$$

$$\sigma^2_{p,x} = \mathbf{x}' \bm{\Sigma} \mathbf{x} = \sigma^0_p = \text{target risk}$$

$$\mathbf{x}' \mathbf{1} = 1$$
Problem 2: find portfolio $\mathbf{x}$ that has the smallest risk, measured by portfolio variance, that achieves a target expected return.

$$\min_{x_A,x_B,x_C} \sigma_{p,x}^2 = \mathbf{x}'\Sigma\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu_p^0 = \text{target return}$$

$$\mathbf{x}'\mathbf{1} = 1$$

Remark: Problem 2 is usually solved in practice by varying the target return between a given range.
Solving for Efficient Portfolios

1. Analytic solution using matrix algebra
2. Numerical solution in Excel using the solver
Analytic solution using matrix algebra

The Lagrangian function associated with Problem 2 is:

\[ L(x, \lambda_1, \lambda_2) = x' \Sigma x + \lambda_1 (x' \mu - \mu_p,0) + \lambda_2 (x'1 - 1) \]

The FOCs are:

\[
\begin{align*}
0_{(3 \times 1)} &= \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial x} = 2 \Sigma x + \lambda_1 \mu + \lambda_2 1, \\
0_{(1 \times 1)} &= \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial \lambda_1} = x' \mu - \mu_p,0, \\
0_{(1 \times 1)} &= \frac{\partial L(x, \lambda_1, \lambda_2)}{\partial \lambda_2} = x'1 - 1.
\end{align*}
\]

These FOCs consist of five linear equations in five unknowns \((x_A, x_B, x_C, \lambda_1, \lambda_2)\).
We can represent the FOCs in matrix notation as:

\[
\begin{pmatrix}
2\Sigma & \mu & 1 \\
\mu' & 0 & 0 \\
1' & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x \\
\lambda_1 \\
\lambda_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\mu_{p,0} \\
1 \\
\end{pmatrix}
\]

or,

\[A_x z_x = b_0\]

where,

\[A_x = \begin{pmatrix}
2\Sigma & \mu & 1 \\
\mu' & 0 & 0 \\
1' & 0 & 0 \\
\end{pmatrix}, \quad z_x = \begin{pmatrix}
x \\
\lambda_1 \\
\lambda_2 \\
\end{pmatrix}\]

and \[b_0 = \begin{pmatrix}
0 \\
\mu_{p,0} \\
1 \\
\end{pmatrix}\]
The solution for $z_x$ is then:

$$z_x = A_x^{-1}b_0.$$  

The first three elements of $z_x$ are the portfolio weights $\mathbf{x} = (x_A, x_B, x_C)'$ for the efficient portfolio with expected return $\mu_{p,x} = \mu_{p,0}$.
Example: Find efficient portfolios with the same expected return as MSFT and SBUX

For MSFT, we solve:

$$\min_{x_A,x_B,x_C} \sigma^2_{p,x} = x'\Sigma x \text{ s.t.}$$

$$\mu_{p,x} = x'\mu = \mu_{MSFT} = 0.0427$$

$$x'1 = 1$$

For SBUX, we solve:

$$\min_{y_A,y_B,y_C} \sigma^2_{p,x} = y'\Sigma y \text{ s.t.}$$

$$\mu_{p,y} = y'\mu = \mu_{SBUX} = 0.0285$$

$$y'1 = 1$$
Using the matrix algebra formulas (see R code in PowerPoint slides) we get:

\[
\begin{pmatrix}
  x_{msft} \\
x_{nord} \\
x_{sbux}
\end{pmatrix}
= 
\begin{pmatrix}
  0.8275 \\
-0.0908 \\
  0.2633
\end{pmatrix},
\begin{pmatrix}
  y_{msft} \\
y_{nord} \\
y_{sbux}
\end{pmatrix}
= 
\begin{pmatrix}
  0.5194 \\
  0.2732 \\
  0.2075
\end{pmatrix}
\]

Also,

\[
\mu_{p,x} = x'\mu = 0.0427, \quad \mu_{p,y} = y'\mu = 0.0285
\]

\[
\sigma_{p,x} = (x'\Sigma x)^{1/2} = 0.09166, \quad \sigma_{p,y} = (y'\Sigma y)^{1/2} = 0.07355
\]

\[
\sigma_{xy} = x'\Sigma y = 0.005914, \quad \rho_{xy} = \sigma_{xy}/(\sigma_{p,x}\sigma_{p,y}) = 0.8772
\]
Portfolios with Three Risky Assets

- Portfolio characteristics using matrix notation
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- Finding efficient portfolios
- **Computing the efficient frontier**
- Mutual fund separation theorem again
Computing the Portfolio Frontier

Result: The portfolio frontier can be represented as convex combinations of any two frontier portfolios. Let $x$ be a frontier portfolio that solves:

$$\min_x \sigma^2_{p,x} = x'\Sigma x \ \text{s.t.}$$

$$\mu_{p,x} = x'\mu = \mu_p^0$$

$$x'1 = 1$$

Let $y \neq x$ be another frontier portfolio that solves:

$$\min_y \sigma^2_{p,y} = y'\Sigma y \ \text{s.t.}$$

$$\mu_{p,y} = y'\mu = \mu_p^1 \neq \mu_p^0$$

$$y'1 = 1$$
Computing the Portfolio Frontier cont.

Let $\alpha$ be any constant. Then the portfolio:

$$z = \alpha \cdot x + (1 - \alpha) \cdot y$$

is a frontier portfolio. Furthermore,

$$\mu_{p,z} = z'\mu = \alpha \cdot \mu_{p,x} + (1 - \alpha) \mu_{p,y}$$

$$\sigma^2_{p,z} = z'\Sigma z$$

$$= \alpha^2 \sigma^2_{p,x} + (1 - \alpha)^2 \sigma^2_{p,y} + 2\alpha(1 - \alpha)\sigma_{x,y}$$

$$\sigma_{x,y} = \text{cov}(R_{p,x}, R_{p,y}) = x'\Sigma y$$
Example: 3 asset case

\[ z = \alpha \cdot x + (1 - \alpha) \cdot y \]

\[ \begin{align*}
\alpha \cdot \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} + (1 - \alpha) \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} &= \\
\begin{pmatrix} \alpha x_A + (1 - \alpha) y_A \\ \alpha x_B + (1 - \alpha) y_B \\ \alpha x_C + (1 - \alpha) y_C \end{pmatrix} &= \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}
\end{align*} \]
Example: Compute efficient portfolio as convex combination of efficient portfolio with same mean as MSFT and efficient portfolio with same mean as SBUX.

Let $x$ denote the efficient portfolio with the same mean as MSFT, $y$ denote the efficient portfolio with the same mean as SBUX, and let $\alpha = 0.5$. Then,

$$z = \alpha \cdot x + (1 - \alpha) \cdot y$$

$$= 0.5 \cdot \begin{pmatrix} 0.82745 \\ -0.09075 \\ 0.26329 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix}$$

$$= \begin{pmatrix} (0.5)(0.82745) \\ (0.5)(-0.09075) \\ (0.5)(0.26329) \end{pmatrix} + \begin{pmatrix} (0.5)(0.5194) \\ (0.5)(0.2732) \\ (0.5)(0.2075) \end{pmatrix} = \begin{pmatrix} 0.6734 \\ 0.0912 \\ 0.2354 \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}.$$
The mean of this portfolio can be computed using:

$$\mu_{p,z} = z'\mu = (0.6734, 0.0912, 0.2354)' \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix} = 0.0356$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,x} + (1 - \alpha)\mu_{p,y} = 0.5(0.0427) + (0.5)(0.0285) = 0.0356$$

The variance can be computed using:

$$\sigma_{p,z}^2 = z'\Sigma z = 0.00641$$

$$\sigma_{p,z}^2 = \alpha^2\sigma_{p,x}^2 + (1 - \alpha)^2\sigma_{p,y}^2 + 2\alpha(1 - \alpha)\sigma_{xy}$$

$$= (0.5)^2(0.09166)^2 + (0.5)^2(0.07355)^2 + 2(0.5)(0.5)(0.005914)$$

$$= 0.00641$$
**Example:** Find efficient portfolio with expected return 0.05 from two efficient portfolios. Use,

\[ 0.05 = \mu_{p,z} = \alpha \cdot \mu_{p,x} + (1 - \alpha) \mu_{p,y} \]

to solve for \( \alpha \):

\[ \alpha = \frac{0.05 - \mu_{p,y}}{\mu_{p,x} - \mu_{p,y}} = \frac{0.05 - 0.0285}{0.0427 - 0.0285} = 1.514 \]

Then, solve for portfolio weights using:

\[ z = \alpha \cdot x + (1 - \alpha) \cdot y \]

\[ = 1.514 \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix} - 0.514 \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} = \begin{pmatrix} 0.9858 \\ -0.2778 \\ 0.2920 \end{pmatrix} \]
Set global minimum variance portfolio = first frontier portfolio

\[
\min_{m} \sigma_{p,m}^2 = m' \Sigma m \quad \text{s.t.} \quad m' 1 = 1
\]

and compute \( \mu_{p,m} = m' \mu \)

Find asset \( i \) that has highest expected return. Set target return to \( \mu^0 = \max(\mu) \) and solve:

\[
\min_{x} \sigma_{p,x}^2 = x' \Sigma x \quad \text{s.t.} \quad \\
\mu_{p,x} = x' \mu = \mu^0_p = \max(\mu) \\
x' 1 = 1
\]
Create grid of $\alpha$ values, initially between 1 and $-1$, and compute

$$z = \alpha \cdot m + (1 - \alpha) \cdot x$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,m} + (1 - \alpha) \mu_{p,x}$$

$$\sigma_{p,z}^2 = \alpha^2 \sigma_{p,m}^2 + (1 - \alpha)^2 \sigma_{p,x}^2 + 2\alpha(1 - \alpha)\sigma_{m,x}$$

$$\sigma_{m,x} = m' \Sigma x$$

Plot $\mu_{p,z}$ against $\sigma_{p,z}$. Expand or contract the grid of $\alpha$ values if necessary to improve the plot.
The tangency portfolio $t$ is the portfolio of risky assets that maximizes Sharpe’s slope:

$$\max_t \text{ Sharpe's ratio} = \frac{\mu_{p,t} - r_f}{\sigma_{p,t}}$$

subject to,

$$t'1 = 1$$

In matrix notation,

$$\text{Sharpe's ratio} = \frac{t'\mu - r_f}{(t'\Sigma t)^{1/2}}$$
Solving for Efficient Portfolios

1. Analytic solution using matrix algebra
2. Numerical solution in Excel using the solver
Analytic solution using matrix algebra

The Lagrangian for this problem is:

\[ L(t, \lambda) = (t' \mu - r_f) (t' \Sigma t)^{-\frac{1}{2}} + \lambda (t' 1 - 1) \]

Using the chain rule, the first order conditions are:

\[
\begin{align*}
0 &= \frac{\partial L(t, \lambda)}{\partial t} = \mu(t' \Sigma t)^{-\frac{1}{2}} - (t' \mu - r_f) (t' \Sigma t)^{-\frac{3}{2}} \Sigma t + \lambda 1 \\
0 &= \frac{\partial L(t, \lambda)}{\partial \lambda} = t' 1 - 1 = 0
\end{align*}
\]

After much tedious algebra, it can be shown that the solution for \( t \) is:

\[
\begin{align*}
t &= \frac{\Sigma^{-1} (\mu - r_f \cdot 1)}{1' \Sigma^{-1} (\mu - r_f \cdot 1)}
\end{align*}
\]
If the risk free rate, $r_f$, is less than the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio has a positive Sharpe slope.

If the risk free rate, $r_f$, is equal to the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio is not defined.

If the risk free rate, $r_f$, is greater than the expected return on the global minimum variance portfolio, $\mu_{g\min}$, then the tangency portfolio has a negative Sharpe slope.
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Mutual Fund Separation Theorem Again

Efficient Portfolios of T-bills and Risky assets are combinations of two portfolios (mutual funds).

- T-bills
- Tangency portfolio

Efficient Portfolios:

\[ x_t = \text{share of wealth in tangency portfolio } t \]

\[ x_f = \text{share of wealth in T-bills} \]

\[ x_t + x_f = 1 \Rightarrow x_f = 1 - x_t \]

\[ \mu_p^{e} = r_f + x_t(\mu_{p,t} - r_f), \quad \mu_{p,t} = t'\mu \]

\[ \sigma_p^{e} = x_t\sigma_{p,t}, \quad \sigma_{p,t} = (t'\Sigma t)^{1/2} \]
The weights $x_t$ and $x_f$ are determined by an investor’s risk preferences.

- Risk averse investors hold mostly T-Bills ($x_t \approx 0$)
- Risk tolerant investors hold mostly tangency portfolio ($x_t \approx 1$)
- If Sharpe’s slope for the tangency portfolio is negative then the efficient portfolio involve shorting the tangency portfolio
Example: Find efficient portfolio with target risk (SD) equal to 0.02

Solve,

\[ 0.02 = \sigma_p^e = x_t \sigma_{p,t} = x_t (0.1116) \]

\[ \Rightarrow x_t = \frac{0.02}{0.1116} = 0.1792 \]

\[ x_f = 1 - x_t = 0.8208 \]

Also,

\[ \mu_p^e = r_f + x_t (\mu_{p,t} - r_f) = 0.005 + (0.1116) (0.05189 - 0.005) = 0.0134 \]

\[ \sigma_p^e = x_t \sigma_{p,t} = (0.1792)(0.1116) = 0.02 \]
Example: Find efficient portfolio with target ER equal to 0.07 Solve,

\[ 0.07 = \mu_p^e = r_f + x_t (\mu_{p,t} - r_f) \]

\[ \Rightarrow x_t = \frac{0.07 - r_f}{\mu_{p,t} - r_f} = \frac{0.07 - 0.005}{0.05189 - 0.005} = 1.386 \]

Also,

\[ \sigma_p^e = x_t \sigma_{p,t} = (1.386)(0.1116) = 0.1547 \]
Let \( \mathbf{x} = (x_1, \ldots, x_n)' \) denote a vector of asset share for a portfolio. Portfolio risk is measured by \( \text{var}(R_{p,x}) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \). Alternatively, portfolio risk can be measured using Value-at-Risk:

\[
\text{VaR}_\alpha = W_0 q^R_\alpha
\]

\( W_0 = \text{initial investment} \)

\( q^R_\alpha = 100 \cdot \alpha\% \) Simple return quantile

\( \alpha = \text{loss probability} \)
If returns are normally distributed then:

\[ q_\alpha = \mu_{p,x} + \sigma_{p,x} q_Z^{\alpha} \]

\[ \mu_{p,x} = x' \mu \]

\[ \sigma_{p,x} = (x' \Sigma x)^{1/2} \]

\[ q_Z^{\alpha} = 100 \cdot \alpha \% \text{ quantile from } N(0, 1) \]
Example: Using VaR to evaluate an efficient portfolio

Invest in 3 risky assets (Microsoft, Starbucks, Nordstrom) and T-bills. Assume $r_f = 0.005$.

1. Determine efficient portfolio that has same expected return as Starbucks

2. Compare VaR_{0.05} for Starbucks and efficient portfolio based on $100,000 investment
Solution for 1

\[ \mu_{SBUX} = 0.0285 \]

\[ \mu_p^e = r_f + x_t(\mu_{p,t} - r_f) \]

\[ r_f = 0.005 \]

\[ \mu_{p,t} = t'\mu = .05186, \sigma_{p,t} = 0.111 \]

Solve,

\[ 0.0285 = 0.005 + x_t(0.05186 - 0.005) \]

\[ x_t = \frac{0.0285 - .005}{0.05186 - .005} = 0.501 \]

\[ x_f = 1 - 0.501 = 0.499 \]
Solution for 1 cont.

Note:

$$\mu_p^e = 0.005 + 0.501 \cdot (0.05186 - 0.005) = 0.0285$$

$$\sigma_p^e = x_t \sigma_{p,t} = (0.501)(0.111) = 0.057$$
Solution for 2

\[ q_{0.05}^{SBUX} = \mu_{SBUX} + \sigma_{SBUX} \cdot (-1.645) \]

\[ = 0.0285 + (0.141) \cdot (-1.645) \]

\[ = -0.203 \]

\[ q_{0.05}^{e} = \mu_{p}^{e} + \sigma_{p}^{e} \cdot (-1.645) \]

\[ = .0285 + (.057) \cdot (-1.645) \]

\[ = 0.063 \]
Then,

\[ \text{VaR}_{0.05}^{SBUX} = 100,000 \cdot q_{0.05}^{SBUX} \]

\[ = 100,000 \cdot (-0.203) = -20,300 \]

\[ \text{VaR}_{0.05}^{e} = 100,000 \cdot q_{0.05}^{e} \]

\[ = 100,000 \cdot (-0.063) = -6,300 \]