Econ 424
Review of Matrix Algebra

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Matrices and Vectors

Matrix

$$\underset{(n\times m)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$n =$ # of rows, $m =$ # of columns

Square matrix : $n = m$

Vector

$$\underset{(n\times 1)}{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Remarks

- R is a matrix oriented programming language

- Excel can handle matrices and vectors in formulas and some functions

- Excel has special functions for working with matrices. There are called *array* functions. Must use

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  to evaluate array function

Transpose of a Matrix

Interchange rows and columns of a matrix

$$\underset{(m\times n)}{\mathbf{A}}{}' = \text{transpose of } \underset{(n\times m)}{\mathbf{A}}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ \mathbf{A}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{x}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

R function

`t(A)`

Excel function

`TRANSPOSE`(matrix)

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Symmetric Matrix

A square matrix $\mathbf{A}$ is symmetric if

$$\mathbf{A} = \mathbf{A}'$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \ \mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Remark: Covariance and correlation matrices are symmetric

Basic Matrix Operations

Addition and Subtraction (element-by-element)

$$\begin{aligned} \begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} &= \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} &= \begin{bmatrix} 4-2 & 9-0 \\ 2-0 & 1-7 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 9 \\ 2 & -6 \end{bmatrix} \end{aligned}$$

Scalar Multiplication (element-by-element)

$$c = 2 = \text{scalar}$$

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}$$

$$2 \cdot A = \begin{bmatrix} 2 \cdot 3 & 2 \cdot (-1) \\ 2 \cdot 0 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 0 & 10 \end{bmatrix}$$

Matrix Multiplication (not element-by-element)

$$\underset{(3\times 2)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad \underset{(2\times 3)}{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Note: $\mathbf{A}$ and $\mathbf{B}$ are comformable matrices: $\#$ of columns in $A = \#$ of rows in B

$$\underset{(3\times 2)}{\mathbf{A}} \cdot \underset{(2\times 3)}{\mathbf{B}}$$
$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}$$

Remark: In general,

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

R operator

```
A%*%B
```

Excel function

`MMULT`(matrix1, matrix2)

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Identity Matrix

The $n-$ dimensional identity matrix has all diagonal elements equal to 1, and all off diagonal elements equal to 0.

Example

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: The identity matrix plays the roll of "1" in matrix algebra

$$\begin{aligned} \mathbf{I}_2 \cdot \mathbf{A} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}+0 & a_{12}+0 \\ 0+a_{21} & 0+a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \mathbf{A} \end{aligned}$$

Similarly

$$\mathbf{A} \cdot \mathbf{I}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}$$

R function

```
diag(n)
```

creates $n-$ dimensional identity matrix

Matrix Inverse

Let $\underset{(n\times n)}{\mathbf{A}}$ = square matrix. $\mathbf{A}^{-1}$ = "inverse of $\mathbf{A}$" satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$
$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$$

Remark: $\mathbf{A}^{-1}$ is similar to the inverse of a number:

$$a = 2,\ a^{-1} = \frac{1}{2}$$
$$a \cdot a^{-1} = 2 \cdot \frac{1}{2} = 1$$
$$a^{-1} \cdot a = \frac{1}{2} \cdot 2 = 1$$

R function

`solve(A)`

Excel function

`MINVERSE`(matrix)

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Representing Systems of Linear Equations Using Matrix Algebra

Consider the system of two linear equations

$$\begin{aligned} x + y &= 1 \\ 2x - y &= 1 \end{aligned}$$

The equations represent two straight lines which intersect at the point

$$x = \frac{2}{3},\ y = \frac{1}{3}$$

Matrix algebra representation:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

or

$$\mathbf{A} \cdot \mathbf{z} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can solve for z by multiplying both sides by $\mathbf{A}^{-1}$

$$\begin{aligned} \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{z} &= \mathbf{A}^{-1} \cdot \mathbf{b} \\ &\implies \mathbf{I} \cdot \mathbf{z} = \mathbf{A}^{-1} \cdot \mathbf{b} \\ &\implies \mathbf{z} = \mathbf{A}^{-1} \cdot \mathbf{b} \end{aligned}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Remark: As long as we can determine the elements in $\mathbf{A}^{-1}$, we can solve for the values of x and y in the vector $\mathbf{z}$. Since the system of linear equations has a solution as long as the two lines intersect, we can determine the elements in $\mathbf{A}^{-1}$ provided the two lines are not parallel.

There are general numerical algorithms for finding the elements of $\mathbf{A}^{-1}$ and programs like Excel and R have these algorithms available. However, if $\mathbf{A}$ is a (2×2) matrix then there is a simple formula for $\mathbf{A}^{-1}$. Let

$$\mathbf{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right].$$

Then

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \left[\begin{array}{cc} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{array} \right].$$

where

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12} \neq 0$$

Let's apply the above rule to find the inverse of $\mathbf{A}$ in our example:

$$\mathbf{A}^{-1} = \frac{1}{-1-2}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}.$$

Notice that

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Our solution for z is then

$$\begin{aligned} \mathbf{z} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

so that $x = \frac{2}{3}$ and $y = \frac{1}{3}$.

In general, if we have n linear equations in n unknown variables we may write the system of equations as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

which we may then express in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or

$$\underset{(n\times n)}{\mathbf{A}} \cdot \underset{(n\times 1)}{\mathbf{x}} = \underset{(n\times 1)}{\mathbf{b}}.$$

The solution to the system of equations is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

where $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{I}$ is the $(n \times n)$ identity matrix. If the number of equations is greater than two, then we generally use numerical algorithms to find the elements in $\mathbf{A}^{-1}$.

Representing Summation Using Matrix Notation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\underset{(n\times 1)}{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \underset{(n\times 1)}{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then

$$\mathbf{x}'\mathbf{1} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

Equivalently

$$\mathbf{1}'\mathbf{x} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

Sum of Squares

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\begin{aligned}\mathbf{x}'\mathbf{x} &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 = \sum_{i=1}^{n} x_i^2\end{aligned}$$

Sums of cross products

$$\sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\begin{aligned}
\mathbf{x}'\mathbf{y} &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\
&= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \\
&= \mathbf{y}'\mathbf{x}
\end{aligned}$$

R function

```
t(x)%*%y, t(y)%*%x
crossprod(x,y)
```

Excel function

```
MMULT(TRANSPOSE(x),y)
MMULT(TRANSPOSE(y),x)
<ctrl>-<shift>-<enter>
```

Portfolio Math with Matrix Algebra

Three Risky Asset Example

Let R_i denote the return on asset $i = A, B, C$ and assume that R_A, R_B and R_C are jointly normally distributed with means, variances and covariances:

$$\mu_i = E[R_i],\ \sigma_i^2 = \mathsf{var}(R_i),\ \mathsf{cov}(R_i, R_j) = \sigma_{ij}$$

Portfolio "$\mathbf{x}$"

$$x_i = \text{share of wealth in asset } i$$

$$x_A + x_B + x_C = 1$$

Portfolio return

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$

Portfolio expected return

$$\mu_{p,x} = E[R_{p,x}] = x_A\mu_A + x_B\mu_B + x_C\mu_C$$

Portfolio variance

$$\begin{aligned}\sigma_{p,x}^2 = \mathsf{var}(R_{p,x}) &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\ &+ 2x_A x_B\sigma_{AB} + 2x_A x_C\sigma_{AC} + 2x_B x_C\sigma_{BC}\end{aligned}$$

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2)$$

Matrix Algebra Representation

$$\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}, \ \mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}$$

Portfolio weights sum to 1

$$\begin{aligned} \mathbf{x}'\mathbf{1} &= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= x_A + x_B + x_C = 1 \end{aligned}$$

Digression on Covariance Matrix

Using matrix algebra, the variance-covariance matrix of the $N \times 1$ return vector $\mathbf{R}$ is defined as

$$\underset{N\times N}{\mathsf{var}(\mathbf{R})} = \mathsf{cov}(\mathbf{R}) = E[(\mathbf{R} - \mu)(\mathbf{R} - \mu)'] = \mathbf{\Sigma}$$

Because $\mathbf{R}$ has N elements, $\mathbf{\Sigma}$ is the $N \times N$ matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{pmatrix}$$

For the case $N = 2$, we have

$$
\begin{aligned}
E[(\underset{2\times 1}{\mathbf{R}-\mu})(\underset{1\times 2}{\mathbf{R}-\mu})'] &= E\left[\begin{pmatrix} R_1-\mu_1 \\ R_2-\mu_2 \end{pmatrix} \cdot (R_1-\mu_1, R_2-\mu_2)\right] \\
&= E\left[\begin{pmatrix} (R_1-\mu_1)^2 & (R_1-\mu_1)(R_2-\mu_2) \\ (R_2-\mu_2)(R_1-\mu_1) & (R_2-\mu_2)^2 \end{pmatrix}\right] \\
&= \begin{pmatrix} E[(R_1-\mu_1)^2] & E[(R_1-\mu_1)(R_2-\mu_2)] \\ E[(R_2-\mu_2)(R_1-\mu_1)] & E[(R_2-\mu_2)^2] \end{pmatrix} \\
&= \begin{pmatrix} \mathsf{var}(R_1) & \mathsf{cov}(R_1, R_2) \\ \mathsf{cov}(R_2, R_1) & \mathsf{var}(R_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \mathbf{\Sigma}.
\end{aligned}
$$

Portfolio return

$$R_{p,x} = \mathbf{x}'\mathbf{R} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}$$
$$= x_A R_A + x_B R_B + x_C R_C$$
$$= \mathbf{R}'\mathbf{x}$$

Portfolio expected return

$$\mu_{p,x} = \mathbf{x}'\mu = \begin{pmatrix} x_A & x_B & x_X \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}$$
$$= x_A \mu_A + x_B \mu_B + x_C \mu_C$$
$$= \mu'\mathbf{x}$$

Excel formula

```
MMULT(transpose(xvec),muvec)
<ctrl>-<shift>-<enter>
```

R formula

```
crossprod(x,mu)
t(x)%*%mu
```

Portfolio variance

$$
\begin{aligned}
\sigma^2_{p,x} &= \mathrm{var}(\mathbf{x}'\mathbf{R}) = E[\left(\mathbf{x}'\mathbf{R} - \mathbf{x}'\mu\right)^2] = E[\left(\mathbf{x}'(\mathbf{R} - \mu)\right)^2] \\
&= E[\mathbf{x}'(\mathbf{R} - \mu)\mathbf{x}'(\mathbf{R} - \mu)] = E[\mathbf{x}'(\mathbf{R} - \mu)(\mathbf{R} - \mu)'\mathbf{x}] = \\
&= \mathbf{x}'E[(\mathbf{R} - \mu)(\mathbf{R} - \mu)']\mathbf{x} = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\
&= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma^2_A & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma^2_B & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma^2_C \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x^2_A\sigma^2_A + x^2_B\sigma^2_B + x^2_C\sigma^2_C \\
&+ 2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}
\end{aligned}
$$

Excel formulas

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))
MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)
<ctrl>-<shift>-<enter>
```

Note: $\mathbf{x}'\mathbf{\Sigma}\mathbf{x} = (\mathbf{x}'\mathbf{\Sigma})\,\mathbf{x} = \mathbf{x}'\,(\mathbf{\Sigma}\mathbf{x})$

R formulas

```
t(x)%*%sigma%*%x
```

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma^2_{p,x})$$

Covariance Between 2 Portfolio Returns

2 portfolios

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix}$$

$$\mathbf{x}'\mathbf{1} = 1, \ \mathbf{y}'\mathbf{1} = 1$$

Portfolio returns

$$\begin{aligned} R_{p,x} &= \mathbf{x}'\mathbf{R} \\ R_{p,y} &= \mathbf{y}'\mathbf{R} \end{aligned}$$

Covariance

$$\begin{aligned} \text{cov}(R_{p,x}, R_{p,y}) &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} \\ &= \mathbf{y}'\mathbf{\Sigma}\mathbf{x} \end{aligned}$$

Derivation

$$\begin{aligned}
\mathsf{cov}(R_{p,x}, R_{p,y}) &= \mathsf{cov}(\mathbf{x}'\mathbf{R}, \mathbf{y}'\mathbf{R}) \\
&= E[(\mathbf{x}'\mathbf{R} - \mathbf{x}'\mu])(\mathbf{y}'\mathbf{R} - \mathbf{y}'\mu])] \\
&= E[\mathbf{x}'(\mathbf{R} - \mu)\mathbf{y}'(\mathbf{R} - \mu)] \\
&= E[\mathbf{x}'(\mathbf{R} - \mu)(\mathbf{R} - \mu)'\mathbf{y}] \\
&= \mathbf{x}'E[(\mathbf{R} - \mu)(\mathbf{R} - \mu)']\mathbf{y} \\
&= \mathbf{x}'\mathbf{\Sigma}\mathbf{y}
\end{aligned}$$

Excel formula

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))
MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))
<ctrl>-<shift>-<enter>
```

R formula

```
t(x)%*%sigma%*%y
```

Bivariate Normal Distribution

Let X and Y be distributed bivariate normal. The joint pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \times$$

$$\exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

where $E[X] = \mu_X$, $E[Y] = \mu_Y$, $\mathsf{sd}(X) = \sigma_X$, $\mathsf{sd}(Y) = \sigma_Y$, and $\rho_{XY} =$ $\mathsf{cor}(X, Y)$.

Define

$$\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}, \; \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \; \mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \; \mathbf{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$$

Then the bivariate normal distribution can be compactly expressed as

$$f(\mathbf{x}) = \frac{1}{2\pi \det(\mathbf{\Sigma})^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}$$

where

$$\begin{aligned} \det(\mathbf{\Sigma}) &= \sigma_X^2\sigma_Y^2 - \sigma_{XY}^2 = \sigma_X^2\sigma_Y^2 - \sigma_X^2\sigma_Y^2\rho_{XY}^2 \\ &= \sigma_X^2\sigma_Y^2(1-\rho_{XY}^2). \end{aligned}$$

We use the shorthand notation

$$\mathbf{X} \sim N(\mu, \mathbf{\Sigma})$$

Derivatives of Simple Matrix Functions

Result: Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be an $n \times 1$ vectors. Then

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1} \mathbf{x}'\mathbf{y} \\ \vdots \\ \frac{\partial}{\partial x_n} \mathbf{x}'\mathbf{y} \end{pmatrix} = \mathbf{y},$$

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1} (\mathbf{A}\mathbf{x})' \\ \vdots \\ \frac{\partial}{\partial x_n} (\mathbf{A}\mathbf{x})' \end{pmatrix} = \mathbf{A},$$

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \mathbf{x}'\mathbf{A}\mathbf{x} \\ \vdots \\ \frac{\partial}{\partial x_n} \mathbf{x}'\mathbf{A}\mathbf{x} \end{pmatrix} = 2\mathbf{A}\mathbf{x}.$$

We will demonstrate these results with simple examples. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

For the first result we have

$$\mathbf{x}'\mathbf{y} = x_1 y_1 + x_2 y_2.$$

Then

$$\frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1} \mathbf{x}'\mathbf{y} \\ \frac{\partial}{\partial x_2} \mathbf{x}'\mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} (x_1 y_1 + x_2 y_2) \\ \frac{\partial}{\partial x_2} (x_1 y_1 + x_2 y_2) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{y}.$$

Next, consider the second result. Note that

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{pmatrix}$$

and

$$(\mathbf{A}\mathbf{x})' = (ax_1 + bx_2, bx_1 + cx_2)$$

Then

$$\frac{\partial}{\partial \mathbf{x}}\mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1}(ax_1 + bx_2, bx_1 + cx_2) \\ \frac{\partial}{\partial x_2}(ax_1 + bx_2, bx_1 + cx_2) \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \mathbf{A}$$

Finally, consider the third result. We have

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2.$$

Then

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{x} &= \begin{pmatrix} \frac{\partial}{\partial x_1}\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right) \\ \frac{\partial}{\partial x_2}\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right) \end{pmatrix} = \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} \\ &= 2 \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\mathbf{A}\mathbf{x}. \end{aligned}$$