Introduction to Computational Finance and Financial Econometrics

Introduction to Portfolio Theory

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Outline

1. Portfolios of Two Risky Assets
2. Efficient Portfolios with Two Risky Asssets
3. Efficient Portfolios with a Risk-free Asset
4. Efficient Portfolios with Two Risky Assets and a Risk-free Asset

Investment in Two Risky Assets

$$R_A = \text{ simple return on asset A}$$

$$R_B = \text{ simple return on asset B}$$

$$W_0 = \text{ initial wealth}$$

Assumptions:

- R_A and R_B are described by the CER model:

$$R_i \sim \text{iid } N(\mu_i, \sigma_i^2), \; i = A, B$$

$$\text{cov}(R_A, R_B) = \sigma_{AB}, \; \text{cor}(R_A, R_B) = \rho_{AB}$$

- Investors like high $E[R_i] = \mu_i$
- Investors dislike high $\text{var}(R_i) = \sigma_i^2$
- Investment horizon is one period (e.g., one month or one year)

Note: Traditionally in portfolio theory, returns are simple and not continuously compounded

Portfolios

$$x_A = \text{share of wealth in asset A} = \frac{\$ \text{ in A}}{W_0}$$

$$x_B = \text{share of wealth in asset B} = \frac{\$ \text{ in B}}{W_0}$$

Long position:

$$x_A,\ x_B > 0$$

Short position:

$$x_A < 0 \text{ or } x_B < 0$$

Assumption: Allocate all wealth between assets A and B:

$$x_A + x_B = 1$$

Portfolio return:

$$R_p = x_A R_A + x_B R_B$$

Portfolios cont.

Portfolio Distribution:

$$\mu_p = E[R_p] = x_A\mu_A + x_B\mu_B$$

$$\sigma_p^2 = \mathrm{var}(R_p) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}$$

$$= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B$$

$$R_p \sim \text{iid } N(\mu_p, \sigma_p^2)$$

End of Period Wealth:

$$W_1 = W_0(1 + R_p) = W_0(1 + x_AR_A + x_BR_B)$$

$$W_1 \sim N(W_0(1 + \mu_p), \sigma_p^2W_0^2)$$

Portfolios cont.

Result: Portfolio SD is not a weighted average of asset SD unless $\rho_{AB} = 1$:

$$\sigma_p = \left(x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \rho_{AB}\sigma_A\sigma_B\right)^{1/2}$$

$$\neq x_A\sigma_A + x_B\sigma_B \text{ for } \rho_{AB} \neq 1$$

If $\rho_{AB} = 1$ then:

$$\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$$

and,

$$\sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B\sigma_A\sigma_B$$

$$= (x_A\sigma_A + x_B\sigma_B)^2$$

$$\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B$$

Example data

$$\mu_A = 0.175,\ \ \mu_B = 0.055$$

$$\sigma_A^2 = 0.067,\ \ \sigma_B^2 = 0.013$$

$$\sigma_A = 0.258,\ \ \sigma_B = 0.115$$

$$\sigma_{AB} = -0.004875,$$

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.164$$

Note: Asset A has higher expected return and risk than asset B.

Example

Example: Long only two asset portfolio

Consider an equally weighted portfolio with $x_A = x_B = 0.5$. The expected return, variance and volatility are:

$$\mu_p = (0.5) \cdot (0.175) + (0.5) \cdot (0.055) = 0.115$$

$$\sigma_p^2 = (0.5)^2 \cdot (0.067) + (0.5)^2 \cdot (0.013)$$

$$+ 2 \cdot (0.5)(0.5)(-0.004875) = 0.01751$$

$$\sigma_p = \sqrt{0.01751} = 0.1323$$

This portfolio has expected return half-way between the expected returns on assets A and B, but the portfolio standard deviation is less than half-way between the asset standard deviations. This reflects risk reduction via diversification.

Example

Example: Long-Short two asset portfolio

Next, consider a long-short portfolio with $x_A = 1.5$ and $x_B = -0.5$. In this portfolio, asset B is sold short and the proceeds of the short sale are used to leverage the investment in asset A. The portfolio characteristics are

$$\mu_p = (1.5) \cdot (0.175) + (-0.5) \cdot (0.055) = 0.235$$

$$\sigma_p^2 = (1.5)^2 \cdot (0.067) + (-0.5)^2 \cdot (0.013)$$

$$+ 2 \cdot (1.5)(-0.5)(-0.004875) = 0.1604$$

$$\sigma_p = \sqrt{0.01751} = 0.4005$$

This portfolio has both a higher expected return and standard deviation than asset A.

Portfolio Value-at-Risk

- Assume an initial investment of $\$W_0$ in the portfolio of assets A and B.
- Given that the simple return $R_p \sim N(\mu_p, \sigma_p^2)$. For $\alpha \in (0, 1)$, the $\alpha \times 100\%$ portfolio value-at-risk is

$$\begin{aligned}\text{VaR}_{p,\alpha} &= q_{p,\alpha}^R W_0 \\ &= \left(\mu_p + \sigma_p q_\alpha^z\right) W_0\end{aligned}$$

where $q_{p,\alpha}^R$ is the α quantile of the distribution of R_p and $q_\alpha^z = \alpha$ quantile of $Z \sim N(0, 1)$.

Relationship between Portfolio VaR and Individual Asset VaR

Result: Portfolio VaR is not a weighted average of asset VaR:

$$\mathrm{VaR}_{p,\alpha} \neq x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha}$$

unless $\rho_{AB} = 1$.

Asset VaRs for A and B are:

$$\mathrm{VaR}_{A,\alpha} = q_{0.05}^{R_A} W_0 = (\mu_A + \sigma_A q_\alpha^z) W_0$$

$$\mathrm{VaR}_{B,\alpha} = q_{0.05}^{R_B} W_0 = (\mu_B + \sigma_B q_\alpha^z) W_0$$

Portfolio VaR is:

$$\mathrm{VaR}_{p,\alpha} = \left(\mu_p + \sigma_p q_\alpha^z\right) W_0$$

$$= \left[(x_A\mu_A + x_B\mu_B) + \left(x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}\right)^{1/2} q_\alpha^z\right] W_0$$

Relationship between Portfolio VaR and Individual Asset VaR cont.

Portfolio weighted asset VaR is:

$$
\begin{aligned}
x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha} &= x_A(\mu_A + \sigma_A q_\alpha^z)W_0 + x_B(\mu_B + \sigma_B q_\alpha^z)W_0 \\
&= \left[(x_A\mu_A + x_B\mu_B) + (x_A\sigma_A + x_B\sigma_B)\, q_\alpha^z\right] W_0 \\
&\neq (\mu_p + \sigma_p q_\alpha^z)\, W_0 = \mathrm{VaR}_{p,\alpha}
\end{aligned}
$$

provided $\rho_{AB} \neq 1$.

If $\rho_{AB} = 1$ then $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$ and:

$$
\sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_A\sigma_B = (x_A\sigma_A + x_B\sigma_B)^2
$$

$$
\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B
$$

and so,

$$
x_A \mathrm{VaR}_{A,\alpha} + x_B \mathrm{VaR}_{B,\alpha} = \mathrm{VaR}_{p,\alpha}
$$

Example

Example: Portfolio VaR and Individual Asset VaR

Consider an initial investment of W_0 =\$100,000. The 5% VaRs on assets A and B are:

$$\text{VaR}_{A,0.05} = q_{0.05}^{R_A} W_0 = (0.175 + 0.258(-1.645)) \cdot 100,000 = -24,937,$$

$$\text{VaR}_{B,0.05} = q_{0.05}^{R_B} W_0 = (0.055 + 0.115(-1.645)) \cdot 100,000 = -13,416.$$

The 5% VaR on the equal weighted portfolio with $x_A = x_B = 0.5$ is:

$$\text{VaR}_{p,0.05} = q_{0.05}^{R_p} W_0 = (0.115 + 0.1323(-1.645)) \cdot 100,000 = -10,268,$$

and the weighted average of the individual asset VaRs is,

$$x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05} = 0.5(-24{,}937) + 0.5(-13{,}416) = -19{,}177.$$

Outline

Portfolio Frontier

Vary investment shares x_A and x_B and compute resulting values of μ_p and σ_p^2. Plot μ_p against σ_p as functions of x_A and x_B.

- Shape of portfolio frontier depends on correlation between assets A and B
- If $\rho_{AB} = -1$ then there exists portfolio shares x_A and x_B such that $\sigma_p^2 = 0$
- If $\rho_{AB} = 1$ then there is no benefit from diversification
- Diversification is beneficial even if $0 < \rho_{AB} < 1$

Efficient Portfolios

Definition: Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios.

- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios
- Which efficient portfolio an investor will hold depends on their risk preferences
    - Very risk averse investors dislike volatility and will hold portfolios near the global minimum variance portfolio. They sacrifice expected return for the safety of low volatility.
    - Risk tolerant investors don't mind volatility and will hold portfolios that have high expected returns. They gain expected return by taking on more volatility.

Globabl Minimum Variance Portfolio

- The portfolio with the smallest possible variance is called the global minimum variance portfolio.
- This portfolio is chosen by the most risk averse individuals
- To find this portfolio, one has to solve the following *constrained minimization problem*

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$

$$s.t. \ x_A + x_B = 1$$

Review of Optimization Techniques: Constrained Optimization

Example: Finding the minimum of a bivariate function subject to a linear constraint

$$y = f(x,z) = x^2 + z^2$$

$$\min_{x,z} \; y = f(x,z)$$

$$s.t. \; x + z = 1$$

Solution methods:

- Substitution
- Lagrange multipliers

Method of Substitution

Substitute $z = x - 1$ in $f(x, z)$ and solve univariate minimization:

$$y = f(x, x-1) = x^2 + (1-x)^2$$

$$\min_x \; f(x, x-1)$$

First order conditions:

$$0 = \frac{d}{dx}(x^2 + (1-x)) = 2x + 2(1-x)(-1)$$

$$= 4x - 2$$

$$\Rightarrow x = 0.5$$

Solving for z:

$$z = 1 - 0.5 = 0.5$$

Method of Lagrange Multipliers

Idea: Augment function to be minimized with extra terms to impose constraints.

1. Put constraints in homogeneous form:

$$x + z = 1 \Rightarrow x + z - 1 = 0$$

2. Form Lagrangian function:

$$L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 1)$$

$$\lambda = \text{Lagrange multiplier}$$

3. Minimize Lagrangian function:

$$\min_{x,z,\lambda} L(x, z, \lambda)$$

Method of Lagrange Multipliers cont.

First order conditions:

$$0 = \frac{\partial L(x, z, \lambda)}{\partial x} = 2 \cdot x + \lambda$$

$$0 = \frac{\partial L(x, z, \lambda)}{\partial z} = 2 \cdot z + \lambda$$

$$0 = \frac{\partial L(x, z, \lambda)}{\partial \lambda} = x + z - 1$$

We have three linear equations in three unknowns. Solving gives:

$$2x = 2z = -\lambda \Rightarrow x = z$$

$$2z - 1 = 0 \Rightarrow z = 0.5,\ x = 0.5$$

Example

Example: Finding the Global Minimum Variance Portfolio

Two methods for solution:

- Analytic solution using Calculus
- Numerical solution
  - use the Solver in Excel
  - use R function `solve.QP()` in package quadprog for quadratic optimization problems with equality and inequality constraints

Calculus Solution

Minimization problem:

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_{AB}$$

$$s.t. \ x_A + x_B = 1$$

Use substitution method with:

$$x_B = 1 - x_A$$

to give the univariate minimization,

$$\min_{x_A} \ \sigma_p^2 = x_A^2\sigma_A^2 + (1 - x_A)^2\sigma_B^2 + 2x_A(1 - x_A)\sigma_{AB}$$

Calculus Solution cont.

First order conditions:

$$0 = \frac{d}{dx_A}\sigma_p^2 = \frac{d}{dx_A}\left(x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A)\sigma_{AB}\right)$$

$$= 2x_A\sigma_A^2 - 2(1-x_A)\sigma_B^2 + 2\sigma_{AB}(1-2x_A)$$

$$\Rightarrow x_A^{\min} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},\ x_B^{\min} = 1 - x_A^{\min}$$

Excel Solver Solution

The Solver is an Excel add-in, that can be used to numerically solve general linear and nonlinear optimization problems subject to equality or inequality constraints.

- The solver is made by FrontLine Systems and is provided with Excel
- The solver add-in may not be installed in a "default installation" of Excel
    - Tools/Add-Ins and check the Solver Add-In box
    - If Solver Add-In box is not available, the Solver Add-In must be installed from original Excel installation CD

Outline

Portfolios with a Risk Free Asset

Risk Free Asset:

- Asset with fixed and known rate of return over investment horizon
- Usually use U.S. government T-Bill rate (horizons < 1 year) or T-Note rate (horizon > 1 year)
- T-Bill or T-Note rate is only nominally risk free

Properties of Risk-Free Asset

$$
\begin{aligned}
R_f &= \text{return on risk-free asset} \\
E[R_f] &= r_f = \text{ constant} \\
\text{var}(R_f) &= 0 \\
\text{cov}(R_f, R_i) &= 0, \; R_i = \text{ return on any asset}
\end{aligned}
$$

Portfolios of Risky Asset and Risk Free Asset:

$$
\begin{aligned}
x_f &= \text{ share of wealth in T-Bills} \\
x_B &= \text{ share of wealth in asset B} \\
x_f + x_B &= 1 \\
x_f &= 1 - x_B
\end{aligned}
$$

Properties of Risk-Free Asset cont.

Portfolio return:

$$\begin{aligned} R_p &= x_f r_f + x_B R_B \\ &= (1 - x_B) r_f + x_B R_B \\ &= r_f + x_B (R_B - r_f) \end{aligned}$$

Portfolio excess return:

$$R_p - r_f = x_B (R_B - r_f)$$

Portfolio Distribution:

$$\mu_p = E[R_p] = r_f + x_B (\mu_B - r_f)$$

$$\sigma_p^2 = \text{var}(R_p) = x_B^2 \sigma_B^2$$

$$\sigma_p = x_B \sigma_B$$

$$R_p \sim N(\mu_p, \sigma_p^2)$$

Risk Premium

$$\mu_B - r_f = \text{excess expected return on asset B}$$

$$= \text{expected return on risky asset over return on safe asset}$$

For the portfolio of T-Bills and asset B:

$$\mu_p - r_f = x_B(\mu_B - r_f)$$

$$= \text{expected portfolio return over T-Bill}$$

The risk premia is an increasing function of the amount invested in asset B.

Leveraged Investment

$$x_f < 0,\ x_B > 1$$

Borrow at T-Bill rate to buy more of asset B.

Result: Leverage increases portfolio expected return and risk.

$$\mu_p = r_f + x_B(\mu_B - r_f)$$

$$\sigma_p = x_B \sigma_B$$

$$x_B \uparrow \;\Rightarrow \mu_p \;\&\; \sigma_p \uparrow$$

Determining Portfolio Frontier

Goal: Plot μ_p vs. σ_p.

$$\sigma_p = x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B}$$

$$\begin{aligned}
\mu_p &= r_f + x_B(\mu_B - r_f) \\
&= r_f + \frac{\sigma_p}{\sigma_B}(\mu_B - r_f) \\
&= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p
\end{aligned}$$

where,

$$\left(\frac{\mu_B - r_f}{\sigma_B}\right) = \mathrm{SR}_B = \text{Asset B Sharpe Ratio}$$

$$= \text{excess expected return per unit risk}$$

Determining Portfolio Frontier cont.

Remarks:

- The Sharpe Ratio (SR) is commonly used to rank assets.
- Assets with high Sharpe Ratios are preferred to assets with low Sharpe Ratios

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Efficient Portfolios with 2 Risky Assets and a Risk Free Asset

Investment in 2 Risky Assets and T-Bill:

$$R_A = \text{ simple return on asset A}$$

$$R_B = \text{ simple return on asset B}$$

$$R_f = r_f = \text{ return on T-Bill}$$

Assumptions:

- R_A and R_B are described by the CER model:

$$R_i \sim iid\ N(\mu_i, \sigma_i^2),\ i = A, B$$

$$\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{corr}(R_A, R_B) = \rho_{AB}$$

Efficient Portfolios with 2 Risky Assets and a Risk Free Asset cont.

Results:

- The best portfolio of two risky assets and T-Bills is the one with the highest Sharpe Ratio
- Graphically, this portfolio occurs at the tangency point of a line drawn from R_f to the risky asset only frontier
- The maximum Sharpe Ratio portfolio is called the "tangency portfolio"

Mutual Fund Separation Theorem

Efficient portfolios are combinations of two portfolios (mutual funds):

- T-Bill portfolio
- Tangency portfolio - portfolio of assets A and B that has the maximum Shape ratio

Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.

Finding the tangency portfolio

$$\max_{x_A,\ x_B} \mathrm{SR}_p = \frac{\mu_p - r_f}{\sigma_p} \text{ subject to}$$

$$\mu_p = x_A\mu_A + x_B\mu_B$$

$$\sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB}$$

$$1 = x_A + x_B$$

Solution can be found analytically or numerically (e.g., using solver in Excel).

Finding the tangency portfolio cont.

Using the substitution method it can be shown that:

$$x_A^{\tan} =$$

$$\frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}$$

$$x_B^{\tan} = 1 - x_A^{\tan}$$

Portfolio characteristics:

$$\mu_p^{\tan} = x_A^{\tan}\mu_A + x_B^{\tan}\mu_B$$

$$\left(\sigma_p^{\tan}\right)^2 = \left(x_A^{\tan}\right)^2\sigma_A^2 + \left(x_B^{\tan}\right)^2\sigma_B^2 + 2x_A^{\tan}x_B^{\tan}\sigma_{AB}$$

Efficient Portfolios: tangency portfolio plus T-Bills

$$x_{\tan} = \text{ share of wealth in tangency portfolio}$$
$$x_f = \text{share of wealth in T-bills}$$
$$x_{\tan} + x_f = 1$$
$$\mu_p^e = r_f + x_{\tan}(\mu_p^{\tan} - r_f)$$
$$\sigma_p^e = x_{\tan}\sigma_p^{\tan}$$

Result: The weights $x_{\tan}$ and x_f are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills
- Risk tolerant investors hold mostly tangency portfolio

Example

For the two asset example, the tangency portfolio is:

$$x_A^{\tan} = .46,\ x_B^{\tan} = 0.54$$

$$\mu_p^{\tan} = (.46)(.175) + (.54)(.055) = 0.11$$

$$\begin{aligned}\left(\sigma_p^{\tan}\right)^2 &= (.46)^2(.067) + (.54)^2(.013) \\ &+ 2(.46)(.54)(-.005) \\ &= 0.015\end{aligned}$$

$$\sigma_p^{\tan} = \sqrt{.015} = 0.124$$

Efficient portfolios have the following characteristics:

$$\begin{aligned}\mu_p^e &= r_f + x_{\tan}(\mu_p^{\tan} - r_f) \\ &= 0.03 + x_{\tan}(0.11 - 0.03)\end{aligned}$$

Problem

Find the efficient portfolio that has the same risk (SD) as asset B? That is, determine x_{tan} and x_f such that

$$\sigma_p^e = \sigma_B = 0.114 = \text{target risk.}$$

Note: The efficient portfolio will have a higher expected return than asset B.

Solution

$$
\begin{aligned}
.114 = \sigma_p^e &= x_{\tan}\sigma_p^{\tan} \\
&= x_{\tan}(.124) \\
&\Rightarrow x_{\tan} = \frac{.114}{.124} = .92
\end{aligned}
$$

$$x_f = 1 - x_{\tan} = .08$$

Efficient portfolio with same risk as asset B has:

$$(.92)(.46) = .42 \text{ in asset A}$$

$$(.92)(.54) = .50 \text{ in asset B}$$

$$.08 \text{ in T-Bills}$$

If $r_f = 0.03$, then expected Return on efficient portfolio is:

$$\mu_p^e = .03 + (.92)(.11 - 0.03) = .104.$$

Problem

Assume that $r_f = 0.03$. Find the efficient portfolio that has the same expected return as asset B. That is, determine $x_{\tan}$ and x_f such that:

$$\mu_p^e = \mu_B = 0.055 = \text{target expected return.}$$

Note: The efficient portfolio will have a lower SD than asset B.

Solution

$$0.055 = \mu_p^e = 0.03 + x_{\tan}(.11 - .03)$$

$$x_{\tan} = \frac{0.055 - 0.03}{.11 - .03} = .31$$

$$x_f = 1 - x_{\tan} = .69$$

Efficient portfolio with same expected return as asset B has:

$(.31)(.46) = .14$ in asset A

$(.31)(.54) = .17$ in asset B

$.69$ in T-Bills

The SD of the efficient portfolio is:

$$\sigma_p^e = .31(.124) = .038.$$

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