Econ 424/CFRM 462
Constant Exected Return Model

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Constant Expected Return (CER) Model

$$r_{it} = \text{ cc return on asset } i \text{ in month } t$$
$$i = 1, \cdots, N \text{ assets; } t = 1, \cdots, T \text{ months}$$

Assumptions (normal distribution and covariance stationarity)

$$r_{it} \sim iid\ N(\mu_i,\ \sigma_i^2) \text{ for all } i \text{ and } t$$
$$\mu_i = E[r_{it}] \ \text{ (constant over time)}$$
$$\sigma_i^2 = \text{var}(r_{it}) \ \text{ (constant over time)}$$
$$\sigma_{ij} = \text{cov}(r_{it},\ r_{jt}) \ \text{ (constant over time)}$$
$$\rho_{ij} = \text{cor}(r_{it},\ r_{jt}) \text{ (constant over time)}$$

Regression Model Representation (CER Model)

$$r_{it} = \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1, \cdots N$$
$$\epsilon_{it} \sim \text{iid } N(0, \sigma_i^2) \text{ or } \epsilon_{it} \sim GWN(0, \sigma_i^2)$$
$$\text{cov}(\epsilon_{it},\ \epsilon_{jt}) = \sigma_{ij},\ \rho_{ij} = \text{cor}(\epsilon_{it},\ \epsilon_{jt})$$
$$\text{cov}(\epsilon_{it},\ \epsilon_{js}) = 0 \quad t \neq s,\ \text{for all } i,\ j$$

If $r_{it} \sim N(\mu_i, \sigma_i^2)$ then I can express r_{it} as

$$\left.\begin{aligned} r_{it} &= \mu_i + \epsilon_{it} \\ \epsilon_{it} &\sim N(0, \sigma_i^2) \end{aligned}\right]$$

$E[r_{it}] = \mu_i$

$\text{var}(r_{it}) = \sigma_i^2$

$r_{it} \sim N(\mu_i, \sigma_i^2)$

$r_{it} = \mu_i + \epsilon_{it}$

expected return

unexpected return.

Interpretation

- ϵ_{it} represents random news that arrives in month t
- News affecting asset i may be correlated with news affecting asset j
- News is uncorrelated over time

$r_{it} = \mu_i + \epsilon_{it}$

ϵ_{it}	$=$	r_{it}	$-$	μ_i
unexpected news		Actual return		expected return

No news $\epsilon_{it} = 0 \implies r_{it} = \mu_i$

Good news $\epsilon_{it} > 0 \implies r_{it} > \mu_i$

Bad news $\epsilon_{it} < 0 \implies r_{it} < \mu_i$

CER Model Regression with Standardized News Shocks

$q^r_\alpha = \mu + \sigma \cdot q^z_\alpha$

$$
\begin{aligned}
r_{it} &= \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \quad i = 1, \cdots N \\
&= \mu_i + \sigma_i \times z_{it} \\
z_{it} &\sim \text{iid } N(0, 1) \\
\text{cov}(z_{it},\ z_{jt}) &= \text{cor}(z_{it},\ z_{jt}) = \rho_{ij} \\
\text{cov}(z_{it},\ z_{js}) &= 0 \quad t \neq s, \text{ for all } i,\ j
\end{aligned}
$$

Here, $z_{it} \sim$ iid $N(0, 1)$ is a standardized news shock and σ_i is the volatility of "news".

$\epsilon_{it} \sim N(0, \sigma_i^2)$

$\Rightarrow \quad \epsilon_{it} = \sigma_i \times z_{it}, \quad z_{it} \sim N(0,1)$

$\text{var}(\epsilon_{it}) = \text{var}(\sigma_i \cdot z_{it}) = \sigma_i^2 \cdot \underbrace{\text{var}(z_{it})}_{=1}$

Implied Model for Simple Returns

$$\begin{aligned} R_{it} &= \mathsf{exp}(r_{it}) - 1 \\ &\Rightarrow 1 + R_{it} \sim \mathsf{lognormal}(\mu_i, \sigma_i^2) \end{aligned}$$

Recall

$$\begin{aligned} E[R_{it}] &= \mathsf{exp}\left(\mu_i + \frac{1}{2}\sigma_i^2\right) - 1 \\ \mathsf{var}(R_{it}) &= \mathsf{exp}(2\mu_i + \sigma_i^2)(\mathsf{exp}(\sigma_i^2) - 1) \end{aligned}$$

Value-at-Risk in the CER Model

For an initial investment of $\$W$ for one month, we have

$$VaR_\alpha = \$W_0 \times (e^{q_\alpha^r} - 1)$$
$$q_\alpha^r = \alpha \times 100\% \text{ quantile of } r_t$$

Result: In the CER model with $r = \mu + \sigma \times z$ where $z \sim N(0,1)$

$$q_\alpha^r = \mu + \sigma \times q_\alpha^z$$
$$q_\alpha^Z = \alpha \times 100\% \text{ quantile of } z \sim N(0,1)$$

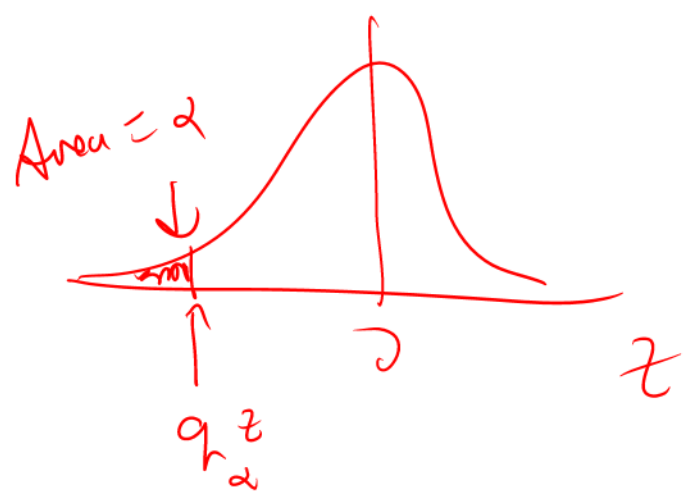
$$VaR_\alpha = \$W_0 \times \left(e^{[\mu + \sigma \cdot q_\alpha^z]} - 1\right)$$

$r = \mu + \sigma * z$

Derivation of $q_\alpha^r = \mu + \sigma \times q_\alpha^z$

Let $z \sim N(0,1)$. Then, by the definition of q_α^z we have

$$\begin{aligned}
&\Pr(z \leq q_\alpha^z) = \alpha \\
&\quad \Rightarrow \Pr(\sigma \times z \leq \sigma \times q_\alpha^Z) = \alpha \\
&\quad \Rightarrow \Pr(\mu + \sigma \times z \leq \mu + \sigma \times q_\alpha^Z) = \alpha \\
&\quad \Rightarrow \Pr(r \leq \mu + \sigma \times q_\alpha^Z) = \alpha \\
&\quad \Rightarrow \mu + \sigma \times q_\alpha^Z = q_\alpha^r
\end{aligned}$$



CER Model in Matrix Notation

Define the $N \times 1$ vectors $r_t = (r_{1t}, \ldots, r_{Nt})'$, $\mu = (\mu_1, \ldots, \mu_N)'$, $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'$ and the $N \times N$ symmetric covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

Then the CER model matrix notation is

$$\begin{aligned} \underset{N\times 1}{\mathbf{r}_t} &= \underset{N\times 1}{\mu} + \underset{N\times 1}{\varepsilon_t}, \\ \varepsilon_t &\sim GWN(\mathbf{0}, \mathbf{\Sigma}), \end{aligned}$$

$\varepsilon_t \sim iid\ N(0, \Sigma)$

which implies that $r_t \sim iid\ N(\mu, \mathbf{\Sigma})$.

Monte Carlo Simulation

Use computer random number generator to create simulated values from assumed model

- Reality check on proposed model
- Create "what if?" scenarios
- Study properties of statistics computed from proposed model

$\longrightarrow$ Helps us understand "bootstrapping"

Simulating Random Numbers from a Distribution

Goal: simulate random number x from pdf $f(x)$ with CDF $F_X(x)$

- Generate $U \sim$ Uniform $[0, 1]$

- Generate $X \sim F_X(x)$ using inverse CDF technique:

$$
\begin{aligned}
x &= F_X^{-1}(u) \\
F_X^{-1} &= \text{inverse CDF function (quantile function)} \\
F_X^{-1}(F_X(x)) &= x
\end{aligned}
$$

Example - Simulate monthly returns on Microsoft from CER Model

- Specify parameters based on sample statistics (use monthly data from June 1992 - Oct 2000)

$$
\begin{aligned}
\mu_i &= 0.03 \text{ (monthly expected return)} \\
\sigma_i &= 0.10 \text{ (monthly SD)} \\
r_{it} &= 0.03 + \varepsilon_{it},\ t = 1, \ldots, 100 \\
\varepsilon_{it} &\sim \text{iid } N(0, (0.10)^2)
\end{aligned}
$$

- Simulation requires generating random numbers from a normal distribution. In R use `rnorm()`.

Monte Carlo Simulation: Multivariate Returns

Example: Simulating observations from CER model for three assets

- Specify parameters based on sample statistics (e.g., use monthly data from June 1992 - Oct 2000)

$$\mu_{SBUX} = .03,\ \mu_{MSFT} = .03,\ \mu_{SP500} = .01$$

$$\mathbf{\Sigma} = \begin{pmatrix} .018 & .004 & .002 \\ & .011 & .002 \\ & & .001 \end{pmatrix}$$

$$r_{it} = \mu_i + \varepsilon_{it},\ t = 1, \ldots, 100$$

$$\varepsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

$$\text{cov}(\varepsilon_{it}, \varepsilon_{jt}) = \sigma_{ij}$$

- Simulation requires generating random numbers from a multivariate normal distribution.

- R package `mvtnorm` has function `mvnorm()` for simulating data from a multivariate normal distribution.

CER Model and Multi-period cc Returns

$$r_t = \mu + \varepsilon_t,\ \varepsilon_t \sim GWN(0, \sigma^2)$$

$$\begin{aligned} r_t(k) &= r_t + r_{t-1} + \cdots + r_{t-k+1} = \sum_{j=0}^{k-1} r_{t-j} \\ &= (\mu + \varepsilon_t) + (\mu + \varepsilon_{t-1}) + \cdots + (\mu + \varepsilon_{t-k+1}) \\ &= k\mu + \sum_{j=0}^{k-1} \varepsilon_{t-j} \\ &= \mu(k) + \varepsilon_t(k) \end{aligned}$$

where

$$\begin{aligned} \mu(k) &= k\mu \\ \varepsilon_t(k) &= \sum_{j=0}^{k-1} \varepsilon_{t-j} \sim GWN\left(0, k\sigma^2\right) \end{aligned}$$

$cov(\varepsilon_t, \varepsilon_s) = 0$

$$var(\varepsilon_t(k)) = var\left(\sum_{j=0}^{k-1} \varepsilon_{t-j}\right) = var(\varepsilon_t) + var(\varepsilon_{t-1}) + \cdots + var(\varepsilon_{t-k+1})$$

$$= k \cdot \sigma^2$$

Result: In the CER model

$$
\begin{aligned}
E[r_t(k)] &= \mu(k) = k\mu \\
\text{var}\,(r_t(k)) &= \sigma^2(k) = \overset{k}{\cancel{12}}\sigma^2 \\
\text{SD}\,(r_t(k)) &= \sigma_k(k) = \sqrt{\underset{k}{\cancel{12}}}\,\sigma
\end{aligned}
\Bigg\}
$$

square root of time rule.

and

$$
\varepsilon_t(k) = \sum_{j=0}^{k-1} \varepsilon_{t-j} = \text{accumulated news shocks}
$$

The Random Walk Model

The CER model for cc returns is equivalent to the random walk (RW) model for log stock prices

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln P_t - \ln P_{t-1}$$
$$= \ln P_t - \ln P_{t-1}$$

which implies

$$\ln P_t = \ln P_{t-1} + r_t$$

Recursive substitution starting at $t = 1$ gives

$$\begin{aligned}
\ln P_1 &= \ln P_0 + r_1 \\
\ln P_2 &= \ln P_1 + r_2 \\
&= \ln P_0 + r_1 + r_2 \\
&\vdots \\
\ln P_t &= \ln P_{t-1} + r_t \\
&= \ln P_0 + \sum_{s=1}^{t} r_s
\end{aligned}$$

Interpretation: Price at t equals initial price plus accumulation of cc returns

In CER model, $r_s = \mu + \varepsilon_s$ so that

$$\begin{aligned}
\ln P_t &= \ln P_0 + \sum_{s=1}^{t} r_s \\
&= \ln P_0 + \sum_{s=1}^{t} (\mu + \varepsilon_s) \\
&= \ln P_0 + t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s
\end{aligned}$$

Interpretation: Log price at t equals initial price $\ln P_0$, plus expected growth in prices $E[\ln P_t] = t \cdot \mu$, plus accumulation of news $\sum_{s=1}^{t} \varepsilon_s$.

The price level at time t is

$$P_t = P_0 \exp\left(t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s\right) = P_0 \exp(t \cdot \mu) \exp\left(\sum_{s=1}^{t} \varepsilon_s\right)$$

$$\exp(t \cdot \mu) = \text{expected growth in price}$$

$$\exp\left(\sum_{s=1}^{t} \varepsilon_s\right) = \text{unexpected growth in price}$$

$r_t = \ln(1 + R_t)$
$\approx R_t$ if $R_t \simeq 0$

CER Model for Simple Returns

- CER Model can also be used for simple returns

$$R_t = \mu + \varepsilon_t$$
$$\varepsilon_t \sim GWN(0, \sigma^2)$$

$R_t(k) \approx N(k \cdot \mu, k \cdot \sigma^2)$

$R_t = 0.01$
$R_{t-1} = 0.02$
$R_t \cdot R_{t-1} = (0.01)(0.02) = 0.0002$

- Main drawbacks: (1) Normal distribution allows $R_t < -1$; (2) Multi-period returns are not normally distributed

$$R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) - 1$$
$$\nsim N(k\mu, k\sigma^2)$$

$$\boxed{\approx R_t + R_{t-1} + \cdots + R_{t-k+1}}$$

$$\Big(1 + R_t + R_{t-1} + \cdots + R_{t-k-1}$$
$$+ \underbrace{R_t \cdot R_{t-1}}_{\approx 0} + \underbrace{R_t \cdot R_{t-2}}_{\approx 0} \cdots \underbrace{R_{t-k+2} \cdot R_{t-k+1}}_{\approx 0} - 1$$

- However, it can be shown that

$$E[R_t(k)] = (1+\mu)^k - 1 \quad \approx k \cdot \mu$$
$$\text{var}(R_t(k)) = (1+\sigma^2+2\mu+\mu^2)^k - (1+\mu)^{2k}$$

$\approx k\sigma^2$

if $\mu \simeq 0$

- An advantage of simple returns is that the portfolio return is a weighted avg. of simple returns

$R_p = x_1 R_1 + \cdots + x_N R_N$, $R_i = \mu + \epsilon$

$\Rightarrow E[R_p] = x_1\mu_1 + \cdots + x_N \mu_N = \mu_p$ $\quad R_p = \mu_p + \epsilon_p$

$\text{var}(R_p) = x' \Sigma x = \sigma_p^2$ $\quad \epsilon_p = x_1 \epsilon_1 + \cdots + x_N \epsilon_N$

$R_{it} = \mu_i + \epsilon_{it}$
$i = 1, \ldots, N$
$t = 1, \ldots, T$

Estimating Parameters of CER model

Parameters of CER Model

$$
\begin{aligned}
\mu_i &= E[r_{it}] \\
\sigma_i^2 &= \mathsf{var}(r_{it}) \\
\sigma_{ij} &= \mathsf{cov}(r_{it}, r_{jt}) \\
\rho_{ij} &= \mathsf{cor}(r_{it}, r_{jt})
\end{aligned}
$$

are not known with certainty

First Econometric Task

- Estimate $\mu_i,\ \sigma_i^2,\ \sigma_{ij},\ \rho_{ij}$ using observed sample of historical monthly returns ex post

Estimators and Estimates

Definition: An estimator is a rule or algorithm (mathematical formula) for computing an *ex ante* estimate of a parameter based on a random sample.

Example: Sample mean as estimator of $E[r_{it}] = \mu_i$

$$\begin{aligned} \{r_{i1}, \ldots, r_{iT}\} &= \text{ covariance stationary time series} \\ &= \text{ collection of random variables} \\ \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it} &= \text{ sample mean} \\ &= \text{ random variable} \end{aligned}$$

Definition: An estimate of a parameter is simply the *ex post* value (numerical value) of an estimator based on observed data

Example: Sample mean from an observed sample

$$\begin{aligned}
\{r_{i1} = .02, r_{i2} = .01, r_{i3} = -.01, \ldots, r_{iT} = .03\} &= \text{observed sample} \\
\hat{\mu}_i &= \frac{1}{T}(.02 + .01 - .01 + \cdots + .03) \\
&= \text{ number } = 0.01 \text{ (say)}
\end{aligned}$$

Estimators of CER Model Parameters: Plug-in Principle

Plug-in principle: Estimate model parameters using appropriate sample statistics

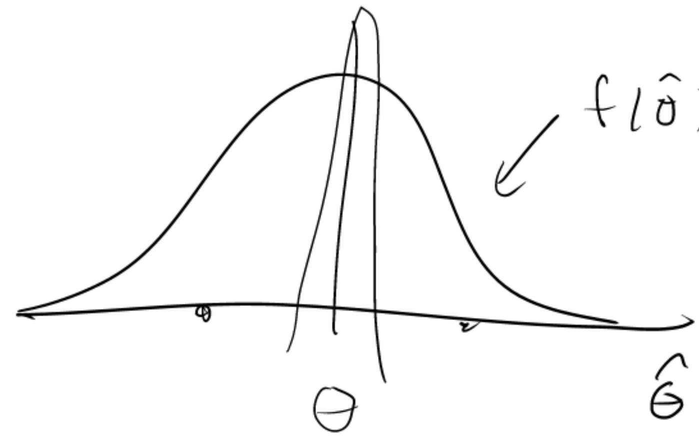
$$\mu_i = E[r_{it}] : \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

$$\sigma_i^2 = E[(r_{it} - \mu_i)^2] : \hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \hat{\mu}_i)^2$$

$$\sigma_i = \sqrt{\sigma_i^2} : \hat{\sigma}_i = \sqrt{\hat{\sigma}_i^2}$$

$$\sigma_{ij} = E[(r_{it} - \mu_i)(r_{jt} - \mu_j)] : \hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \hat{\mu}_i)(r_{jt} - \hat{\mu}_j)$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} : \hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{\hat{\sigma}_i \cdot \hat{\sigma}_j}$$



Properties of Estimators

$$\theta = \text{parameter to be estimated}$$
$$\hat{\theta} = \text{estimator of } \theta \text{ from random sample}$$

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample

- $f(\hat{\theta})$ = pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample

- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Estimation Error

$$error(\hat{\theta}, \theta) = \hat{\theta} - \theta$$

Bias

$$\textbf{bias}(\hat{\theta}, \theta) = E\left[error(\hat{\theta}, \theta)\right] = E\left[\hat{\theta}\right] - \theta$$

$$\hat{\theta} \text{ is unbiased if } E[\hat{\theta}] = \theta \Rightarrow \textbf{bias}(\hat{\theta}, \theta) = 0$$

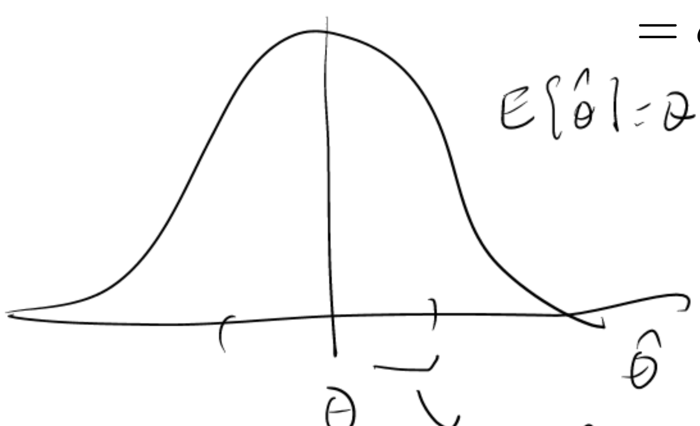
Remark: An unbiased estimator is "on average" correct, where "on average" means over many hypothetical samples. It most surely will not be exactly correct for the sample at hand!

Precision

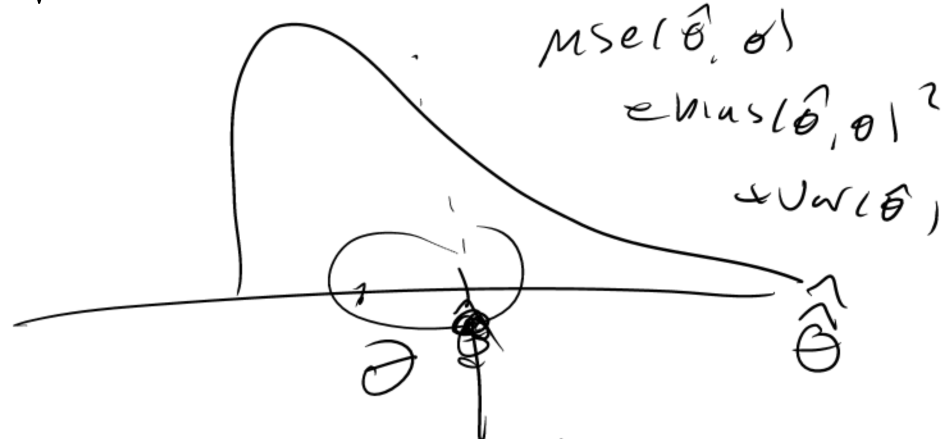
$$
\begin{aligned}
mse(\hat\theta,\theta) &= E\left[error(\hat\theta,\theta)^2\right] = E\left[\left(\hat\theta-\theta\right)^2\right]\\
&= \text{bias}(\hat\theta,\theta)^2 + \text{var}(\hat\theta)\\
\text{var}(\hat\theta) &= E[(\hat\theta - E[\hat\theta])^2
\end{aligned}
$$

Remark: If bias$(\hat\theta,\theta)\approx 0$ then precision is typically measured by the *standard error* of $\hat\theta$ defined by

$$
\begin{aligned}
\text{SE}(\hat\theta) &= \text{ standard error of } \hat\theta\\
&= \sqrt{\text{var}(\hat\theta)} = \sqrt{E[(\hat\theta - E[\hat\theta])^2]}\\
&= \sigma_{\hat\theta}
\end{aligned}
$$



$mse(\hat\theta,\theta) = var(\hat\theta)$



$E[\hat\theta] \Rightarrow bias(\hat\theta,\theta) > 0$

Bias of CER Model Estimates

- $\hat{\mu}_i, \hat{\sigma}_i^2$ and $\hat{\sigma}_{ij}$ are unbiased estimators:

$$E\left[\hat{\mu}_i\right] = \mu_i \Rightarrow \text{bias}(\hat{\mu}_i, \mu_i) = 0$$
$$E\left[\hat{\sigma}_i^2\right] = \sigma_i^2 \Rightarrow \text{bias}(\hat{\sigma}_i^2, \sigma_i^2) = 0$$
$$E\left[\hat{\sigma}_{ij}\right] = \sigma_{ij} \Rightarrow \text{bias}(\hat{\sigma}_{ij}, \sigma_{ij}) = 0$$

- $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are biased estimators

$$E[\hat{\sigma}_i] \neq \sigma_i \Rightarrow \text{bias}(\hat{\sigma}_i, \sigma_i) \neq 0$$
$$E[\hat{\rho}_{ij}] \neq \rho_{ij} \Rightarrow \text{bias}(\hat{\rho}_{ij}, \rho_{ij}) \neq 0$$

  but bias is very small except for very small samples and disappears as sample size T gets large.

Remarks

- "On average" being correct doesn't mean the estimate is any good for your sample!
- The value of SE($\hat{\theta}$) will tell you how far from θ the estimate $\hat{\theta}$ typically will be.
- Good estimators $\hat{\theta}$ have small bias and small SE($\hat{\theta}$)

Proof that $E\left[\hat{\mu}_i\right] = \mu_i$

Recall,

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

$$r_{it} = \mu_i + \epsilon_{it},\ \ \epsilon_{it} \sim \text{iid } N(0, \sigma^2)$$

Now

$$E[r_{it}] = \mu_i + E[\epsilon_{it}] = \mu_i$$

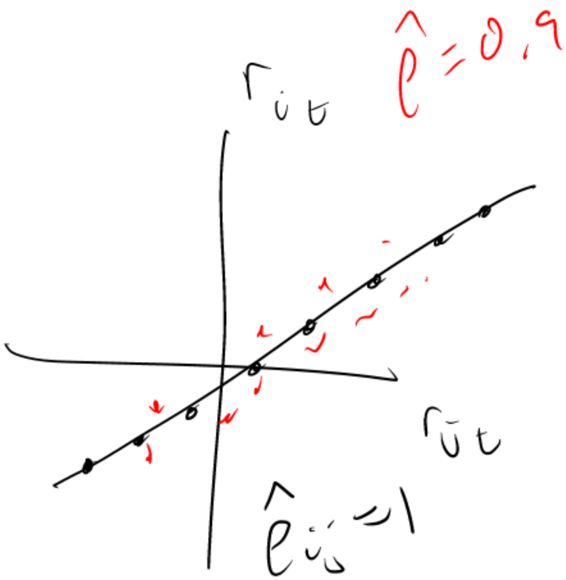
since $E[\epsilon_{it}] = 0$.

Therefore,

$$
\begin{aligned}
E[\hat{\mu}_i] &= \frac{1}{T}\sum_{t=1}^{T} E[r_{it}] \\
&= \frac{1}{T}\sum_{t=1}^{T} \mu_i \\
&= \frac{1}{T} T\mu_i = \mu_i
\end{aligned}
$$

$\hat{\sigma}_i^2$

Standard Error formulas for $\hat{\mu}_i$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$



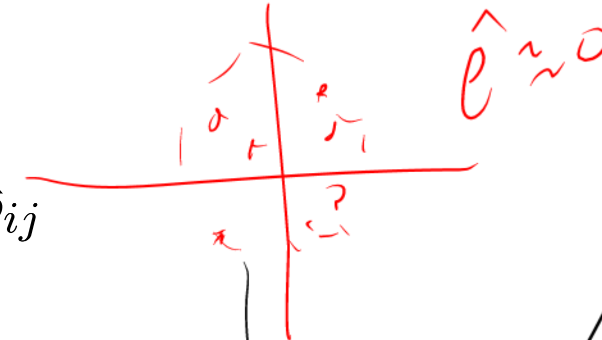
$$\mathrm{SE}(\hat{\mu}_i) = \frac{\sigma_i}{\sqrt{T}}$$

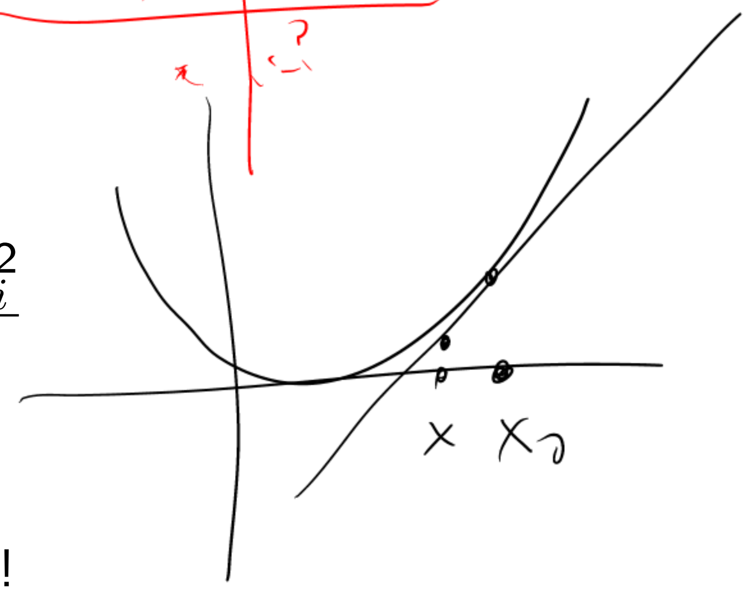
$$\mathrm{SE}(\hat{\sigma}_i^2) \approx \frac{\sigma_i^2}{\sqrt{T/2}} = \frac{\sqrt{2}\sigma_i^2}{\sqrt{T}}$$

$$\mathrm{SE}(\hat{\sigma}_i) \approx \frac{\sigma_i}{\sqrt{2T}}$$

$\mathrm{SE}(\hat{\sigma}_{ij})$: no easy formula!

$$\mathrm{SE}(\hat{\rho}_{ij}) \approx \frac{(1-\rho_{ij}^2)}{\sqrt{T}}$$





Note: "$\approx$" denotes "approximately equal to", where approximation error $\longrightarrow 0$ as $T \longrightarrow \infty$ for normally distributed data.

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$+ \text{Remainder}$$

Remarks

- Large SE$\Longrightarrow$ imprecise estimate; Small SE$\Longrightarrow$ precise estimate
- Precision increases with sample size: SE$\longrightarrow$ 0 as $T \longrightarrow \infty$
- $\hat{\sigma}_i$ is generally a more precise estimate than $\hat{\mu}_i$ or $\hat{\rho}_{ij}$
- SE formulas for $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are approximations based on the Central Limit Theorem. Monte Carlo simulation and bootstrapping can be used to get better approximations
- SE formulas depend on unknown values of parameters $\Rightarrow$ formulas are not practically useful

- Practically useful formulas replace unknown values with estimated values:

$$\widehat{\mathsf{SE}}(\hat{\mu}_i) = \frac{\hat{\sigma}_i}{\sqrt{T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathsf{SE}}(\hat{\sigma}_i^2) \approx \frac{\hat{\sigma}_i^2}{\sqrt{T/2}},\ \hat{\sigma}_i^2 \text{ replaces } \sigma_i^2$$

$$\widehat{\mathsf{SE}}(\hat{\sigma}_i) \approx \frac{\hat{\sigma}_i}{\sqrt{2T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathsf{SE}}(\hat{\rho}_{ij}) \approx \frac{(1-\hat{\rho}_{ij}^2)}{\sqrt{T}},\ \hat{\rho}_{ij} \text{ replaces } \rho_{ij}$$

$$\frac{1}{(\sqrt{T})^2}\left(\frac{\hat{\sigma}_i}{\sqrt{2T}}\right)^2$$

$$SE(\hat{SE}(\hat{\mu}_i)) = ?$$

$$= SE\left(\frac{\hat{\sigma}_i}{\sqrt{T}}\right)$$

$$= \sqrt{var\left(\frac{\hat{\sigma}_i}{\sqrt{T}}\right)} = \frac{1}{(\sqrt{T})^2} var(\hat{\sigma}_i)$$

Deriving SE($\hat{\mu}_i$)

$$\text{var}(\hat{\mu}_i) = \text{var}\left(\frac{1}{T}\sum_{t=1}^{T} r_{it}\right)$$

$$= \frac{1}{T^2}\left[\text{var}(r_{it}) + \text{var}(r_{it-1}) + \cdots + 2\,\text{cov}(r_{it}, r_{it-1}) + \cdots\right.$$

$$= \frac{1}{T^2}\sum_{t=1}^{T} \text{var}(r_{it}) \text{ (since } r_{it} \text{ are independent)}$$

$$= \frac{1}{T^2}\sum_{t=1}^{T} \sigma_i^2 = \frac{\sigma_i^2}{T} \text{ (since var}(r_{it}) = \sigma^2)$$

$$\text{SE}(\hat{\mu}_i) = \sqrt{\text{var}(\hat{\mu}_i)} = \frac{\sigma_i}{\sqrt{T}}$$

0

b/c

$r_{it} \sim \text{iid} \; N(\mu_i, \sigma_i^2)$

Consistency

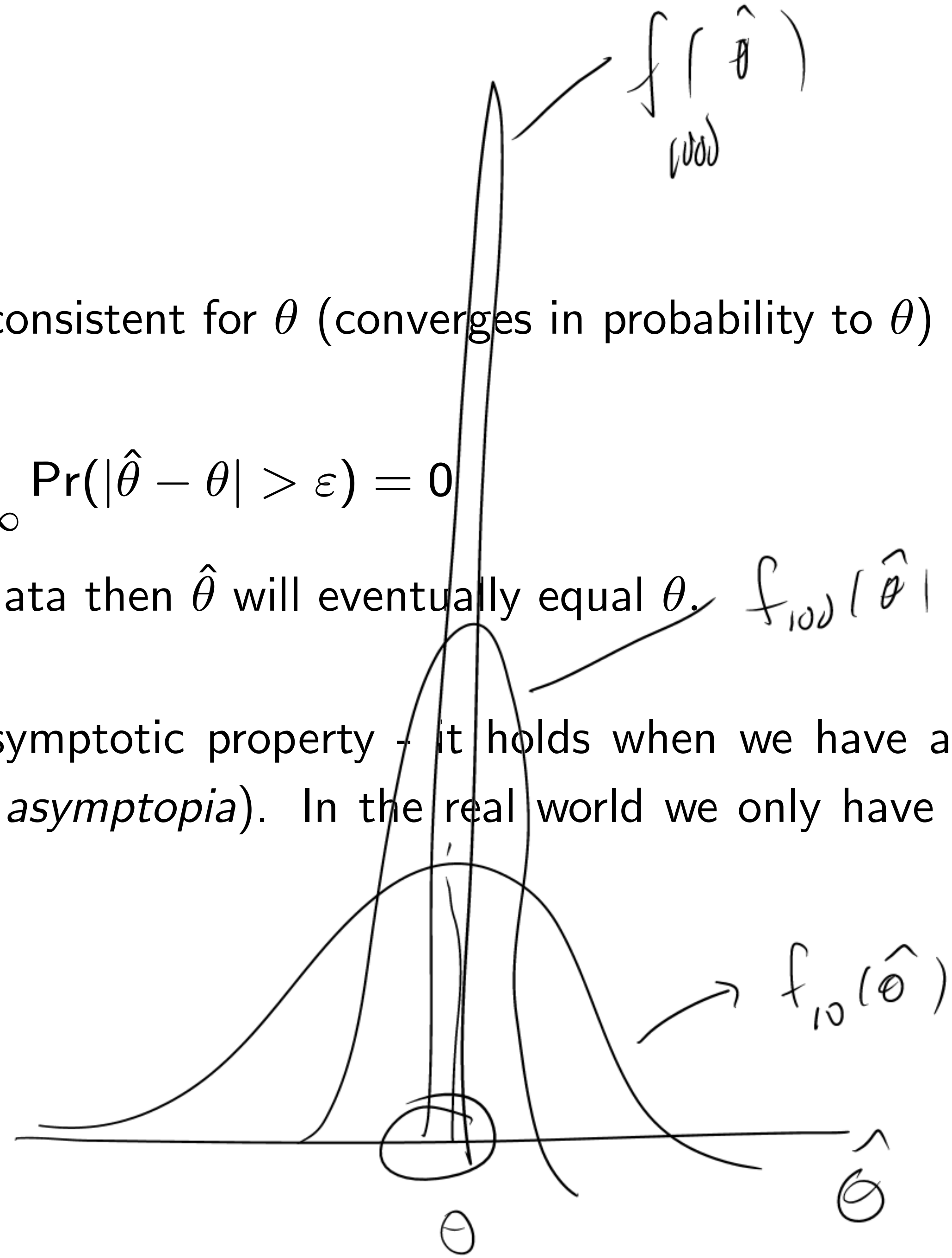
Definition: An estimator $\hat{\theta}$ is consistent for θ (converges in probability to θ) if for any $\varepsilon > 0$

$$\lim_{T \to \infty} \Pr(|\hat{\theta} - \theta| > \varepsilon) = 0$$

Intuitively, as we get enough data then $\hat{\theta}$ will eventually equal θ.

Remark: Consistency is an asymptotic property - it holds when we have an infinitely large sample (i.e, in *asymptopia*). In the real world we only have a finite amount of data!

$$\hat{\theta} \xrightarrow{p} \theta \quad \text{as } T \to \infty$$



Result: An estimator $\hat{\theta}$ is consistent for θ if

- bias$(\hat{\theta}, \theta) = 0$ as $T \rightarrow \infty$
- SE$\left(\hat{\theta}\right) = 0$ as $T \rightarrow \infty$

Result: In the CER model, the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i,$ $\hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ are consistent.

Distribution of CER Model Estimators

$$\theta = \text{parameter to be estimated}$$
$$\hat{\theta} = \text{estimator of } \theta \text{ from random sample}$$

KEY POINTS

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample

- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample

- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Example: Distribution of $\hat{\mu}$ in CER Model

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it},\ r_{it} = \mu_i + \epsilon_{it},\ \epsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

Result:

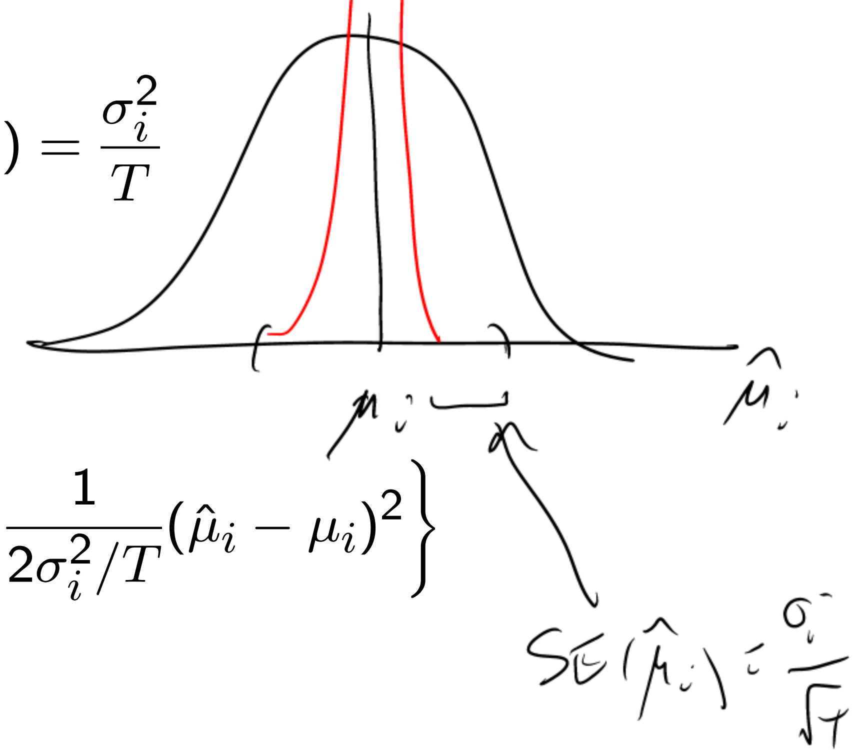
$\hat{\mu}_i$ is $\frac{1}{T}$ times the sum of T normally distributed random variables $\Rightarrow \hat{\mu}_i$ is also normally distributed with

$$E[\hat{\mu}_i] = \mu_i,\ \text{var}(\hat{\mu}_i) = \frac{\sigma_i^2}{T}$$

That is,

$$\hat{\mu}_i \sim N\left(\mu_i, \frac{\sigma_i^2}{T}\right)$$

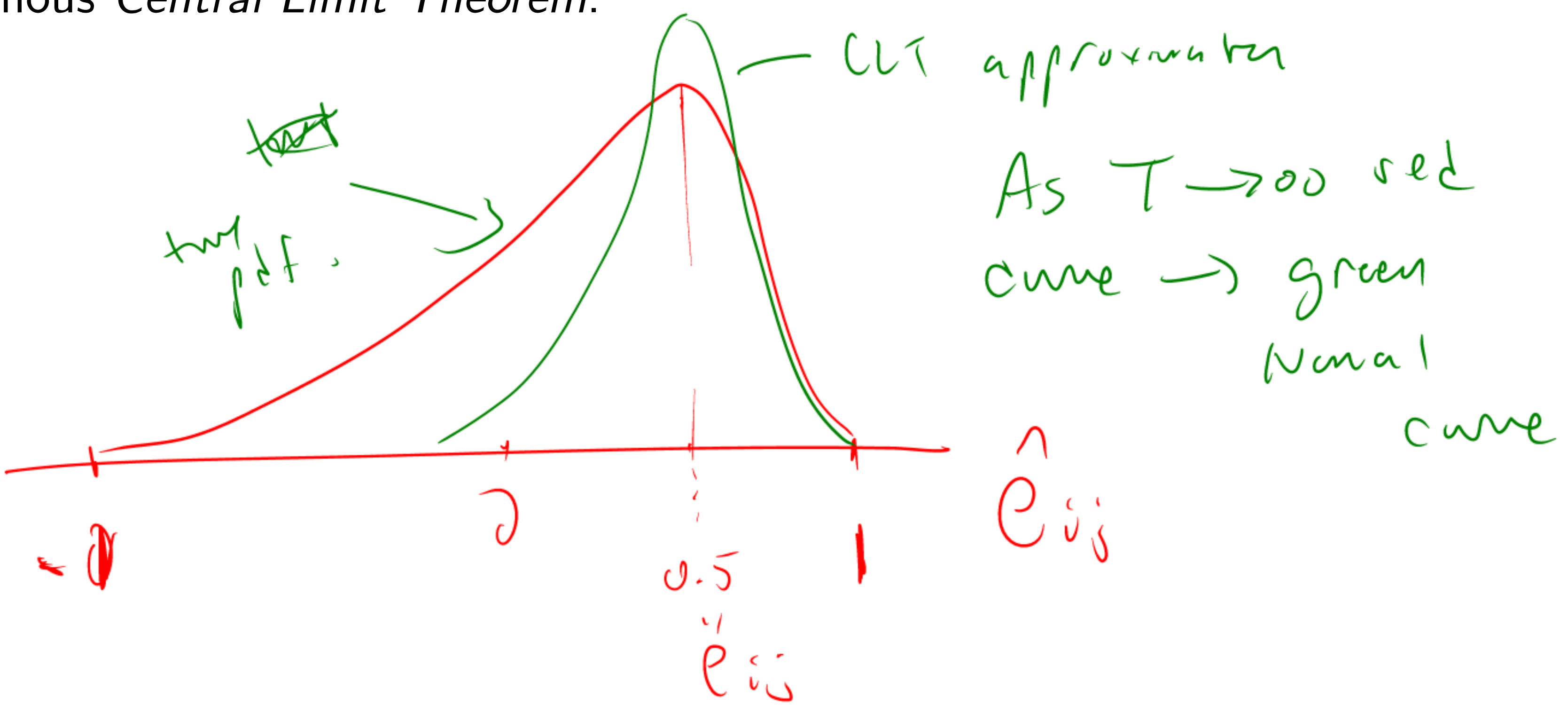
$$f(\hat{u}_i) = (2\pi\sigma_i^2/T)^{-1/2} \exp\left\{-\frac{1}{2\sigma_i^2/T}(\hat{\mu}_i - \mu_i)^2\right\}$$



Distribution of $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$

Result: The exact distributions (for finite sample size T) of $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are not normal.

However, as the sample size T gets large the exact distributions of $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ get closer and closer to the normal distribution. This is the due to the famous *Central Limit Theorem*.



Central Limit Theorem (CLT)

Let $X_1, \ldots, X_T$ be a iid random variables with $E[X_t] = \mu$ and $\mathsf{var}(X_t) = \sigma^2$. Then

$$\frac{\bar{X} - \mu}{\mathsf{SE}(\bar{X})} = \frac{\bar{X} - \mu}{\sigma/\sqrt{T}} = \sqrt{T}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0, 1) \text{ as } T \to \infty$$

Equivalently,

$$\bar{X} \sim N\left(\mu, \mathsf{SE}(\bar{X})^2\right) \sim N\left(\mu, \frac{\sigma^2}{T}\right)$$

for large enough T

We say that $\bar{X}$ is asymptotically normally distributed with mean μ and variance $\mathsf{SE}(\bar{X})^2$.

Definition: An estimator $\hat{\theta}$ is asymptotically normally distributed if

$$\hat{\theta} \sim N(\theta, \mathsf{SE}(\hat{\theta})^2)$$

for large enough T

Result: An implication of the CLT is that the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are asymptotically normally distributed under the CER model.

$$\hat{\rho}_{ij} \sim N\left(\rho_{ij}, \left(\frac{1-\rho_{ij}^2}{\sqrt{T}}\right)^2\right)$$

asymptotic normal distn for $\hat{\rho}_{ij}$

95% CI for θ

Confidence Intervals

$$\hat{\theta} \pm 2 * SE(\hat{\theta})$$

$$[\hat{\theta} - 2 * SE(\hat{\theta}), \hat{\theta} + 2 SE(\hat{\theta})]$$

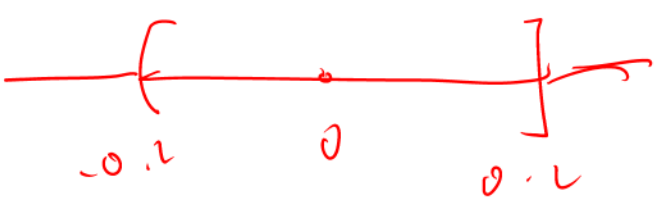
$$\begin{aligned}\hat{\theta} &= \text{estimate of } \theta \\ &= \text{best guess for unknown value of } \theta\end{aligned}$$

Idea: A confidence interval for θ is an interval estimate of θ that covers θ with a stated probability

Intuition: think of a confidence interval like a "horse shoe". For a given sample, there is stated probability that the confidence interval (horse shoe thrown at θ) will cover θ.

Ex: $\hat{\theta} = 0$, $SE[\hat{\theta}] = 0.1$

$2 * SE[\hat{\theta}] = 0.2$



95% CI $[-0.2, 0.2]$

Result: Let $\hat{\theta}$ be an asymptotically normal estimator for θ. Then

- An approximate 95% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right]$$
$$\hat{\theta} \pm 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)$$

  that covers θ with probability approximately equal to 0.95. That is

$$\mathsf{Pr}\left\{\hat{\theta} - 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right) \leq \theta \leq \hat{\theta} + 2 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right\} \approx 0.95$$

- An approximate 99% confidence interval for θ is an interval estimate of the form

$$\left[\hat{\theta} - 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)\right]$$

$$\hat{\theta} \pm 3 \cdot \widehat{\mathsf{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 0.99.

Remarks

- 99% confidence intervals are wider than 95% confidence intervals

- For a given confidence level the width of a confidence interval depends on the size of $\widehat{\mathsf{SE}}(\hat{\theta})$

In the CER model, 95% Confidence Intervals for $\mu_i,\ \sigma_i,$ and ρ_{ij} are:

$$
\begin{aligned}
&\hat{\mu}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{T}} \\
&\hat{\sigma}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{2T}} \\
&\hat{\rho}_{ij} \pm 2 \cdot \frac{(1 - \hat{\rho}_{ij}^2)}{\sqrt{T}}
\end{aligned}
$$

Using Monte Carlo Simulation to Evaluate Bias, Standard Error and Confidence Interval Coverage

- Create many simulated samples from CER model
- Compute parameter estimates for each simulated sample
- Compute mean and sd of estimates over simulated samples
- Compute 95% confidence interval for each sample
- Count number of intervals that cover true parameter

Value-at-Risk in the CER Model

In the CER model

$$r_{it} \sim iid\ N(\mu_i, \sigma_i^2) \Rightarrow r_{it} = \mu_i + \sigma_i \times z_{it},\ z_{it} \sim iid\ N(0,1)$$

The $\alpha \cdot 100\%$ quantile q_α^r may be expressed as

$$q_\alpha^r = \mu_i + \sigma_i \times q_\alpha^Z$$
$$q_\alpha^Z = \text{standard Normal quantile}$$

Then

$$VaR_\alpha = (\exp(q_\alpha^r) - 1) \cdot W_0$$

Example: $r_t \sim N(0.02, (0.10)^2)$ and $W_0 = \$10,000$. Here, $\mu_r = 0.02$ and $\sigma_r = 0.10$ are known values. Then

$$
\begin{aligned}
q^Z_{.05} &= -1.645 \\
q_{.05} &= 0.02 + (0.10)(-1.645) = -0.1445 \\
\text{VaR}_{.05} &= (\exp(-0.1145) - 1) \cdot \$10,000 = -\$1,345
\end{aligned}
$$

Estimating Quantiles from CER Model

$$\hat{q}_\alpha^r = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z$$
$$q_\alpha^Z = \text{standard Normal quantile}$$

Estimating Value-at-Risk from CER Model

$$\widehat{\mathsf{VaR}}_\alpha = (\exp(\hat{q}_\alpha^r) - 1) \cdot W_0$$
$$\hat{q}_\alpha^r = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z$$
$$W_0 = \text{initial investment in \$}$$

Q: What is $E\left[\widehat{\mathsf{VaR}}_\alpha\right]$ and $\mathsf{SE}\left(\widehat{\mathsf{VaR}}_\alpha\right)$?

Computing Standard Errors for VaR

- We can compute $\mathsf{SE}(\hat{q}^r_\alpha)$ using

$$\begin{aligned}\mathsf{var}(\hat{q}^r_\alpha) &= \mathsf{var}\left(\hat{\mu}_i + \hat{\sigma}_i q^Z_\alpha\right)\\ &= \mathsf{var}(\hat{\mu}_i) + \left(q^Z_\alpha\right)^2 \mathsf{var}(\hat{\sigma}_i) + 2q^Z_\alpha \mathsf{cov}(\hat{\mu}_i, \hat{\sigma}_i)\\ &= \mathsf{var}(\hat{\mu}_i) + \left(q^Z_\alpha\right)^2 \mathsf{var}(\hat{\sigma}_i),\ \text{since } \mathsf{cov}(\hat{\mu}_i, \hat{\sigma}_i) = 0\end{aligned}$$

  Then

$$\mathsf{SE}(\hat{q}^r_\alpha) = \sqrt{\mathsf{var}(\hat{\mu}_i) + \left(q^Z_\alpha\right)^2 \mathsf{var}(\hat{\sigma}_i)}$$

- However, computing $\mathsf{SE}(\widehat{\mathsf{VaR}}_\alpha)$ is not straightforward since

$$\mathsf{var}\left(\widehat{\mathsf{VaR}}_\alpha\right) = \mathsf{var}\left((\exp(\hat{q}^r_\alpha) - 1) \cdot W_0\right)$$