Introduction to Computational Finance and Financial Econometrics

Constant Expected Return (CER) Model

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Outline

1. Constant Expected Return (CER) Model Assumptions
2. Monte Carlo Simulation of the CER Model
3. Estimation of the CER Model

Constant Expected Return (CER) Model

$$r_{it} = \text{ cc return on asset } i \text{ in month } t$$

$$i = 1, \cdots, N \text{ assets; } t = 1, \cdots, T \text{ months}$$

Assumptions (normal distribution and covariance stationarity):

$$r_{it} \sim iid\ N(\mu_i,\ \sigma_i^2) \text{ for all } i \text{ and } t$$

$$\mu_i = E[r_{it}] \text{ (constant over time)}$$

$$\sigma_i^2 = \text{var}(r_{it}) \text{ (constant over time)}$$

$$\sigma_{ij} = \text{cov}(r_{it},\ r_{jt}) \text{ (constant over time)}$$

$$\rho_{ij} = \text{cor}(r_{it},\ r_{jt}) \text{ (constant over time)}$$

Regression Model Representation (CER Model)

$$r_{it} = \mu_i + \epsilon_{it} \;\; t = 1, \cdots, T; \;\; i = 1, \cdots N$$

$$\epsilon_{it} \sim \text{iid } N(0, \sigma_i^2) \text{ or } \epsilon_{it} \sim GWN(0, \sigma_i^2)$$

$$\text{cov}(\epsilon_{it}, \; \epsilon_{jt}) = \sigma_{ij}, \;\; \rho_{ij} = \text{cor}(\epsilon_{it}, \; \epsilon_{jt})$$

$$\text{cov}(\epsilon_{it}, \; \epsilon_{js}) = 0, \;\; t \neq s, \;\; \text{for all } i, \; j$$

Interpretation

$$r_{it} = \mu_i + \epsilon_{it} \Rightarrow r_{it} - \mu_i = \epsilon_{it}$$

- ϵ_{it} represents random news that arrives in month t
- News affecting asset i may be correlated with news affecting asset j
- News is uncorrelated over time

$$\underset{\substack{\text{unexpected}\\\text{news}}}{\epsilon_{it}} = \underset{\substack{\text{Actual}\\\text{return}}}{r_{it}} - \underset{\substack{\text{expected}\\\text{return}}}{\mu_i}$$

$$\text{No news } \epsilon_{it} = 0 \Longrightarrow r_{it} = \mu_i$$

$$\text{Good news } \epsilon_{it} > 0 \Longrightarrow r_{it} > \mu_i$$

$$\text{Bad news } \epsilon_{it} < 0 \Longrightarrow r_{it} < \mu_i$$

CER Model Regression with Standardized News Shocks

$X \sim N(0, \sigma^2)$, $\frac{1}{\sigma} \cdot X = Z$, $var\left(\frac{1}{\sigma} X\right) = \frac{\sigma^2}{\sigma^2}$
$\Rightarrow X = \sigma \cdot Z$
$= 1$

$$
\begin{aligned}
r_{it} &= \mu_i + \epsilon_{it} \quad t = 1, \cdots, T; \;\; i = 1, \cdots N \\
&= \mu_i + \underline{\sigma_i} \times z_{it} \\
z_{it} &\sim \text{iid } N(0, 1) \\
\text{cov}(z_{it},\; z_{jt}) &= \text{cor}(z_{it},\; z_{jt}) = \rho_{ij} \\
\text{cov}(z_{it},\; z_{js}) &= 0, \;\; t \neq s, \;\; \text{for all } i, \;\; j
\end{aligned}
$$

Here, $z_{it} \sim \text{iid } N(0, 1)$ is a standardized news shock and σ_i is the volatility of "news".

Implied Model for Simple Returns

$$R_{it} = \exp(r_{it}) - 1$$

$$\Rightarrow 1 + R_{it} \sim \text{lognormal}(\mu_i, \sigma_i^2)$$

Recall,

$$E[R_{it}] = \exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right) - 1$$

$$\text{var}(R_{it}) = \exp(2\mu_i + \sigma_i^2)(\exp(\sigma_i^2) - 1)$$

Value-at-Risk in the CER Model

For an initial investment of $\$\,W$ for one month, we have:

$$VaR_\alpha = \$\,W_0 \times (e^{q_\alpha^r} - 1)$$

$$q_\alpha^r = \alpha \times 100\% \text{ quantile of } r_t$$

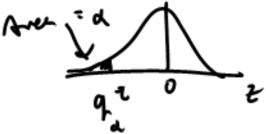
Result: In the CER model with $r = \mu + \sigma \times z$ where $z \sim N(0,1)$.

$$q_\alpha^r = \mu + \sigma \times q_\alpha^z$$

$$q_\alpha^Z = \alpha \times 100\% \text{ quantile of } z \sim N(0,1)$$

Value-at-Risk in the CER Model cont.

Derivation of $q_\alpha^r = \mu + \sigma \times q_\alpha^z$



Let $z \sim N(0,1)$. Then, by the definition of q_α^z we have:

$$\Pr(z \leq q_\alpha^z) = \alpha$$

$$\Rightarrow \Pr(\sigma \times z \leq \sigma \times q_\alpha^Z) = \alpha$$

$$\Rightarrow \Pr(\mu + \sigma \times z \leq \mu + \sigma \times q_\alpha^Z) = \alpha$$

$$\Rightarrow \Pr(r \leq \mu + \sigma \times q_\alpha^Z) = \alpha$$

$$\Rightarrow \mu + \sigma \times q_\alpha^Z = q_\alpha^r$$

CER Model in Matrix Notation

Define the $N \times 1$ vectors $r_t = (r_{1t}, \ldots, r_{Nt})'$, $\mu = (\mu_1, \ldots, \mu_N)'$, $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'$ and the $N \times N$ symmetric covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

Then the CER model matrix notation is:

$$\mathbf{r}_t = \mu + \varepsilon_t,$$

$$\varepsilon_t \sim GWN(\mathbf{0}, \mathbf{\Sigma}),$$

which implies that $r_t \sim iid\ N(\mu, \mathbf{\Sigma})$.

Outline

Monte Carlo Simulation

Use computer random number generator to create simulated values from assumed model.

- Reality check on proposed model
- Create "what if?" scenarios
- Study properties of statistics computed from proposed model

Simulating Random Numbers from a Distribution

Goal: simulate random number x from pdf $f(x)$ with CDF $F_X(x)$.

- Generate $U \sim$ Uniform $[0,1]$
- Generate $X \sim F_X(x)$ using inverse CDF technique:

$$x = F_X^{-1}(u)$$

$$F_X^{-1} = \text{inverse CDF function (quantile function)}$$

$$F_X^{-1}\left(F_X(x)\right) = x$$

Example

Example: Simulate monthly returns on Microsoft from CER Model

- Specify parameters based on sample statistics (use monthly data from January 1998 - May 2012)

$$\mu_i = 0.004 \text{ (monthly expected return)}$$

$$\sigma_i = 0.10 \text{ (monthly SD)}$$

$$r_{it} = 0.004 + \varepsilon_{it}, \;\; t = 1, \ldots, 172$$

$$\varepsilon_{it} \sim \text{iid } N(0, (0.10)^2)$$

- Simulation requires generating random numbers from a normal distribution. In R use `rnorm()`.

Example: Simulating observations from CER model for three assets

- Specify parameters based on sample statistics (e.g., use monthly data from January 1998 - May 2012)

$$\mu_{MSFT} = .004,\ \mu_{SBUX} = .015,\ \mu_{SP500} = .002$$

$$\Sigma = \begin{pmatrix} .010 & .004 & .003 \\ & .012 & .002 \\ & & .002 \end{pmatrix}$$

$$\underset{N\times 1}{\underset{\sim}{r}_t} = \underset{\sim}{\mu} + \underset{\sim}{\varepsilon}_t$$

$$r_{it} = \mu_i + \varepsilon_{it},\ t = 1, \ldots, 172$$

$$\underset{\sim}{\varepsilon}_t \sim N(0, \Sigma)$$

$$\varepsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

$$\text{cov}(\varepsilon_{it}, \varepsilon_{jt}) = \sigma_{ij}$$

Monte Carlo Simulation: Multivariate Returns cont.

Example: Simulating observations from CER model for three assets

- Simulation requires generating random numbers from a multivariate normal distribution.
- R package `mvtnorm` has function `mvnorm()` for simulating data from a multivariate normal distribution.

CER Model and Multi-period cc Returns

$$
\begin{aligned}
r_t &= \mu + \varepsilon_t, \ \varepsilon_t \sim GWN(0, \sigma^2) \\
r_t(k) &= r_t + r_{t-1} + \cdots + r_{t-k+1} = \sum_{j=0}^{k-1} r_{t-j} \\
&= (\mu + \varepsilon_t) + (\mu + \varepsilon_{t-1}) + \cdots + (\mu + \varepsilon_{t-k+1}) \\
&= k\mu + \sum_{j=0}^{k-1} \varepsilon_{t-j} \\
&= \mu(k) + \varepsilon_t(k)
\end{aligned}
$$

where,

$$\mu(k) = k\mu$$

$$\varepsilon_t(k) = \sum_{j=0}^{k-1} \varepsilon_{t-j} \sim GWN\left(0, k\sigma^2\right)$$

Result: In the CER model,

$$
\begin{aligned}
E[r_t(k)] &= \mu(k) = k\mu \\
\mathrm{var}\left(r_t(k)\right) &= \sigma^2(k) = k\sigma^2 \\
\mathrm{SD}\left(r_t(k)\right) &= \sigma_k(k) = \sqrt{k}\sigma
\end{aligned}
$$

and,

$$
\varepsilon_t(k) = \sum_{j=0}^{k-1} \varepsilon_{t-j} = \text{accumulated news shocks}
$$

The Random Walk Model

The CER model for cc returns is equivalent to the random walk (RW) model for log stock prices:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln P_t - \ln P_{t-1}$$

$$= \ln P_t - \ln P_{t-1}$$

which implies,

$$\ln P_t = \ln P_{t-1} + r_t.$$

The Random Walk Model cont.

Recursive substitution starting at $t = 1$ gives:

$$
\begin{aligned}
\ln P_1 &= \ln P_0 + r_1 \\
\ln P_2 &= \ln P_1 + r_2 \\
&= \ln P_0 + r_1 + r_2 \\
&\vdots \\
\ln P_t &= \ln P_{t-1} + r_t \\
&= \ln P_0 + \sum_{s=1}^{t} r_s
\end{aligned}
$$

Interpretation: Price at t equals initial price plus accumulation of cc returns.

The CER Model

In CER model, $r_s = \mu + \varepsilon_s$ so that:

$$\begin{aligned}
\ln P_t &= \ln P_0 + \sum_{s=1}^{t} r_s \\
&= \ln P_0 + \sum_{s=1}^{t} (\mu + \varepsilon_s) \\
&= \ln P_0 + t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s
\end{aligned}$$

RW + drift model
for log prices

Interpretation: Log price at t equals initial price $\ln P_0$, plus expected growth in prices $E[\ln P_t] = t \cdot \mu$, plus accumulation of news $\sum_{s=1}^{t} \varepsilon_s$.

The CER Model cont.

The price level at time t is:

$$P_t = P_0 \exp\left(t \cdot \mu + \sum_{s=1}^{t} \varepsilon_s\right) = P_0 \exp\left(t \cdot \mu\right) \exp\left(\sum_{s=1}^{t} \varepsilon_s\right)$$

$$\exp\left(t \cdot \mu\right) = \text{expected growth in price}$$

$$\exp\left(\sum_{s=1}^{t} \varepsilon_s\right) = \text{unexpected growth in price}$$

CER Model for Simple Returns

- CER Model can also be used for simple returns

$$R_t = \mu + \varepsilon_t$$

$$\varepsilon_t \sim GWN(0, \sigma^2)$$

$$R_t(2) = (1+R_t)(1+R_{t-1})$$
$$= 1 + \underline{R_{t-1}} + \underline{R_t} + R_t \cdot R_{t-1}$$

- Main drawbacks: (1) Normal distribution allows $R_t < -1$; (2) Multi-period returns are not normally distributed

$$R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) - 1$$

$$\nsim N(k\mu, k\sigma^2)$$

$$X \sim N(0,1)$$
$$X \cdot X = X^2 > 0$$

- However, it can be shown that:

$$E[R_t(k)] = (1 + \mu)^k - 1$$

$$\text{var}(R_t(k)) = (1 + \sigma^2 + 2\mu + \mu^2)^k - (1 + \mu)^{2k}$$

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Estimating Parameters of CER model

Parameters of CER Model:

$$\mu_i = E[r_{it}]$$

$$\sigma_i^2 = \mathrm{var}(r_{it})$$

$$\sigma_{ij} = \mathrm{cov}(r_{it}, r_{jt})$$

$$\rho_{ij} = \mathrm{cor}(r_{it}, r_{jt})$$

$r_{it} = \mu_i + \epsilon_{it}$

are not known with certainty.

First Econometric Task:

- Estimate μ_i, σ_i^2, σ_{ij}, ρ_{ij} using observed sample of historical monthly returns

Estimators and Estimates

Definition: An estimator is a rule or algorithm (mathematical formula) for computing an *ex ante* estimate of a parameter based on a random sample.

Example: Sample mean as estimator of $E[r_{it}] = \mu_i$

$$
\begin{aligned}
\{r_{i1}, \ldots, r_{iT}\} &= \text{ covariance stationary time series} \\
&= \text{ collection of random variables} \\
\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it} &= \text{ sample mean} \\
&= \text{ random variable}
\end{aligned}
$$

Estimators and Estimates cont.

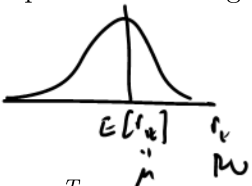
Definition: An estimate of a parameter is simply the *ex post* value (numerical value) of an estimator based on observed data.

Example: Sample mean from an observed sample

$$\{r_{i1} = .02, r_{i2} = .01, r_{i3} = -.01, \ldots, r_{iT} = .03\} = \text{observed sample}$$

$$\begin{aligned} \hat{\mu}_i &= \frac{1}{T}(.02 + .01 - .01 + \cdots + .03) \\ &= \text{ number } = 0.01 \text{ (say)} \end{aligned}$$

Plug-in principle: Estimate model parameters using appropriate sample statistics.



$$\mu_i = E[r_{it}] : \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

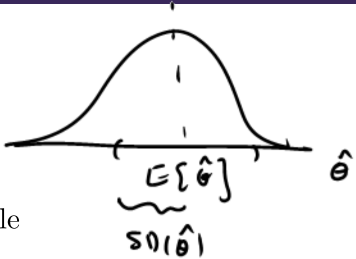
$$\sigma_i^2 = E[(r_{it} - \mu_i)^2] : \hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \hat{\mu}_i)^2$$

$$\sigma_i = \sqrt{\sigma_i^2} : \hat{\sigma}_i = \sqrt{\hat{\sigma}_i^2}$$

$$\sigma_{ij} = E[(r_{it} - \mu_i)(r_{jt} - \mu_j)] : \hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \hat{\mu}_i)(r_{jt} - \hat{\mu}_j)$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} : \hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{\hat{\sigma}_i \cdot \hat{\sigma}_j}$$

Properties of Estimators



$\theta =$ parameter to be estimated

$\hat{\theta} =$ estimator of θ from random sample

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample
- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample
- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Properties of Estimators cont.

Estimation Error:

$$error(\hat{\theta}, \theta) = \hat{\theta} - \theta$$

Bias:

$$\text{bias}(\hat{\theta}, \theta) = E\left[error(\hat{\theta}, \theta)\right] = E\left[\hat{\theta}\right] - \theta$$

$\hat{\theta}$ is unbiased if $E[\hat{\theta}] = \theta \Rightarrow \text{bias}(\hat{\theta}, \theta) = 0$

Remark: An unbiased estimator is "on average" correct, where "on average" means over many hypothetical samples. It most surely will not be exactly correct for the sample at hand!

Properties of Estimators cont.

Precision:

$$mse(\hat{\theta},\theta) = E\left[error(\hat{\theta},\theta)^2\right] = E\left[\left(\hat{\theta}-\theta\right)^2\right]$$

$$= \text{bias}(\hat{\theta},\theta)^2 + \text{var}(\hat{\theta})$$

$$\text{var}(\hat{\theta}) = E[(\hat{\theta} - E[\hat{\theta}])^2$$

Remark: If $\text{bias}(\hat{\theta},\theta) \approx 0$ then precision is typically measured by the *standard error* of $\hat{\theta}$ defined by:

$$\text{SE}(\hat{\theta}) = \text{ standard error of } \hat{\theta}$$

$$= \sqrt{\text{var}(\hat{\theta})} = \sqrt{E[(\hat{\theta} - E[\hat{\theta}])^2]}$$

$$= \sigma_{\hat{\theta}}$$

Bias of CER Model Estimates

- $\hat{\mu}_i, \hat{\sigma}_i^2$ and $\hat{\sigma}_{ij}$ are unbiased estimators:

$$E\left[\hat{\mu}_i\right] = \mu_i \Rightarrow \text{bias}(\hat{\mu}_i, \mu_i) = 0$$

$$E\left[\hat{\sigma}_i^2\right] = \sigma_i^2 \Rightarrow \text{bias}(\hat{\sigma}_i^2, \sigma_i^2) = 0$$

$$E\left[\hat{\sigma}_{ij}\right] = \sigma_{ij} \Rightarrow \text{bias}(\hat{\sigma}_{ij}, \sigma_{ij}) = 0$$

- $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are biased estimators

$$E[\hat{\sigma}_i] \neq \sigma_i \Rightarrow \text{bias}(\hat{\sigma}_i, \sigma_i) \neq 0$$

$$E[\hat{\rho}_{ij}] \neq \rho_{ij} \Rightarrow \text{bias}(\hat{\rho}_{ij}, \rho_{ij}) \neq 0$$

but bias is very small except for very small samples and disappears as sample size T gets large.

Remarks

- "On average" being correct doesn't mean the estimate is any good for your sample!
- The value of $\mathrm{SE}(\hat{\theta})$ will tell you how far from θ the estimate $\hat{\theta}$ typically will be.
- Good estimators $\hat{\theta}$ have small bias and small $\mathrm{SE}(\hat{\theta})$.

Proof

Proof: $E\left[\hat{\mu}_i\right] = \mu_i$

Recall,

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

$$r_{it} = \mu_i + \epsilon_{it},\ \ \epsilon_{it} \sim \text{iid } N(0, \sigma^2)$$

Now,

$$E[r_{it}] = \mu_i + E[\epsilon_{it}] = \mu_i$$

since $E[\epsilon_{it}] = 0$.

Proof cont.

Therefore,

$$
\begin{aligned}
E[\hat{\mu}_i] &= \frac{1}{T}\sum_{t=1}^{T} E[r_{it}] \\
&= \frac{1}{T}\sum_{t=1}^{T} \mu_i \\
&= \frac{1}{T} T\mu_i = \mu_i
\end{aligned}
$$

Standard Error formulas for $\hat{\mu}_i$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$

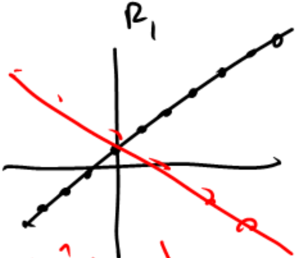
$$\mathrm{SE}(\hat{\mu}_i) = \frac{\sigma_i}{\sqrt{T}}$$

$$\mathrm{SE}(\hat{\sigma}_i^2) \approx \frac{\sigma_i^2}{\sqrt{T/2}} = \frac{\sqrt{2}\sigma_i^2}{\sqrt{T}}$$

$$\mathrm{SE}(\hat{\sigma}_i) \approx \frac{\sigma_i}{\sqrt{2T}}$$

$\mathrm{SE}(\hat{\sigma}_{ij})$: no easy formula!

$$\mathrm{SE}(\hat{\rho}_{ij}) \approx \frac{(1-\rho_{ij}^2)}{\sqrt{T}}$$



$\hat{\rho} = 1$

$\mathrm{SE}(\hat{\rho}) = \frac{1-1}{\sqrt{T}} = 0$

$\rho_{ij} = 0$, $\mathrm{SE}(\hat{\rho}_{ij}) = \frac{1}{\sqrt{T}}$

Note: "≈" denotes "approximately equal to", where approximation error $\longrightarrow 0$ as $T \longrightarrow \infty$ for normally distributed data.

Remarks

- Large SE$\Longrightarrow$ imprecise estimate; Small SE$\Longrightarrow$ precise estimate
- Precision increases with sample size: SE$\longrightarrow 0$ as $T \longrightarrow \infty$
- $\hat{\sigma}_i$ is generally a more precise estimate than $\hat{\mu}_i$ or $\hat{\rho}_{ij}$
- SE formulas for $\hat{\sigma}_i$ and $\hat{\rho}_{ij}$ are approximations based on the Central Limit Theorem. Monte Carlo simulation and bootstrapping can be used to get better approximations.
- SE formulas depend on unknown values of parameters $\Rightarrow$ formulas are not practically useful

Standard Error formulas cont.

- Practically useful formulas replace unknown values with estimated values:

$$\widehat{\mathrm{SE}}(\hat{\mu}_i) = \frac{\hat{\sigma}_i}{\sqrt{T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathrm{SE}}(\hat{\sigma}_i^2) \approx \frac{\hat{\sigma}_i^2}{\sqrt{T/2}},\ \hat{\sigma}_i^2 \text{ replaces } \sigma_i^2$$

$$\widehat{\mathrm{SE}}(\hat{\sigma}_i) \approx \frac{\hat{\sigma}_i}{\sqrt{2T}},\ \hat{\sigma}_i \text{ replaces } \sigma_i$$

$$\widehat{\mathrm{SE}}(\hat{\rho}_{ij}) \approx \frac{(1-\hat{\rho}_{ij}^2)}{\sqrt{T}},\ \hat{\rho}_{ij} \text{ replaces } \rho_{ij}$$

Deriving $\mathrm{SE}(\hat{\mu}_i)$

$\mathrm{SE}(\hat{\mu}_i) = \sigma_i/\sqrt{T}$, $\mathrm{var}(\hat{\mu}_i) = \frac{\sigma_i^2}{T}$

$$\mathrm{var}(\hat{\mu}_i) = \mathrm{var}\left(\frac{1}{T}\sum_{t=1}^{T} r_{it}\right)$$

$$= \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{var}(r_{it}) \text{ (since } r_{it} \text{ are independent)}$$

$$= \frac{1}{T^2}\sum_{t=1}^{T}\sigma_i^2 = \frac{\sigma_i^2}{T} \text{ (since } \mathrm{var}(r_{it}) = \sigma^2\text{)}$$

$$\mathrm{SE}(\hat{\mu}_i) = \sqrt{\mathrm{var}(\hat{\mu}_i)} = \frac{\sigma_i}{\sqrt{T}}$$

Consistency

Definition: An estimator $\hat{\theta}$ is consistent for θ (converges in probability to θ) if for any $\varepsilon > 0$.

$$\lim_{T\to\infty} \Pr(|\hat{\theta} - \theta| > \varepsilon) = 0$$

Intuitively, as we get enough data then $\hat{\theta}$ will eventually equal θ.

Remark: Consistency is an asymptotic property - it holds when we have an infinitely large sample (i.e, in *asymptopia*). In the real world we only have a finite amount of data!

Consistency cont.

Result: An estimator $\hat{\theta}$ is consistent for θ if:

- $\text{bias}(\hat{\theta}, \theta) = 0$ as $T \to \infty$
- $\text{SE}\left(\hat{\theta}\right) = 0$ as $T \to \infty$

Result: In the CER model, the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are consistent.

Distribution of CER Model Estimators

$\theta =$ parameter to be estimated

$\hat{\theta} =$ estimator of θ from random sample

KEY POINTS:

- $\hat{\theta}$ is a random variable – its value depends on realized values of random sample
- $f(\hat{\theta}) =$ pdf of $\hat{\theta}$ - depends on pdf of random variables in random sample
- Properties of $\hat{\theta}$ can be derived analytically (using probability theory) or by using Monte Carlo simulation

Example

Example: Distribution of $\hat{\mu}$ in CER Model

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it},\ r_{it} = \mu_i + \epsilon_{it},\ \epsilon_{it} \sim \text{iid } N(0, \sigma_i^2)$$

Result:

$\hat{\mu}_i$ is $\frac{1}{T}$ times the sum of T normally distributed random variables
$\Rightarrow \hat{\mu}_i$ is also normally distributed with:

$$E[\hat{\mu}_i] = \mu_i,\ \text{var}(\hat{\mu}_i) = \frac{\sigma_i^2}{T}$$

That is,

$$\hat{\mu}_i \sim N\left(\mu_i, \frac{\sigma_i^2}{T}\right)$$

$$f(\hat{u}_i) = (2\pi\sigma_i^2/T)^{-1/2} \exp\left\{-\frac{1}{2\sigma_i^2/T}(\hat{\mu}_i - \mu_i)^2\right\}$$

Example cont.

Distribution of $\hat{\sigma}_i,\ \hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$

Result: The exact distributions (for finite sample size T) of $\hat{\sigma}_i,\ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ are not normal.

However, as the sample size T gets large the exact distributions of $\hat{\sigma}_i,\ \hat{\sigma}_{ij},$ and $\hat{\rho}_{ij}$ get closer and closer to the normal distribution. This is the due to the famous *Central Limit Theorem*.

Central Limit Theorem (CLT)

Let $X_1, \ldots, X_T$ be a iid random variables with $E[X_t] = \mu$ and $\text{var}(X_t) = \sigma^2$. Then,

$$\frac{\bar{X} - \mu}{\text{SE}(\bar{X})} = \frac{\bar{X} - \mu}{\sigma / \sqrt{T}} = \sqrt{T}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0, 1) \text{ as } T \to \infty$$

Equivalently,

$$\bar{X} \sim N\left(\mu, \text{SE}(\bar{X})^2\right) \sim N\left(\mu, \frac{\sigma^2}{T}\right)$$

for large enough T

We say that $\bar{X}$ is asymptotically normally distributed with mean μ and variance $\text{SE}(\bar{X})^2$.

Central Limit Theorem (CLT) cont.

Definition: An estimator $\hat{\theta}$ is asymptotically normally distributed if:

$$\hat{\theta} \sim N(\theta, \mathrm{SE}(\hat{\theta})^2)$$

for large enough T

Result: An implication of the CLT is that the estimators $\hat{\mu}_i$, $\hat{\sigma}_i^2$, $\hat{\sigma}_i$, $\hat{\sigma}_{ij}$, and $\hat{\rho}_{ij}$ are asymptotically normally distributed under the CER model.

$$\hat{\theta} = \text{estimate of } \theta$$
$$= \text{best guess for unknown value of } \theta$$

Idea: A confidence interval for θ is an interval estimate of θ that covers θ with a stated probability.

Intuition: think of a confidence interval like a "horse shoe". For a given sample, there is stated probability that the confidence interval (horse shoe thrown at θ) will cover θ.

Confidence Intervals cont.

Result: Let $\hat{\theta}$ be an asymptotically normal estimator for θ. Then,

- An approximate 95% confidence interval for θ is an interval estimate of the form:

$$\left[\hat{\theta} - 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right]$$

$$\hat{\theta} \pm 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 0.95. That is,

$$\Pr\left\{\hat{\theta} - 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right) \leq \theta \leq \hat{\theta} + 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right\} \approx 0.95$$

Confidence Intervals cont.

- An approximate 99% confidence interval for θ is an interval estimate of the form:

$$\left[\hat{\theta} - 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right),\ \hat{\theta} + 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)\right]$$

$$\hat{\theta} \pm 3 \cdot \widehat{\mathrm{SE}}\left(\hat{\theta}\right)$$

that covers θ with probability approximately equal to 0.99.

Remarks

- 99% confidence intervals are wider than 95% confidence intervals
- For a given confidence level the width of a confidence interval depends on the size of $\widehat{\mathrm{SE}}(\hat{\theta})$

In the CER model, 95% Confidence Intervals for μ_i, σ_i, and ρ_{ij} are:

$$\hat{\mu}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{T}}$$

$$\hat{\sigma}_i \pm 2 \cdot \frac{\hat{\sigma}_i}{\sqrt{2T}}$$

$$\hat{\rho}_{ij} \pm 2 \cdot \frac{(1 - \hat{\rho}_{ij}^2)}{\sqrt{T}}$$

Using Monte Carlo Simulation to Evaluate Bias, Standard Error and Confidence Interval Coverage

- Create many simulated samples from CER model
- Compute parameter estimates for each simulated sample
- Compute mean and sd of estimates over simulated samples
- Compute 95% confidence interval for each sample
- Count number of intervals that cover true parameter

Value-at-Risk in the CER Model

In the CER model:

$$r_{it} \sim iid\ N(\mu_i, \sigma_i^2) \Rightarrow r_{it} = \mu_i + \sigma_i \times z_{it},\ z_{it} \sim iid\ N(0,1)$$

The $\alpha \cdot 100\%$ quantile q_α^r may be expressed as:

$$q_\alpha^r = \mu_i + \sigma_i \times q_\alpha^Z$$

$$q_\alpha^Z = \text{standard Normal quantile}$$

Then,

$$VaR_\alpha = (\exp(q_\alpha^r) - 1) \cdot W_0$$

Example

Example: $r_t \sim N(0.02, (0.10)^2)$ and $W_0 = \$10,000$. Here, $\mu_r = 0.02$ and $\sigma_r = 0.10$ are known values. Then,

$$q^Z_{.05} = -1.645$$

$$q_{.05} = 0.02 + (0.10)(-1.645) = -0.1445$$

$$\text{VaR}_{.05} = (\exp(-0.1145) - 1) \cdot \$10,000 = -\$1,345$$

Estimating Quantiles from CER Model

$$\hat{q}_\alpha^r = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z$$

$$q_\alpha^Z = \text{standard Normal quantile}$$

Estimating Value-at-Risk from CER Model:

$$\widehat{\text{VaR}}_\alpha = (\exp(\hat{q}_\alpha^r) - 1) \cdot W_0$$

$$\hat{q}_\alpha^r = \hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z$$

$$W_0 = \text{initial investment in } \$$$

Q: What is $E\left[\widehat{\text{VaR}}_\alpha\right]$ and $\text{SE}\left(\widehat{\text{VaR}}_\alpha\right)$?

Computing Standard Errors for VaR

- We can compute $\mathrm{SE}(\hat{q}_\alpha^r)$ using:

$$\begin{aligned}\mathrm{var}(\hat{q}_\alpha^r) &= \mathrm{var}\left(\hat{\mu}_i + \hat{\sigma}_i q_\alpha^Z\right) \\ &= \mathrm{var}(\hat{\mu}_i) + \left(q_\alpha^Z\right)^2 \mathrm{var}(\hat{\sigma}_i) + 2q_\alpha^Z \mathrm{cov}(\hat{\mu}_i, \hat{\sigma}_i) \\ &= \mathrm{var}(\hat{\mu}_i) + \left(q_\alpha^Z\right)^2 \mathrm{var}(\hat{\sigma}_i), \text{ since } \mathrm{cov}(\hat{\mu}_i, \hat{\sigma}_i) = 0\end{aligned}$$

Then,

$$\mathrm{SE}(\hat{q}_\alpha^r) = \sqrt{\mathrm{var}(\hat{\mu}_i) + (q_\alpha^Z)^2 \mathrm{var}(\hat{\sigma}_i)}$$

- However, computing $\mathrm{SE}(\widehat{\mathrm{VaR}}_\alpha)$ is not straightforward since:

$$\mathrm{var}\left(\widehat{\mathrm{VaR}}_\alpha\right) = \mathrm{var}\left((\exp(\hat{q}_\alpha^r) - 1) \cdot W_0\right)$$

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