

Economics 424/Applied Mathematics 540

Final Exam Solutions

I. Constant Expected Return Model and Hypothesis Testing (35 points, 5 points each)

Figure 1 shows monthly continuously compounded returns on the Vanguard S&P 500 index (VFINX) and the Vanguard Emerging Markets index (VEIEX) over the 5-year period September 2005, through September 2010. For this period there are $T=60$ monthly observations.

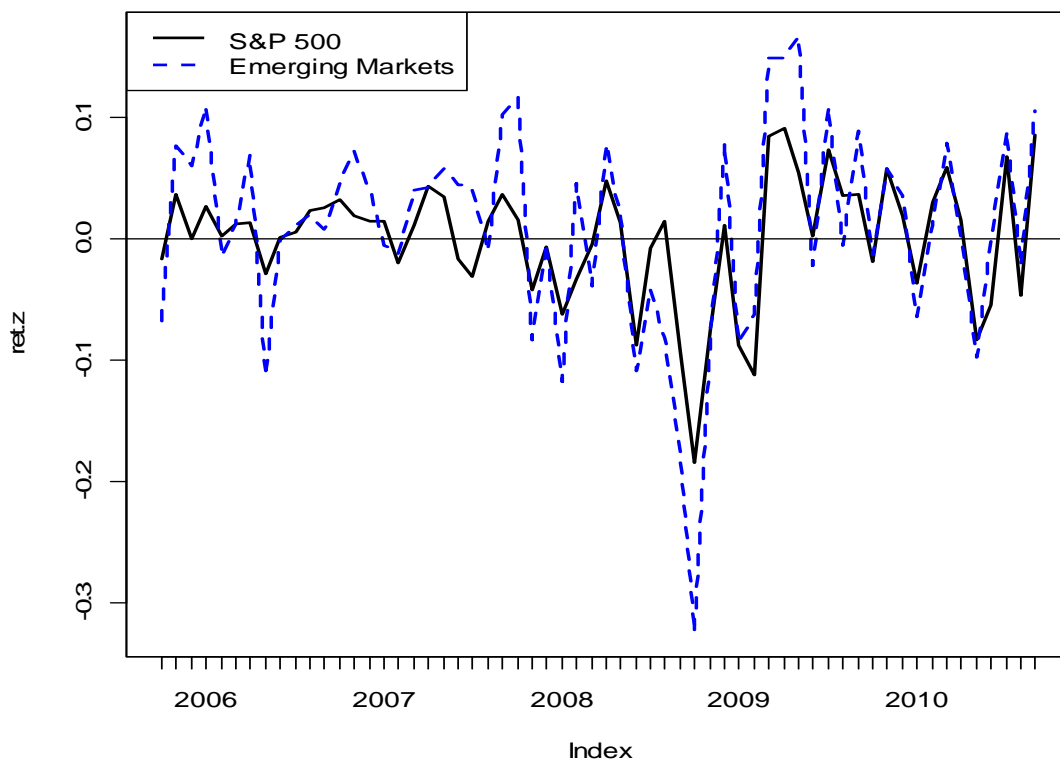


Figure 1 Monthly Continuously Compounded Returns on Two Vanguard Funds

Consider the constant expected return model

$$r_{it} = \mu_i + \varepsilon_{it}, \varepsilon_{it} \sim iid N(0, \sigma_i^2)$$

$$cov(r_{it}, r_{jt}) = \sigma_{ij}, cor(r_{it}, r_{jt}) = \rho_{ij}$$

for the monthly continuously compounded returns on the Vanguard S&P500 index and the Vanguard emerging markets index. The following R output gives the estimates of $\mu_i, \sigma_i, \sigma_{ij}$ and ρ_{ij} for the S&P 500 index (VFINX) and the emerging markets index (VEIEX) from the $T=60$ months of data:

```
> muhat.vals
      vfinx      veiex
0.0004694 0.0094886
```

```
> sigmahat.vals
      vfinx      veiex
0.05195 0.08549
```

```
> covhat.vals
vfinx,veiex
0.003809
```

```
> rhohat.vals
vfinx,veiex
0.8576
```

a. For the S&P 500 index (vfinx), compute 95% confidence intervals for μ, σ . Briefly comment on the precision of the estimates. In particular, note if both positive and negative values are in the respective confidence intervals.

The rule-of-thumb formula for a 95% confidence interval for a parameter θ is

$$\hat{\theta} \pm 2 \times \widehat{SE}(\hat{\theta})$$

For VFINX, the estimated standard errors for μ and σ are

$$\widehat{SE}(\hat{\mu}) = \hat{\sigma} / \sqrt{T} = 0.05195 / \sqrt{60} = 0.0067$$

$$\widehat{SE}(\hat{\sigma}) = \hat{\sigma} / \sqrt{2T} = 0.05195 / \sqrt{120} = 0.0047$$

The 95% confidence intervals for μ and σ are

$$\mu_{vfinx} : 0.0004698 \pm 2 \times 0.0067 = [-0.01294, 0.01388]$$

$$\sigma_{vfinx} : 0.05195 \pm 2 \times 0.0047 = [0.04247, 0.06144]$$

The mean is not estimated very precisely. Both negative and positive values are in the confidence interval and the width of the interval is fairly big. The sd is estimated much more precisely.

b. For the S&P 500 index, test the hypotheses $H_0 : \mu = 0$ vs. $H_1 : \mu \neq 0$ using a 5% significance level.

We can do this test in two equivalent ways. For both tests, we use the asymptotic normal distribution for $\hat{\mu}$ based on the Central Limit Theorem. First, we can see if $\mu = 0$ lies in the 95% confidence interval we computed in part a. Since $\mu = 0 \in [-0.01294, 0.01388]$ we do not reject the null at the 5% level. Alternatively, we can compute the t-ratio

$$t_{\mu=0} = \frac{\hat{\mu}}{SE(\hat{\mu})} = \frac{0.004694}{0.0067} = 0.0706$$

And see if it's absolute value is bigger than 2. Since it is not, we do not reject the null at the 5% level.

c. Compute a 95% confidence interval for $\rho_{vfinx,veiex}$. Use this interval to test the hypotheses $H_0 : \rho_{vfinx,veiex} = 0.5$ vs. $H_1 : \rho_{vfinx,veiex} \neq 0.5$ using a 5% significance level.

For VFINX, the estimated standard error for $\rho_{vfinx,veiex}$ is

$$\widehat{SE}(\hat{\rho}) = \frac{1 - \hat{\rho}^2}{\sqrt{T}} = \frac{1 - (0.8575)^2}{\sqrt{60}} = 0.03417$$

The 95% confidence intervals for $\rho_{vfinx,veiex}$ is

$$\rho_{vfinx,veiex} : \hat{\rho} \pm 2 \times \widehat{SE}(\hat{\rho}) = 0.8575 \pm 0.03417 = [0.7891, 0.9258]$$

This interval is quite tight around 0.8575 and indicates a precise estimate. To test $H_0 : \rho_{vfinx,veiex} = 0.5$ vs. $H_1 : \rho_{vfinx,veiex} \neq 0.5$ we check to see if 0.5 lies in the 95% confidence interval. Since 0.5 is not in the confidence interval we reject at the 5% significance level.

d. For vfinx and veiex Consider testing the hypotheses

$$H_0 : r_t \sim \text{normal} \text{ vs. } H_1 : r_t \text{ is not normally distributed}$$

using a 5% significance level. The JB statistics are $JB_{vfinx} = 15.11$ and $JB_{veiex} = 28.82$. What do you conclude?

Under the null of normality, the JB statistic has a chi-square distribution with 2 degrees of freedom. The 5% critical value is the 95% quantile of the chi-square distribution with 2 degrees of freedom. This quantile is 6. Hence, we reject the null at the 5% significance level if $JB > 6$. Since $JB > 6$ for both assets we reject the normality assumption at the 5% significance level for both assets.

e. Suppose you currently hold \$2M (million) in the S&P 500 Index. That is, your initial wealth at the beginning of the month is $W_0 = \$2M$. Using the estimates from the CER model compute the 5% value-at-risk (VaR) associated with a one-month investment in the S&P 500 index.

$$\widehat{VaR}_{0.05} = \$2M \times (\exp(\hat{q}_{0.05}^R) - 1) = \$2M \times (\exp(-0.08498) - 1) = -0.1629 = \$162,900$$

$$\hat{q}_{0.05}^R = \hat{\mu} + \hat{\sigma} \times (-1.645) = 0.0004698 + 0.05195 \times (-1.645) = -0.08498$$

f. Describe briefly how you could use the bootstrap to compute an estimated standard error and 95% confidence interval for the 5% VaR found in part f above.

Bootstrapping involves sampling with replacement from the original data to create B different samples. On each sample the 5% VaR is computed, and the estimated bootstrap SE is the sample standard deviation of these B 5% VaR values.

g. The 24-month rolling estimates of m and s are illustrated in and below. The CER model assumes that m and s are constant over time. Is this a reasonable assumption? Why or why not.

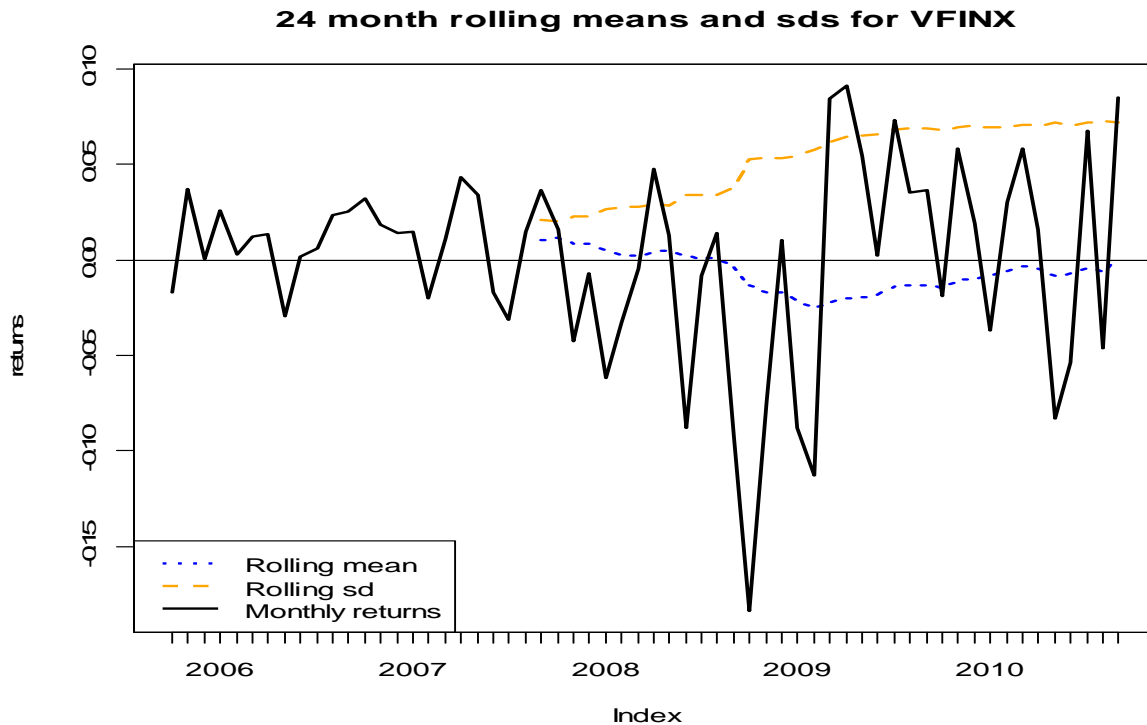
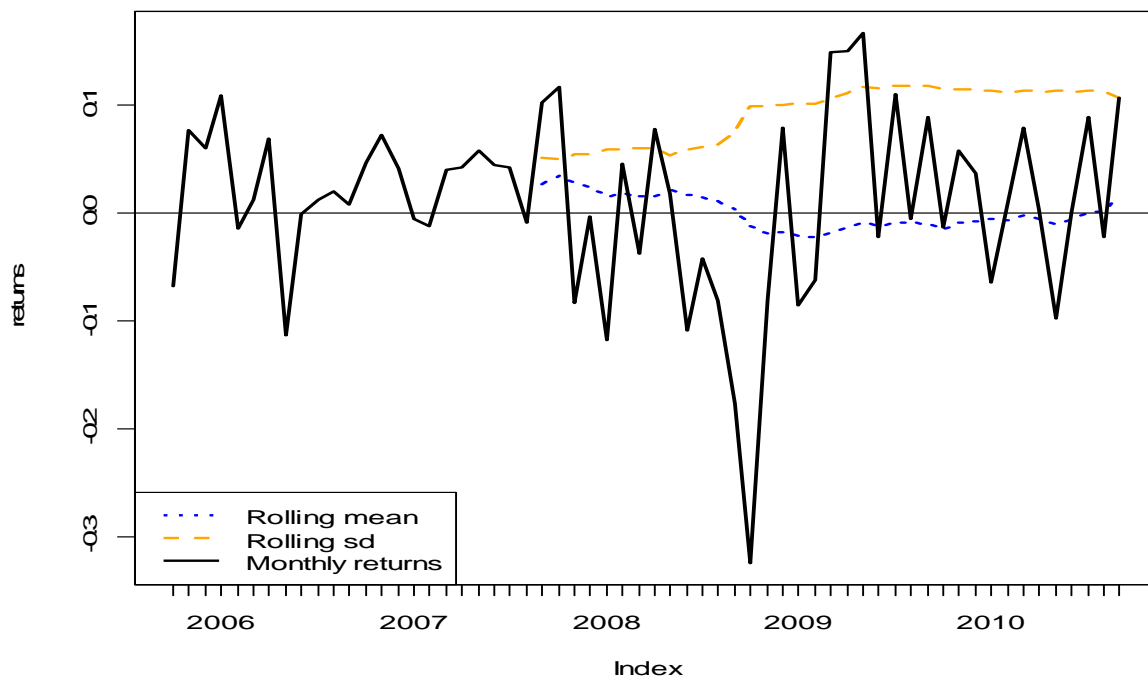


Figure 2 24-month rolling estimates for VFINX

24 month rolling means and sds for VEIEX



The rolling estimates of μ and σ for both assets are very similar. The rolling estimates of μ start out slightly positive and then dip negative during the financial crisis. There is clearly a structural break in the mean from positive values to negative values around the end of 2008. The rolling estimates of σ start to increase at the end of 2008 and then level off at in the middle of 2009. The values at the end of the sample are almost twice as big as the values at the beginning of the sample. The rolling estimates show a non-constant value for σ throughout the sample.

I. Matrix Algebra and Portfolio Math (30 points, 5 points each)

Let R_i denote the continuously compounded return on asset i ($i = 1, \dots, N$) with $E[R_i] = \mu_i$, $\text{var}(R_i) = \sigma_i^2$ and $\text{cov}(R_i, R_j) = \sigma_{ij}$. Define the $(N \times 1)$ vectors $\mathbf{R} = (R_1, \dots, R_N)'$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N)'$, $\mathbf{m} = (m_1, \dots, m_N)'$, $\mathbf{x} = (x_1, \dots, x_N)'$, $\mathbf{y} = (y_1, \dots, y_N)'$, $\mathbf{t} = (t_1, \dots, t_N)'$, $\mathbf{1} = (1, \dots, 1)'$ and the $(N \times N)$ covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors \mathbf{m} , \mathbf{x} , \mathbf{y} and \mathbf{t} contain portfolio weights that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolios defined by the vectors \mathbf{x} and \mathbf{y} give the expression for the portfolio returns, ($R_{p,x}$ and $R_{p,y}$), the portfolio expected returns ($\mu_{p,x}$ and $\mu_{p,y}$), the portfolio variances ($\sigma_{p,x}^2$ and $\sigma_{p,y}^2$), and the covariance between $R_{p,x}$ and $R_{p,y}$ (σ_{xy}).

$$R_{p,x} = \mathbf{R}'\mathbf{x}, \mu_{p,x} = \mathbf{x}'\boldsymbol{\mu}, \sigma_{p,x}^2 = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$$

$$R_{p,y} = \mathbf{R}'\mathbf{y}, \mu_{p,y} = \mathbf{y}'\boldsymbol{\mu}, \sigma_{p,y}^2 = \mathbf{y}'\boldsymbol{\Sigma}\mathbf{y}$$

$$\sigma_{xy} = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{y}$$

b. Write down the optimization problem and give the Lagrangian used to determine the global minimum variance portfolio assuming short sales are allowed. Let \mathbf{m} denote the vector of portfolio weights in the global minimum variance portfolio.

$$\min_{\mathbf{m}} \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

$$L(\mathbf{m}, \lambda) = \mathbf{m}'\boldsymbol{\Sigma}\mathbf{m} + \lambda(\mathbf{m}'\mathbf{1} - 1)$$

c. Write down the optimization problem and give the Lagrangian used to determine an efficient portfolio with target return equal to μ_0 assuming short sales are allowed. Let \mathbf{x} denote the vector of portfolio weights in the efficient portfolio.

$$\min_{\mathbf{x}} \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \text{ s.t. } \mathbf{x}'\mathbf{1} = 1 \text{ and } \mathbf{x}'\boldsymbol{\mu} = \mu_0$$

$$L(\mathbf{x}, \lambda_1, \lambda_2) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} + \lambda_1(\mathbf{x}'\mathbf{1} - 1) + \lambda_2(\mathbf{x}'\boldsymbol{\mu} - \mu_0)$$

d. Briefly describe how you would compute the efficient frontier containing only risky assets (Markowitz bullet) when short sales are allowed.

Because short sales are allowed, the Markowitz bullet can be constructed using a convex combination of any two efficient (frontier) portfolio: $\mathbf{z} = \alpha \cdot \mathbf{m} + (1 - \alpha) \cdot \mathbf{x}$, where \mathbf{m} and \mathbf{x} are any two frontier portfolios. To get a nice picture, choose one efficient portfolio to be the global minimum variance portfolio and choose the other efficient portfolio to be the efficient portfolio with target return equal to the highest expected return of the available assets.

e. Write down the optimization problem used to determine the tangency portfolio, assuming short sales are *not* allowed and the risk free rate is given by r_f . Let \mathbf{t} denote the vector of portfolio weights in the tangency portfolio.

$$\max_{\mathbf{t}} \frac{\mathbf{t}'\boldsymbol{\mu} - r_f}{(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{1/2}} \text{ s.t. } \mathbf{t}'\mathbf{1} = 1 \text{ and } t_i \geq 0, i = 1, \dots, N$$

f. Write down the equations for the expected return (μ_p^e) and standard deviation (σ_p^e) of efficient portfolios consisting of the tangency portfolio and T-bills, where the T-bill rate (risk-free rate) is given by r_f and \mathbf{t} denotes the vector of portfolio weights in the tangency portfolio.

$$\mu_p^e = r_f + x_{\text{tan}} (\mu_{\text{tan}} - r_f), \quad \mu_{\text{tan}} = \mathbf{t}'\boldsymbol{\mu}$$

$$\sigma_p^e = x_{\text{tan}} \sigma_{\text{tan}}, \quad \sigma_{\text{tan}}^2 = \mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}$$

II. Efficient Portfolios (28 points)

The graph below shows the efficient frontier computed from three Vanguard mutual funds: Pacific Stock Index (vpacx), US Long Term Bond Index (vbltx), and Emerging Markets Fund (veiox).

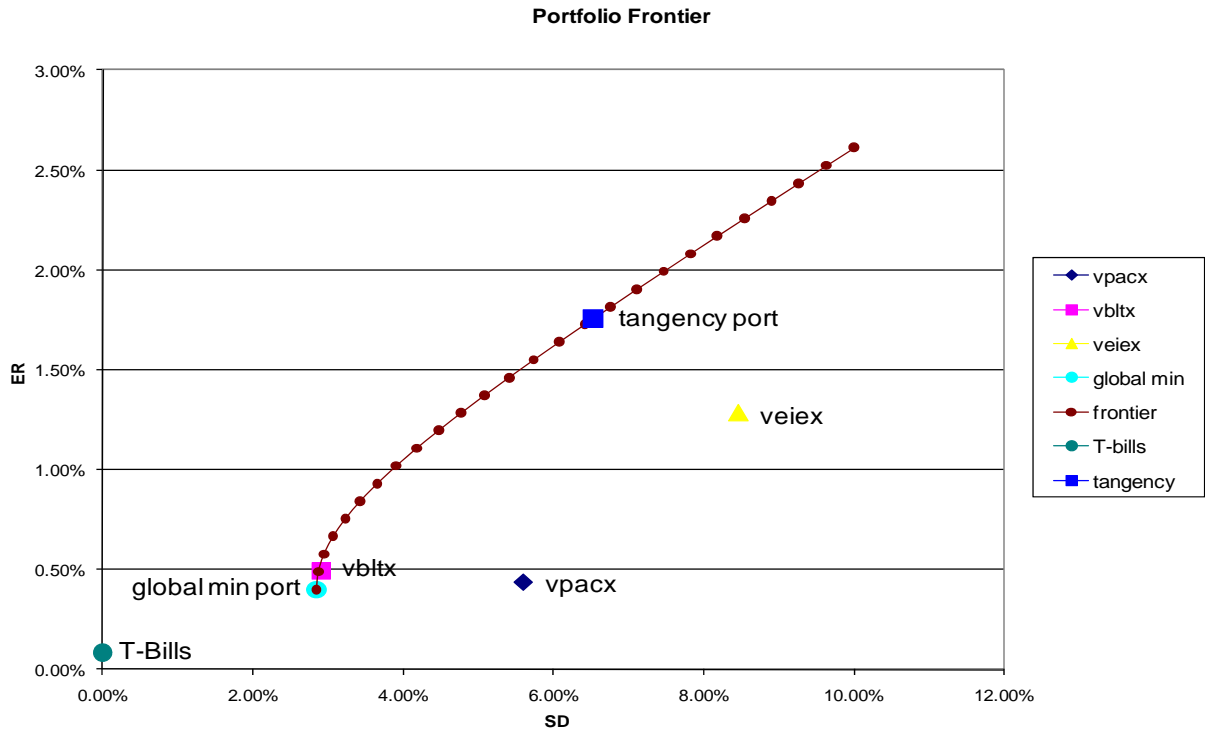


Figure 3 Markowitz Bullet

Expected return and standard deviation estimates for specific assets are summarized in the table below. These estimates are based on monthly *continuously compounded* return data over the five year period September 2004 – September 2009.

Table 1 Portfolio Statistics

| Asset | Mean E[R] | Standard deviation (SD(R)) | Weight in Global Min Portfolio | Weight in Efficient Portfolio Mean = 1.28% | Weight in Tangency portfolio |
|----------------------|-----------|----------------------------|--------------------------------|--|------------------------------|
| VPACX | 0.43% | 5.59% | 23% | -120% | -197% |
| VBLTX | 0.49% | 2.90% | 87% | 129% | 151% |
| VEIEX | 1.28% | 8.45% | -10% | 91% | 145% |
| T-Bills | 0.08% | 0% | | | |
| Global Min Portfolio | 0.40% | 2.84% | | | |

| | | | | | |
|-------------------------------------|-------|-------|--|--|--|
| Efficient Portfolio with Mean=1.28% | 1.28% | 4.77% | | | |
| Tangency Portfolio | 1.76% | 6.53% | | | |

Using the above information, please answer the following questions.

a. Compute annualized means and standard deviations from the monthly statistics in Table 1 for the three portfolios vpacx, vbltx, and veiox (Remember, the returns are continuously compounded)

The annualized cc mean return is $\mu_A = 12 \cdot \mu_m$. For the three portfolios we have

$$\mu_{A,vpacx} = 12 \cdot (.0043) = .0516$$

$$\mu_{A,vbltx} = 12 \cdot (.0049) = .0588$$

$$\mu_{A,veiox} = 12 \cdot (.0128) = .1536$$

The annualized cc standard deviation is $\sigma_A = \sqrt{12} \sigma_m$

$$\sigma_{A,vpacx} = \sqrt{12} \cdot (.0559) = .1936$$

$$\sigma_{A,vbltx} = \sqrt{12} \cdot (.0290) = .1005$$

$$\sigma_{A,veiox} = \sqrt{12} \cdot (.0845) = .2927$$

The annualized T-Bill rate is $r_{A,f} = 12 \cdot 0.0008 = 0.01$

b. Using the annualized information from part a., compute the annualized Sharpe ratios/slopes for each of the three portfolios. Which portfolio is ranked best using the Sharpe ratio?

The annualized Sharpe ratio is

$$SR_A = \frac{\mu_A - r_{f,A}}{\sigma_A}, \quad r_{f,A} = 12 \cdot r_{f,m}$$

Using the results from part a., we get

$$SR_{A,vpacx} = \frac{0.0516 - 0.01}{0.1936} = 0.2149$$

$$SR_{A,vbltx} = \frac{0.0588 - 0.01}{0.1005} = 0.4856$$

$$SR_{A,veiox} = \frac{0.1536 - 0.01}{0.2927} = 0.4906$$

The asset with the highest annual Sharpe ratio is veieix.

c. Find the efficient portfolio of risky assets only (e.g. a portfolio on the Markowitz bullet) that has an expected monthly return equal to 1%. In this portfolio, how much is invested in vpacx, vbltx, and veieix?

Here, we make use of the fact that the global minimum variance portfolio and the tangency portfolio are on the Markowitz bullet. Therefore, the expected return for any portfolio on the Markowitz bullet can be expressed as

$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,t}$$

Setting $\mu_{p,z} = 0.01$ and using $\mu_{p,m} = 0.0040$, $\mu_{p,t} = 0.0176$ we can solve for α :

$$\alpha = \frac{\mu_{p,z} - \mu_{p,t}}{\mu_{p,m} - \mu_{p,t}} = \frac{0.01 - 0.0176}{0.0040 - 0.0176} = 0.56$$

$$1 - \alpha = 0.44$$

The weights on vfinx, veurx and vbltx in this portfolio are

$$\begin{aligned} \mathbf{z} &= \alpha\mathbf{m} + (1-\alpha)\mathbf{t} \\ &= (0.56) \begin{pmatrix} .23 \\ .87 \\ -.10 \end{pmatrix} + (0.44) \begin{pmatrix} -1.97 \\ 1.51 \\ 1.45 \end{pmatrix} = \begin{pmatrix} -.74 \\ 1.16 \\ .59 \end{pmatrix} \end{aligned}$$

We can get the same answer if we use the global minimum variance portfolio and the efficient portfolio with expected return equal to 1.28%. The calculations are essentially the same:

$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,e}$$

Setting $\mu_{p,z} = 0.01$ and using $\mu_{p,m} = 0.0040$, $\mu_{p,e} = 0.0128$ we can solve for α :

$$\alpha = \frac{\mu_{p,z} - \mu_{p,e}}{\mu_{p,m} - \mu_{p,e}} = \frac{0.01 - 0.0128}{0.0040 - 0.0128} = 0.32$$

$$1 - \alpha = 0.68$$

The weights on vfinx, veurx and vbltx in this portfolio are

$$z = \alpha m + (1 - \alpha)x$$

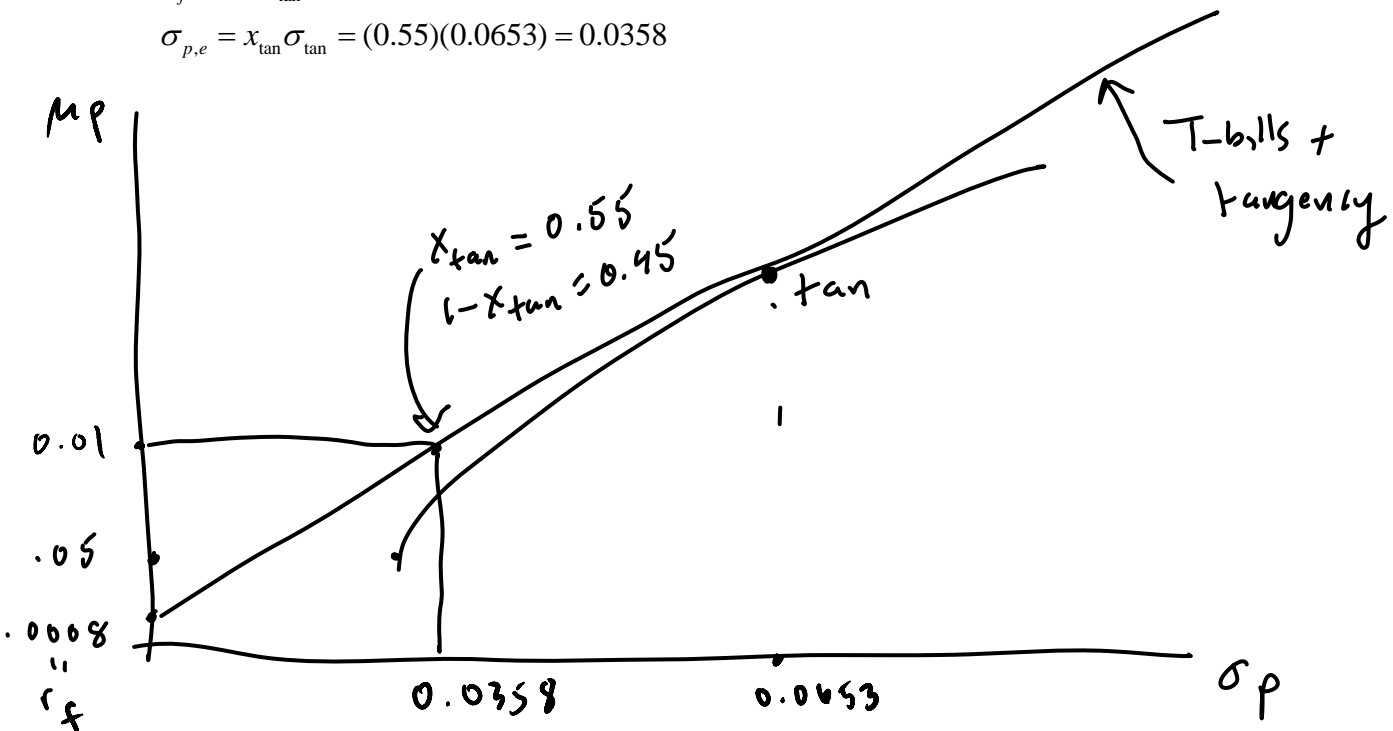
$$= (0.32) \begin{pmatrix} .23 \\ .87 \\ -.10 \end{pmatrix} + (0.68) \begin{pmatrix} -1.20 \\ 1.29 \\ .91 \end{pmatrix} = \begin{pmatrix} -.74 \\ 1.16 \\ 0.59 \end{pmatrix}$$

d. How much should be invested in T-bills and the tangency portfolio to create an efficient portfolio with expected return equal to 1%? What is the standard deviation of this efficient portfolio? Transfer the graph of the Markowitz bullet to your blue book and indicate the location of this efficient portfolio on the graph.

$$x_{\text{tan}} = \frac{.01 - r_f}{\mu_{\text{tan}} - r_f} = \frac{.01 - 0.0008}{.0176 - 0.0008} = 0.55$$

$$x_f = 1 - x_{\text{tan}} = 1 - 0.55 = 0.45$$

$$\sigma_{p,e} = x_{\text{tan}} \sigma_{\text{tan}} = (0.55)(0.0653) = 0.0358$$



e. In the efficient portfolio you found in part d, what are the shares of wealth invested in T-Bills, v_{pacx}, v_{bltx}, and v_{eiex}?

| v _{pacx} | v _{bltx} | v _{eiex} | T-Bills |
|--------------------|-------------------|-------------------|---------|
| 0.55*(-1.97)=-1.08 | 0.55*(1.51)=0.83 | 0.55*(1.45)=0.80 | 0.45 |

f. Assuming an initial \$100,000 investment for one month, compute the 5% value-at-risk on the global minimum variance portfolio.

First we find the 5% quantiles for the global minimum variance portfolio

$$\text{port: } q_{.05} = 0.0004 + (0.0284)(-1.645) = -0.0427$$

Then we compute the 5% VaR

$$\text{port: } \text{VaR}_{.05} = \$100,000 \times (e^{-0.0427} - 1) = -\$4,182.95$$

g. The efficient frontier of risky assets shown in Figure 1 allows for short sales (see the weights in the portfolios listed in Table 1). Transfer this graph to your blue book. On this graph, indicate roughly the location of the efficient frontier of risky assets that *does not* allow short sales.

