University of Washington
Department of Economics

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Economics 424

**Final Exam**

This is a closed book and closed note exam. However, you are allowed one page of handwritten notes. Answer all questions and write all answers in a blue book. Total points = 100.

I. Matrix Algebra and Portfolio Math (25 points)

Let $R_i$ denote the continuously compounded return on asset $i$ ($i$ = 1, …, $N$) with $E[R_i] = \mu_i$, var($R_i$) = $\sigma_i^2$ and cov($R_i$, $R_j$) = $\sigma_{ij}$. Define the ($N \times 1$) vectors $\boldsymbol{R} = \left(R_1, \ldots, R_N\right)'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$, $\boldsymbol{m} = \left(m_1, \ldots, m_N\right)'$, $\boldsymbol{x} = \left(x_1, \ldots, x_N\right)'$, $\boldsymbol{y} = \left(y_1, \ldots, y_N\right)'$, $\boldsymbol{t} = \left(t_1, \ldots, t_N\right)'$, $\mathbf{1} = \left(1, \ldots, 1\right)'$ and the ($N \times N$) covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1N} & \sigma_{2N} & \cdots & \sigma_N^2 \end{pmatrix}.$$

The vectors $\boldsymbol{m}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ contain portfolio weights that sum to one. Using simple matrix algebra, answer the following questions.

a. For the portfolio defined by the vector $\boldsymbol{x}$, give the expression for the portfolio return ($R_{p,x}$), the portfolio expected return ($\mu_{p,x}$), and the portfolio variance ($\sigma_{p,x}^2$).

$$R_{p,x} = \boldsymbol{R'x},\ \mu_{p,x} = \boldsymbol{x'\mu},\ \sigma_{p,x}^2 = \boldsymbol{x'\Sigma x}$$

b. Write down the optimization problem used to determine the global minimum variance portfolio assuming short sales are allowed. Let $\boldsymbol{m}$ denote the vector of portfolio weights in the global minimum variance portfolio.

$$\min_{m}\ \boldsymbol{m'\Sigma m} \text{ s.t. } \boldsymbol{m'1} = 1$$

c. Write down the optimization problem used to determine an efficient portfolio with target return equal to $\mu_0$ assuming short sales are allowed. Let $\boldsymbol{x}$ denote the vector of portfolio weights in the efficient portfolio.

$$\min_{x}\ \boldsymbol{x'\Sigma x} \text{ s.t. } \boldsymbol{x'1} = 1 \text{ and } \boldsymbol{x'\mu} = \mu_0$$

d. Using the results from parts b. and c., briefly describe how you can compute the efficient frontier containing only risky assets (Markowitz bullet) using just two efficient portfolios. Give the expressions for the expected return and variance for portfolios on the efficient frontier.

*Any portfolio on the Markowitz bullet can be computed as a convex combination of any other two portfolios on the bullet. For example, one portfolio could be the global minimum variance portfolio m and the other could be the portfolio x that has target return* $\mu_0$. *Then, any efficient portfolio has the form*

$$z = \alpha \boldsymbol{m} + (1-\alpha)\boldsymbol{x}$$
$$\mu_{p,z} = \alpha \mu_{p,m} + (1-\alpha)\mu_{p,x}$$
$$\sigma^2_{p,z} = \alpha^2 \sigma^2_{p,m} + (1-\alpha)^2 \sigma^2_{p,x} + 2\alpha(1-\alpha)\sigma_{mx}$$
$$\sigma_{mx} = \boldsymbol{m}'\boldsymbol{\Sigma x}$$

e. Write down the optimization problem used to determine the tangency portfolio, assuming short sales are allowed and the risk free rate is give by $r_f$. Let $\boldsymbol{t}$ denote the vector of portfolio weights in the tangency portfolio.

$$\max_{t} \frac{\boldsymbol{t}'\boldsymbol{\mu} - r_f}{(\boldsymbol{t}'\boldsymbol{\Sigma t})^{1/2}} \text{ s.t. } \boldsymbol{t}'\boldsymbol{1} = 1$$

II. Efficient Portfolios (35 points)

The graph below shows the efficient frontier computed from three Vanguard mutual funds: Pacific Stock Index (vpacx), US Long Term Bond Index (vbltx), and Emerging Markets Fund (veiex).

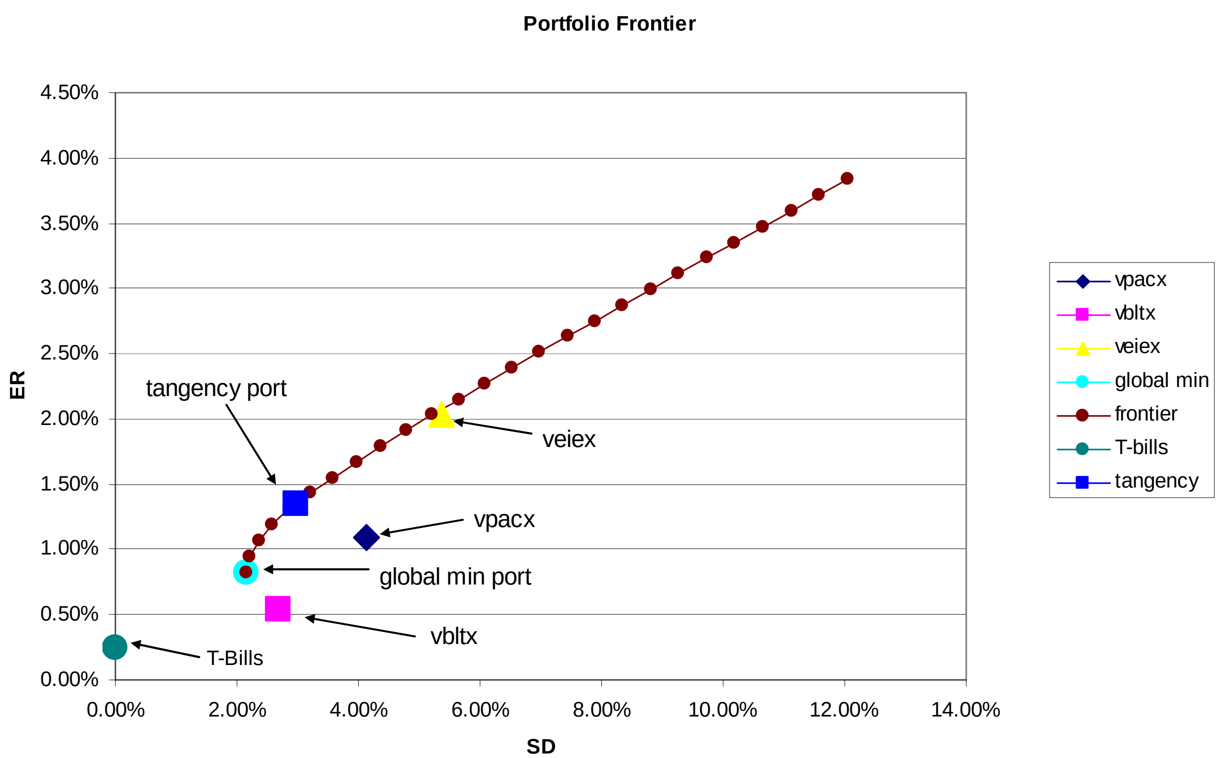



Figure 1 Markowitz Bullet

Expected return and standard deviation estimates for specific assets are summarized in the table below. These estimates are based on monthly continuously compounded return data over the five year period September 2001 – September 2006.

Table 1 Portfolio Statistics

| **Asset** | **Mean (E[R])** | **Standard deviation (SD(R))** | **Weight in Global Min Portfolio** | **Weight in Efficient Portfolio Mean = 2.03%** | **Weight in Tangency portfolio** |
|---|---|---|---|---|---|
| VPACX | 1.09% | 4.14% | 19% | -35% | -4% |
| VBLTX | 0.54% | 2.68% | 69% | 22% | 49% |
| VEIEX | 2.03% | 5.36% | 12% | 113% | 56% |
| T-Bills | 0.25% | 0% | | | |
| Global Min Portfolio | 0.83% | 2.16% | | | |
| Tangency Portfolio | 1.35% | 2.98% | | | |
| Efficient Portfolio with Mean=2.03% | 2.03% | 5.21% | | | |

Using the above information, please answer the following questions.

a. Compute annualized means from the monthly means for the three portfolios vpacx, vbltx, and veiex. For each asset, assuming you get the same annual return for 10 years, how much will \$1 grow to after 10 years?

The annualized cc return is $r_A = 12 \cdot r_m$. For the three portfolios we have

$$r_{A,vpacx} = 12 \cdot (.0109) = .1313$$
$$r_{A,vbltx} = 12 \cdot (.0054) = .0653$$
$$r_{A,veiex} = 12 \cdot (.0203) = .2436$$

The future value after 10 year with continuous compounding is $FV = e^{10 \cdot r_A}$. For the three portfolios we have

$$FV_{vpacx} = \exp(10 \times .1313) = \$3.72$$
$$FV_{vbltx} = \exp(10 \times .0653) = \$1.92$$
$$FV_{veiex} = \exp(10 \times .2436) = \$11.43$$

b. Find the efficient portfolio of risky assets only (e.g. a portfolio on the Markowitz bullet) that has an expected monthly return equal to 3%. In this portfolio, how much is invested in vpacx, vbltx, and veiex?

*Here, we make use of the fact that the global minimum variance portfolio and the tangency portfolio are on the Markowitz bullet. Therefore, the expected return for any portfolio on the Markowitz bullet can be expressed as*

$$\mu_{p,z} = \alpha \mu_{p,m} + (1-\alpha)\mu_{p,t}$$

*Setting* $\mu_{p,z} = 0.03$ *and using* $\mu_{p,m} = 0.0083,\ \mu_{p,t} = 0.0135$ *we can solve for* $\alpha$:

$$\alpha = \frac{\mu_{p,z} - \mu_{p,t}}{\mu_{p,m} - \mu_{p,t}} = \frac{0.03 - 0.0135}{0.0083 - 0.0135} = -3.18$$
$$1 - \alpha = 4.18$$

*The weights on vfinx, veurx and vbltx in this portfolio are*

$$
\begin{aligned}
z &= \alpha m + (1-\alpha) x \\
&= (-3.18)\begin{pmatrix} .19 \\ .69 \\ .12 \end{pmatrix} + (4.18)\begin{pmatrix} -.04 \\ .49 \\ .56 \end{pmatrix} = \begin{pmatrix} -.77 \\ -.16 \\ 1.94 \end{pmatrix}
\end{aligned}
$$

*We can get the same answer if we use the global minimum variance portfolio and the efficient portfolio with expected return equal to 2.03%. The calculations are essentially the same:*

$$\mu_{p,z} = \alpha\mu_{p,m} + (1-\alpha)\mu_{p,e}$$

*Setting* $\mu_{p,z} = 0.03$ *and using* $\mu_{p,m} = 0.0083,\ \mu_{p,e} = 0.0203$ *we can solve for* $\alpha$:

$$
\begin{aligned}
\alpha &= \frac{\mu_{p,z} - \mu_{p,e}}{\mu_{p,m} - \mu_{p,e}} = \frac{0.03 - 0.0203}{0.0083 - 0.0203} = -0.81 \\
1 - \alpha &= 1.81
\end{aligned}
$$

*The weights on vfinx, veurx and vbltx in this portfolio are*

$$
\begin{aligned}
z &= \alpha m + (1-\alpha) x \\
&= (-0.81)\begin{pmatrix} .19 \\ .69 \\ .12 \end{pmatrix} + (1.81)\begin{pmatrix} -.35 \\ .22 \\ .113 \end{pmatrix} = \begin{pmatrix} -.77 \\ -.16 \\ 1.94 \end{pmatrix}
\end{aligned}
$$

c. Transfer the graph of the Markowitz bullet to your blue book. On this graph indicate the efficient combinations of T-bills and risky assets. Also, indicate a portfolio that would be preferred by a very risk averse investor and a portfolio that would be preferred by a risk tolerant investor.

d. How much should be invested in T-bills and the tangency portfolio to create an efficient portfolio with expected return equal to 3%? What is the standard deviation of this efficient portfolio? Transfer the graph of the Markowitz bullet to your blue book and indicate the location of this efficient portfolio on the graph.

$$x_{\tan} = \frac{.03 - r_f}{\mu_{\tan} - r_f} = \frac{.03 - 0.0042}{.0188 - 0.0042} = 2.51$$

$$x_f = 1 - x_{\tan} = 1 - 2.51 = -1.51$$

$$\sigma_{p,e} = x_{\tan}\sigma_{\tan} = (2.51)(0.0298) = 0.0746$$

e. In the efficient portfolio you found in part d, what are the shares of wealth invested in vpacx, vbltx, and veiex?

| vpacx | vbltx | veiex |
|---|---|---|
| 2.51*(-0.04)=-0.11 | 2.51*0.49=1.22 | 2.51*(0.56)=1.39 |

f. Suppose you are not allowed to sell any of the assets short? Can you find an efficient portfolio of the three assets that has an expected return of 3%?

*No. One way to see this is to note that the tangency portfolio, which has an expected return of 1.35%, already has a negative weight in VPACX. Since an efficient portfolio is a convex combination of T-bills and the tangency portfolio and the weight on the tangency portfolio needs to be greater than 1 to achieve an expected return of 3% we cannot find a combination of the three assets to achieve a target return of 3% not allowing short sales.*

g. Assuming an initial $100,000 investment for one month, compute the 5% value-at-risk on the global minimum variance portfolio.

*First we find the 1% quantiles for the global minimum variance portfolio*

$$port: \; q_{.05} = 0.0083 + (0.0216)(-1.645) = -0.0272$$

*Then we compute the 1% VaR*

$$port: \; VaR_{.05} = \$100,000 \times \left(e^{-0.0272} - 1\right) = -\$2,687.61$$

III. Empirical Analysis of the single index model and the CAPM (40 points)

The single index model for asset returns has the form

$$
\begin{aligned}
&R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it} \\
&i = 1, \ldots, n \text{ assets} \\
&t = 1, \ldots, T \text{ time periods}
\end{aligned}
$$

where $R_{it}$ denotes the cc return on asset i at time t, and $R_{Mt}$ denotes the return on a market index portfolio at time t.

a. In the SI model, what is the interpretation of $\beta_i$?

*$\beta_i$ measures the contribution of asset i to the risk of the market index measured by the standard deviation of the market index. Explicitly,*

$$\beta_i = \frac{\text{cov}(R_{it}, R_{Mt})}{\text{var}(R_{Mt})}$$

b. How is the interpretation of $\varepsilon_{it}$ in the SI model different from its interpretation in the constant expected return (CER) model?

*$\varepsilon_{it}$ captures random news that is uncorrelated with the "market" as captured by $R_{Mt}$. In the CER model, $\varepsilon_{it}$ captures all random news.*

The following represents S-PLUS linear regression output from estimating the single index model for the Vanguard Pacific Stock Index (vpacx), the Vanguard long-term bond index (vbltx) and the Vanguard Emerging Markets Fund (veiex) using monthly continuously compounded return data over the 5 year period September 2001 – September 2006. In the regressions, the market index is the Vanguard S&P 500 index (vfinx).

```
> summary(vpacx.fit,cor=F)

Call: lm(formula = vpacx ~ vfinx, data = final.ts)
Residuals:
     Min      1Q   Median     3Q  Max
 -0.0822 -0.0258 -0.00312 0.0262 0.09

Coefficients:
            Value Std. Error t value Pr(>|t|)
(Intercept) 0.008 0.005      1.678   0.099
      vfinx 0.493 0.131      3.758   0.000

Residual standard error: 0.0374 on 58 degrees of freedom
Multiple R-Squared: 0.196
F-statistic: 14.1 on 1 and 58 degrees of freedom, the p-value is 0.000399

> summary(vbltx.fit,cor=F)

Call: lm(formula = vbltx ~ vfinx, data = final.ts)
Residuals:
     Min      1Q  Median     3Q    Max
```

```
 -0.0936 -0.0171 0.00431 0.0151 0.0556

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept)  0.006  0.003      1.891   0.064
      vfinx -0.183  0.092     -1.994   0.051

Residual standard error: 0.0262 on 58 degrees of freedom
Multiple R-Squared: 0.0642
F-statistic: 3.98 on 1 and 58 degrees of freedom, the p-value is 0.0508

> summary(veiex.fit,cor=F)

Call: lm(formula = veiex ~ vfinx, data = final.ts)
Residuals:
     Min      1Q   Median     3Q    Max
 -0.0949 -0.0178 0.000847 0.0274 0.0655

Coefficients:
            Value Std. Error t value Pr(>|t|)
(Intercept) 0.014 0.004      3.150   0.003
      vfinx 1.117 0.120      9.287   0.000

Residual standard error: 0.0343 on 58 degrees of freedom
Multiple R-Squared: 0.598
F-statistic: 86.3 on 1 and 58 degrees of freedom, the p-value is 4.47e-013
```

c. Make a table showing the estimated values of $\beta$, its estimated standard error, the estimate of $\sigma_\varepsilon$, and the $R^2$ values from the three regression equations. How accurate are the estimates of $\beta$?

| | $\hat{\beta}$ | SE($\hat{\beta}$) | $\hat{\sigma}_\varepsilon$ | $R^2$ |
|---|---|---|---|---|
| vpacx | .493 | .131 | .0374 | .196 |
| vbltx | -.183 | .092 | .0262 | .0642 |
| veiex | 1.117 | .120 | .0343 | .596 |

d. What can you say about the risk characteristics of the three assets relative to the S&P 500 index (vfinx)? Which asset is most beneficial to hold in terms of diversification?

*Using $\beta$ as a measure of risk, we see that vbltx has the lowest risk and veiex has the highest risk. The bond index vbltx is the most beneficial asset to hold in terms of diversification since it has the lowest $\beta$ and the lowest $R^2$.*

e. For each asset, test the null hypothesis that $\beta = 1$ against the alternative that $\beta \neq 1$ using a 5% significance level. What do you conclude?

$$vpacx : t_{\beta=1} = \frac{\hat{\beta}-1}{SE(\hat{\beta})} = \frac{.493-1}{.131} = -3.87$$

$$vbltx : t_{\beta=1} = \frac{\hat{\beta}-1}{SE(\hat{\beta})} = \frac{-.183-1}{.092} = -12.86$$

$$veiex : t_{\beta=1} = \frac{\hat{\beta}-1}{SE(\hat{\beta})} = \frac{1.117-1}{.120} = 0.98$$

*We reject the null hypothesis that $\beta = 1$ against the alternative that $\beta \neq 1$ using a 5% significance level whenever $|t_{\beta=1}| > 2$. Therefore, we reject the null for vpacx and vbltx but not for veiex.*

f. Using the information listed in Table 1 (from Portfolio theory section), what is the $\beta$ of the global minimum variance portfolio?

$$\begin{aligned}\beta_p &= m_{vpacx}\beta_{vpacx} + m_{vbltx}\beta_{vbltx} + m_{veiex}\beta_{veiex} \\ &= (.19)(.493) + (.69)(-.183) + (.12)(1.117) = 0.10\end{aligned}$$

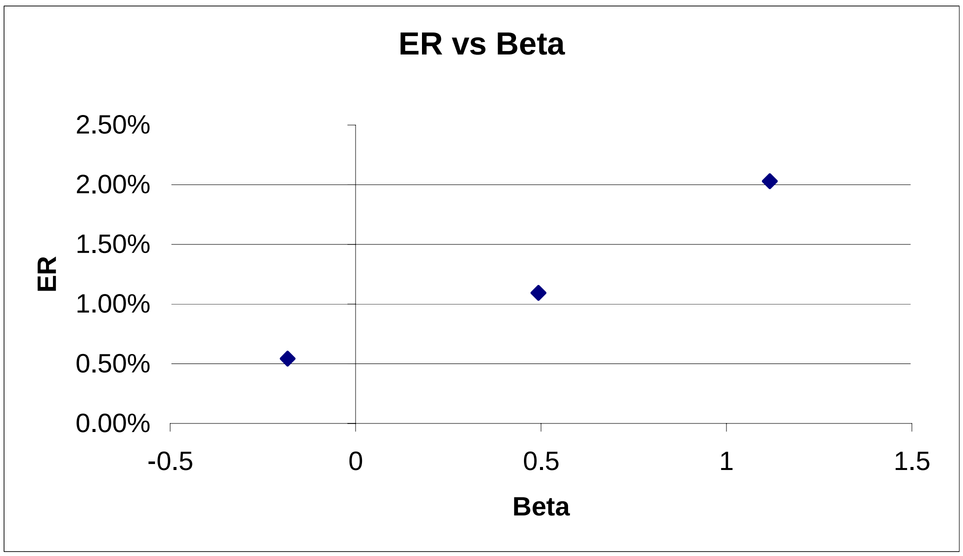
g. If the monthly risk free rate is 0.25% (0.0025) and the monthly risk premium on the S&P 500 is index is .5% (0.005), what are the monthly expected returns predicted by the capital asset pricing model (CAPM) for the three assets?
The CAPM pricing relationship is

$$E[R_i] = r_f + \beta_i(E[R_M] - r_f)$$

Using $r_f = 0.0025$ and $E[R_M] - r_f = .005$ we have

$$E[R_{vpacx}] = 0.0025 + (.493)(.005) = .00497$$
$$E[R_{vbltx}] = 0.0025 + (-.183)(.005) = .00159$$
$$E[R_{veiex}] = 0.0025 + (1.117)(.005) = .00809$$

h. Using the information listed in Table 1 (from Portfolio theory section) and the regression output, make a plot of the average return for against the estimate of $\beta$ for the three assets. What does the CAPM predict about the relationship between average return and $\beta$? Does this relationship appear to hold for the three assets?




*There appears to be a strong upward sloping linear relationship between the average return and $\beta$ as predicted by the CAPM.*