

Portfolio Theory

Econ 422: Investment, Capital & Finance
University of Washington
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Forming Combinations of Assets or Portfolios

- Portfolio Theory dates back to the late 1950s and the seminal work of Harry Markowitz and is still heavily relied upon today by Portfolio Managers
- We want to understand the characteristics of portfolios formed from combining assets
- Given our understanding of portfolio characteristics, how does an individual investor form optimal portfolios, i.e., consistent within the economic models presented to date?
- What useful generalities or properties can we derive?
- How does this theory apply to the economy or capital markets (investors in the aggregate)?
- Is this theory consistent with behavior we observe in financial markets?

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Preliminaries: Portfolio Weights

- Portfolio weights indicate the fraction of the portfolio's total value held in each asset, i.e.

$$x_i = (\text{value held in the } i^{\text{th}} \text{ asset}) / (\text{total portfolio value})$$

- Portfolio composition can be described by its portfolio weights:

$$x = \{x_1, x_2, \dots, x_n\} \text{ and the set of assets } \{A_1, A_2, \dots, A_n\}$$

- By definition, portfolio weights must sum to one:

$$x_1 + x_2 + \dots + x_n = 1$$

- Initially we will assume the weights are non-negative ($x_i \geq 0$), but later we will relax this assumption. Negative portfolio weights allow us to deal with borrowing and short selling assets.

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Data Needed for Portfolio Calculations

$E(r_i)$ Expected returns for all assets i

$V(r_i)$ or $SD(r_i)$ Variances or standard deviations of return for all assets i

$\text{Cov}(r_i, r_j)$ Covariances of returns for all pairs of assets i and j

Where do we obtain this data ?

- Estimate them from historical sample data using statistical techniques (*sample statistics*). This is the most common approach.

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Portfolio Inputs in Greek

- $\mu = E[R]$
- $\sigma^2 = \text{var}(R)$
- $\sigma = \text{SD}(R)$
- $\sigma_{ij} = \text{Cov}(R_i, R_j)$
- $\rho_{ij} = \text{Cor}(R_i, R_j)$
- Note: $\sigma_{ij} = \rho_{ij} * \sigma_i * \sigma_j$

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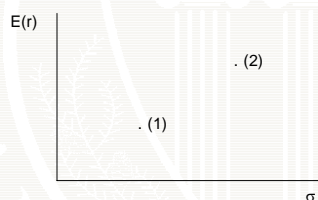
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A Portfolio of Two Risky Assets

Real world relevance:

1. Client looking to diversify single concentrated holding in one particular asset.
2. Portfolio Manager looking to add an additional asset to a pre-existing portfolio.



Points (1) and (2) show the expected return and standard deviation characteristics for each of the risky assets.

- *What are the characteristics of a portfolio that is composed of these two assets with portfolio weights x_1 and x_2 of asset 1 and 2, respectively?*

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Portfolio Characteristics n = 2 Case

As you hold x_1 of asset 1 and x_2 of asset 2, you will receive x_1 of the return of asset 1 plus x_2 of the return of asset 2:

$$r_p = x_1 r_1 + x_2 r_2$$

Find expected return and variance of return.

$$E(r_p) = x_1 E(r_1) + x_2 E(r_2)$$

$$V(r_p) = x_1^2 V(r_1) + x_2^2 V(r_2) + 2x_1 x_2 \text{Cov}(r_1, r_2)$$

- The portfolio's expected return is a weighted sum of the expected returns of assets 1 and 2.
- The variance is the square-weighted sum of the variances plus twice the cross-weighted covariance.

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Calculating Portfolio Variance Matrix Approach n=2

1. Set up a 2x2 matrix, using the respective asset portfolio weights as the heading.

	x_1	x_2
x_1	σ_{11}	σ_{12}
x_2	σ_{21}	σ_{22}

2. Fill the 2x2 matrix with the variance and covariance information.

Notation: $\sigma_{ii} = \sigma_i^2 = V(r_i)$ = variance of return for asset i
 $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$ = covariance of returns for assets i and j
 $\sigma_{ij} = \sigma_{ji}$ the covariances are symmetric

3. For each cell, multiply the row weight by the column weight by the cell entry. Do for all four inner cells and add. The result:

$$\begin{aligned} \sigma_p^2 &= x_1^2 \sigma_{11} + x_1 x_2 \sigma_{12} + x_2 x_1 \sigma_{21} + x_2^2 \sigma_{22} \\ &= x_1^2 \sigma_{11} + 2 x_1 x_2 \sigma_{12} + x_2^2 \sigma_{22} \end{aligned}$$

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Example: Portfolio Characteristics (n=2)

Suppose two assets, 1 and 2, respectively have the following characteristics:

- Expected returns: Standard deviations

$E(r_1) = 0.12$	$\sigma_1 = 0.20$
$E(r_2) = 0.17$	$\sigma_2 = 0.30$

- Correlation coefficient:

$$\rho_{12} = .4$$

- Portfolio weights:

$$x_1 = 0.25$$

$$x_2 = 0.75$$

Find $E(r_p)$ and $V(r_p)$.

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Diversification & Portfolio Effect

- Portfolio diversification results from holding two or more assets in a portfolio.
- Generally the more different the assets are, the greater the diversification.
- The diversification effect is the reduction in portfolio standard deviation, compared with a simple linear combination of the standard deviations, that comes from holding two or more assets in the portfolio (provided their returns are not perfectly, positively correlated).

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Diversification & Portfolio Effect

- The size of the diversification effect depends on the degree of correlation among the assets' returns.

Recall: $\sigma_p^2 = x_1^2 \sigma_{11} + x_2^2 \sigma_{22} + 2x_1 x_2 \sigma_{12}$

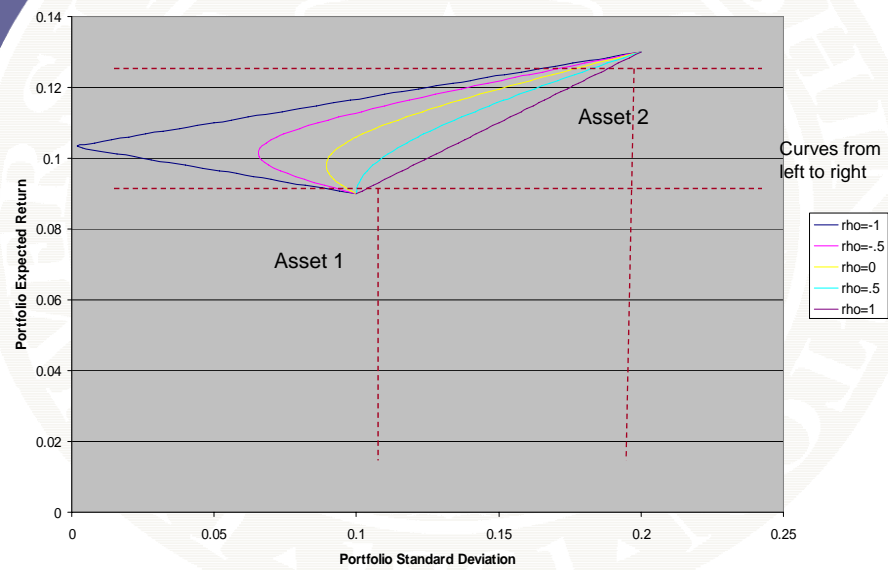
and $\sigma_{12} = \rho_{12} \sigma_1 \sigma_2$

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Portfolio Characteristics Depend on the Correlation of Returns



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Portfolio Characteristics (General n asset case)

- The portfolio expected return is always the share-weighted sum of the expected returns for the assets included in the portfolio.

$$E(r_p) = \sum_{\substack{i \text{ for all} \\ \text{assets in} \\ \text{portfolio}}} x_i E(r_i)$$

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To Calculate Portfolio Variance (n > 2)

	X ₁	X ₂	X ₃	...	X _n
X ₁	σ ₁₁	σ ₁₂	σ ₁₃	...	σ _{1n}
X ₂	σ ₂₁	σ ₂₂	σ ₂₃	...	σ _{2n}
X ₃	σ ₃₁	σ ₃₂	σ ₃₃	...	σ _{3n}

X _n	σ _{n1}	σ _{n2}	σ _{n3}	...	σ _{nn}

Given a vector of portfolio weights and the matrix of variances and covariances, the portfolio variance is computed by adding for all cells the product of the row weight, the column weight, and the cell variance or covariance.

We can write this succinctly as follows:

$$V(R_p) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}$$

Q: How many variances and covariances are there in the matrix?

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Example

- 3 asset portfolio: $x_1 = 0.2$, $x_2 = 0.5$, $x_3 = 0.3$
- $E[R_1] = 0.10$, $E[R_2] = 0.05$, $E[R_3] = 0.20$
- Covariance matrix is given below
- Find $E[R_p]$ and $V(R_p)$

$$\Sigma =$$

.011	.003	.002
.003	.020	.001
.002	.001	.010

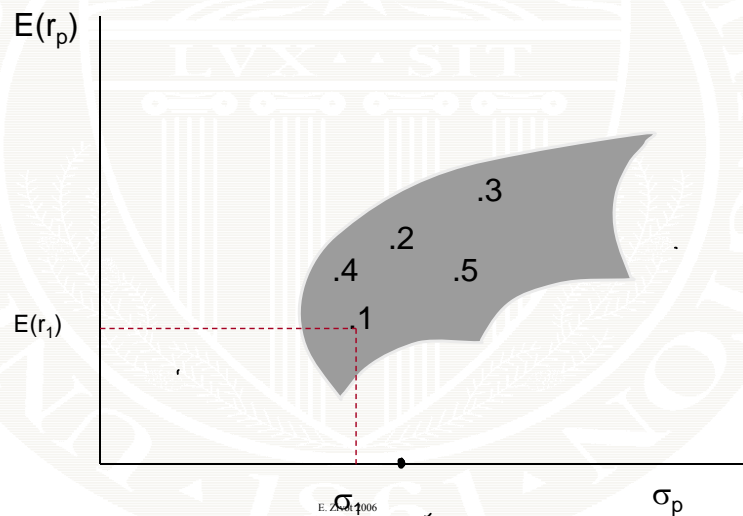
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The Set of All Portfolios of Risky Assets

Each labeled point in the shaded area represents the characteristics of a risky asset. Points in the shaded area represent the characteristics of all the portfolios that can be constructed by combining the risky assets. This will be discussed later on.



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Portfolio Characteristics: Effect of Increasing # of Assets in Equal Weighted Portfolio

n = the number of assets in the portfolio

$x_i = \frac{1}{n}$ = equally weighted portfolios

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} = \sum_{i=1}^n \frac{1}{n^2} \sigma_{ii} + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{n^2} \sigma_{ij}$$

$$= \frac{1}{n} \bar{\sigma}_{ii} + \left(1 - \frac{1}{n}\right) \bar{\sigma}_{ij}$$

$$\bar{\sigma}_{ii} = 1/n \sum_{i=1}^n \sigma_{ii} \quad \text{for all } i=1 \text{ to } n$$

$$\bar{\sigma}_{ij} = 1/n(n-1) \sum_{i \neq j} \sigma_{ij} \quad \text{for all } i, j$$

where $i \neq j$ and each $i, j, = 1$ to n

where $\bar{\sigma}_{ii}$ is the average variance and $\bar{\sigma}_{ij}$ is the average covariance

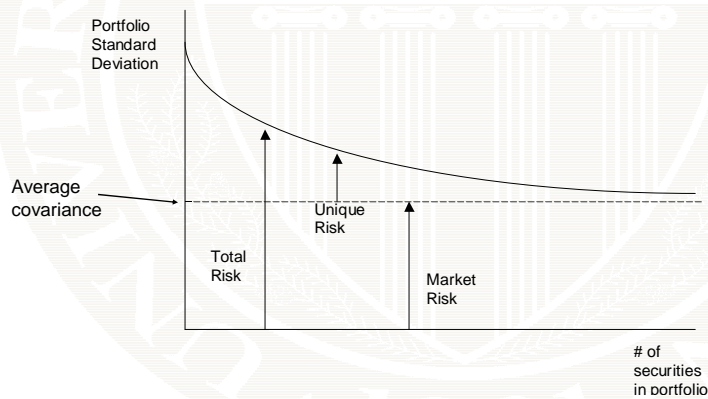
If both these averages are bounded, then as n increases the contribution to portfolio variance made by the variance diminishes and the portfolio variance converges to the average covariance.

\Rightarrow *Covariances prove more important than variances in determining the portfolio variance.*

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Diversification Eliminates Asset Specific Risk



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Empirical Example: The Diversification Effect of Increasing the # of Assets in Equal-Weighted Portfolio

Eugene Fama's example (*Foundations of Finance text*)

- Fama selects stock for the portfolio at random.
- Weights are chosen (for simplicity) as $1/n$, where n is the number of assets in the portfolio. Up to 50 stocks are added, one by one.
- Characteristics of individual stocks ($E(r_i)$, s_i , and s_{ij}) are estimated from an out-of-sample prior 5-year sample of monthly returns.
- Characteristics of the portfolio are computed using the techniques that we have discussed.
- No material diversification benefit beyond the first 20 or 30 stocks.

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Risk-free Borrowing and Lending

- When you purchase U.S. Treasury securities you are lending money to the US Government.
- Investors can lend risklessly by investing a portion of the portfolio in Treasury bills.
- Investors can also borrow money and use it to expend their holdings of risky assets.
- *We want to know how using these two approaches—borrowing and lending—affects the characteristics of portfolios.*

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Portfolio Characteristics: Lending

Let x_1 = the share of the portfolio invested in a risk-free asset (T-bill)
 r_f = the return on the risk-free asset = constant (not random!)
 $x_2 = 1 - x_1$ = the share of the portfolio invested in a risky asset, asset 2

The risky asset is described by:

Expected return $E(r_2)$ & Standard deviation σ_2

Using our standard formulas, we can compute the expected return and variance for the **portfolio**:

$$E(r_p) = x_1 r_f + x_2 E(r_2)$$

$$V(r_p) = x_1^2 V(r_f) + x_2^2 V(r_2) + 2x_1 x_2 \text{Cov}(r_f, r_2)$$

but $V(r_f) = \text{Cov}(r_f, r_2) = 0$ hence

$$V(r_p) = x_2^2 V(r_2) \text{ or}$$

$$\sigma_p = x_2 \sigma_2$$

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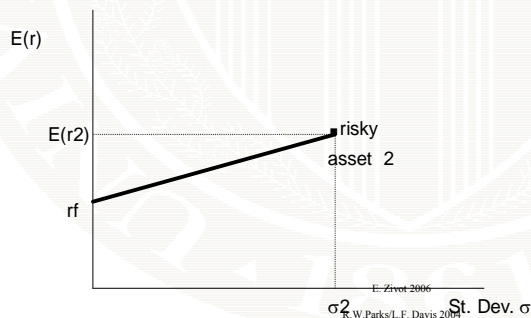
Portfolio Characteristics: Lending

The characteristics of portfolios—expected return and standard deviation—that combine lending risk-free (lending the risk-free asset) with a risky asset plot on a straight line connecting the risky and risk-free points:

$$E(r_p) = x_1 r_f + x_2 E(r_2)$$

$$\sigma(r_p) = x_2 \sigma_2$$

To plot the locus vary the pairs (x_1, x_2) but holding $x_1 + x_2 = 1$.



Q: What is the slope of the line?

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Sharpe's Slope

$$E[r_p] = x_1 r_f + x_2 E[r_2]$$

$$x_1 + x_2 = 1 \Rightarrow x_1 = 1 - x_2$$

$$\Rightarrow E[r_p] = (1 - x_2)r_f + x_2 E[r_2] = r_f + x_2(E[r_2] - r_f)$$

$$\sigma_p = x_2 \sigma_2 \Rightarrow x_2 = \frac{\sigma_p}{\sigma_2}$$

$$\Rightarrow E[r_p] = r_f + \frac{\sigma_p}{\sigma_2}(E[r_2] - r_f) = r_f + \frac{E[r_2] - r_f}{\sigma_2} \sigma_p$$

$$\text{Sharpe's slope} = \frac{E[r_2] - r_f}{\sigma_2}$$

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Financial Leverage

- Leverage involves borrowing in order to hold a risky asset.

Example: You have \$100,000 in your investment portfolio. Suppose you borrow \$50,000 at 6% and invest the entire \$150,000 in a risky asset with an expected return of 15%.

- Your expected dollar return = \$150,000 * 15% = \$22,500
- Your required interest payment = 6% * \$50,000 = **\$3,000**
- Expected net return = expected return less interest = \$19,500
- Expected rate of return for portfolio = \$19,500/\$100,000=19.5%

Note: Expected rate of return for portfolio with leverage exceeds expected rate of return for fully invested portfolio without leverage (borrowing to invest in the risky asset), i.e., 19.5% versus 15%.

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Leveraged Portfolio Share Computation

- Borrowing is represented by a negative share associated with the risk-free asset. You essentially sell the risk-less asset to hold more of the risky asset.
- You are holding more than 100% of your portfolio's net value in the risky asset.

Leverage Example continued:

- Recall the portfolio is initially worth \$100,000.
- You borrow \$50,000. This borrowing represents 50% of your initial portfolio ($-\$50,000/\$100,000$); thus, the share of the risk-less asset is -0.5.
- You hold \$150,000 in the risky asset or $\$150,000/\$100,000 = 1.5$ shares in the risky asset.

Note: The portfolio shares still sum to unity: $-0.5 + 1.5 = 1.0$

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Leveraged Portfolio Expected Return Computation

$$E[r_p] = r_f + x_2(E[r_2] - r_f)$$

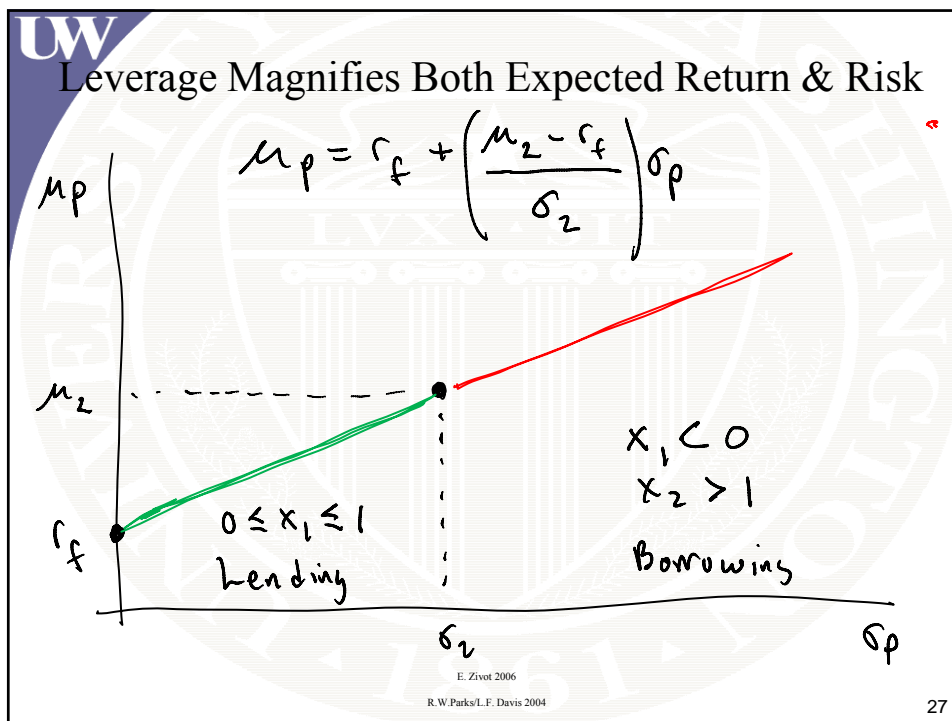
$$r_f = 0.06, x_2 = 1.5, E[r_2] = 0.15$$

$$E[r_p] = 0.06 + 1.5(0.15 - 0.06) = 0.195$$

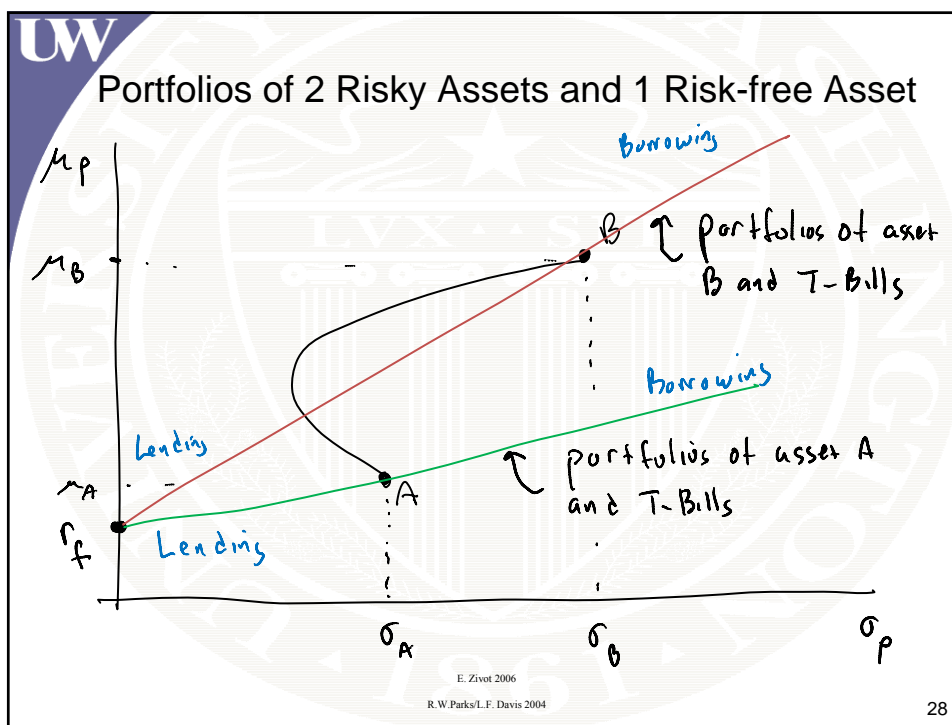
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Portfolios of 2 Risky Assets and 1 Risk-free Asset

- Sharpe's slope for asset B and T-Bills is larger than Sharpe's slope for asset A and T-Bills

$$\frac{\mu_B - r_f}{\sigma_B} > \frac{\mu_A - r_f}{\sigma_A}$$

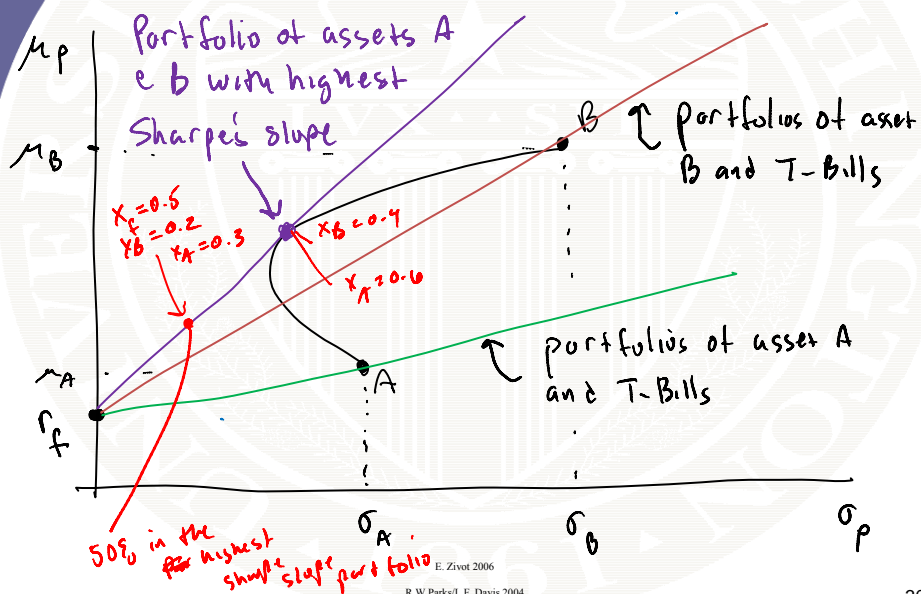
- Portfolios of asset B and T-Bills are efficient relative to portfolios of asset A and T-Bills

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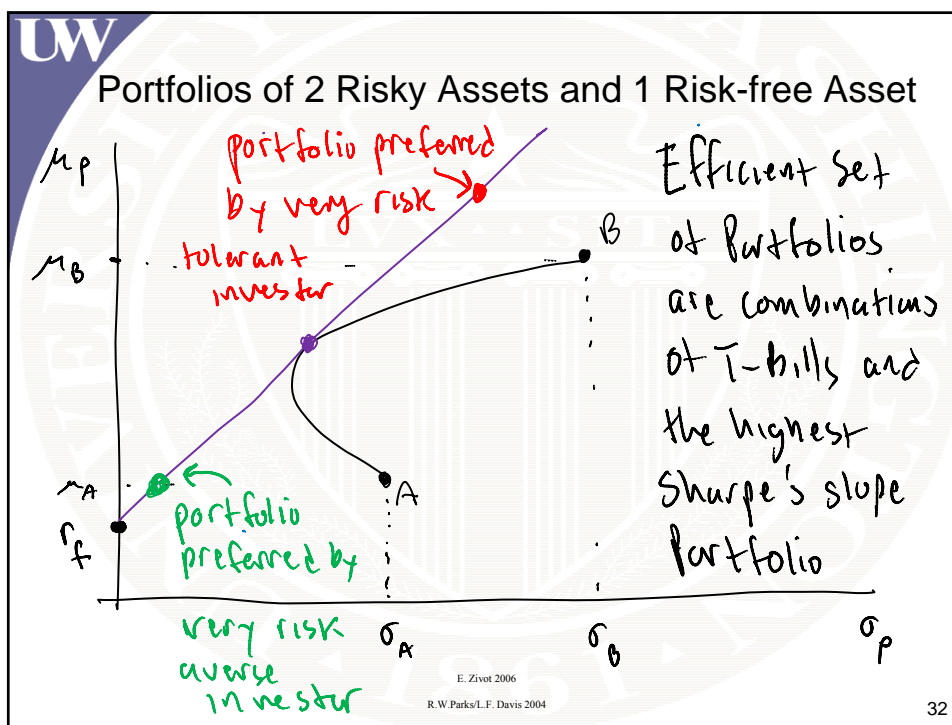
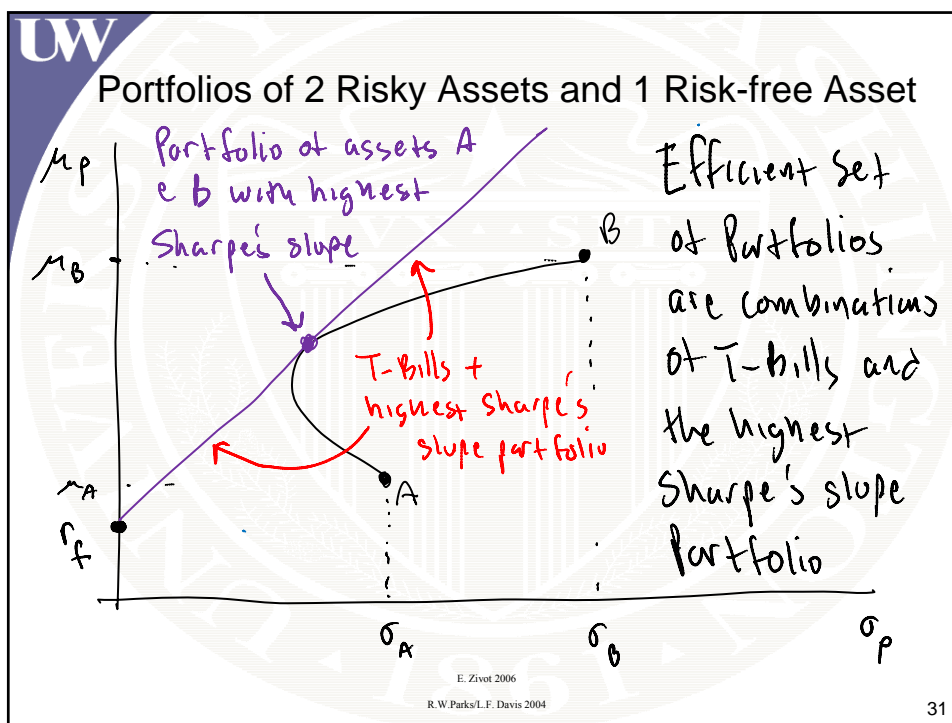
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Portfolios of 2 Risky Assets and 1 Risk-free Asset



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The Consumption-Investment Choice

Putting It All Together:

- Expected Utility maximizing consumer
- Inter-temporal choice: consumption today versus next period
- Borrowing/Lending possible

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The Consumption-Investment Choice

- Consider a consumer/investor with preferences given by: $U(C_0, C_1)$ and initial wealth W_0 .
- The consumer chooses C_0 , leaving $W_0 - C_0 = I_0$ available to invest for tomorrow's consumption.
- The rate of return earned on I_0 is a random variable r
- Consumption next period is the future value of the investment:
- $$C_1 = (W_0 - C_0)(1 + r)$$
- Future consumption; therefore, is a random variable.
- The consumer's choice problem is to choose the level of consumption C_0 to maximize expected utility:
- **maximize $E[U(C_0, C_1)] = E[U(C_0, (1 + r)(W_0 - C_0))]$**

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Preferences Over Portfolio Characteristics

- Suppose the return on the investment, r , is normally distributed with mean $E(r)$ and standard deviation σ_r . From the symmetric nature of the normal distribution, we can write any return, r , as a linear combination of the expected return and standard deviation:

- $$r = E(r) + z \sigma_r$$
- $$z \sim N(0,1)$$

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Preferences Over Portfolio Characteristics

- Substituting $r = E(r) + z \sigma_r$ for r in the consumer's objective function gives:
- $\max E[U(C_0, (1 + E(r) + z \sigma_r)(W_0 - C_0))]$
- *The consumer/investor can be thought of as choosing C_0 and by choosing the composition of the investment portfolio via $E(r)$ and σ_r .*

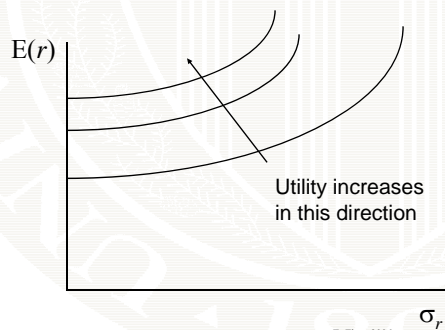
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Portfolio Characteristic Preferences

- By choosing the portfolio composition the investor determines $E(r)$ and σ_r .
- For risk averse investors $E(r)$ is a “good” and σ_r is a “bad.”
- Indifference curves will look like this:

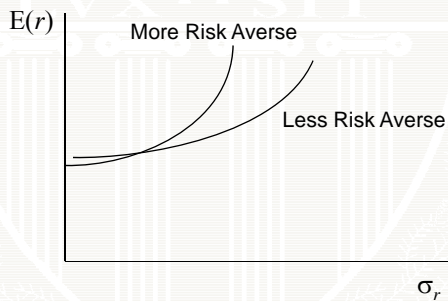


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Degrees of Risk Aversion—Portfolio Characteristics



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The 'Efficient' Set of Risky Portfolios

- Recall that investors like expected return but dislike standard deviation.
- Among the set of all possible portfolios constructed from the risky assets, the [efficient portfolios](#) give the minimum risk for given expected return or the maximum expected return for given risk.
- Harry Markowitz developed a mathematical algorithm based on quadratic programming to determine the set of efficient portfolios. This algorithm is described in detail in econ 424

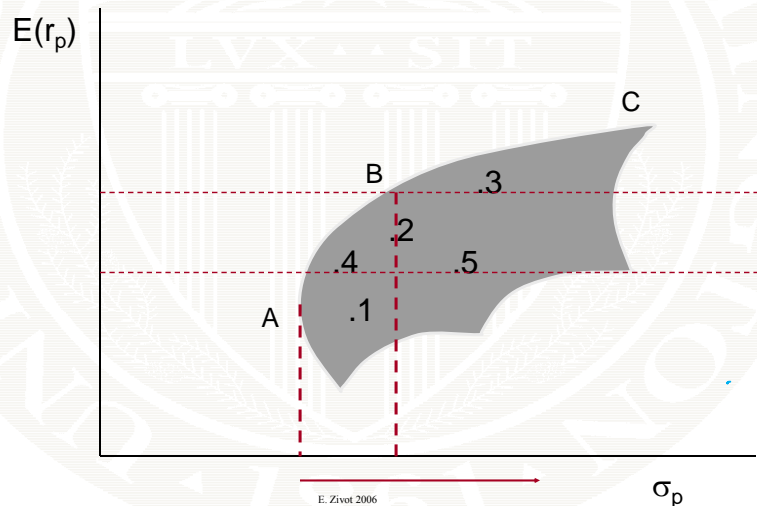
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Efficient Risky Portfolios

The northwest boundary of the set, above and to the right of A, are efficient portfolios. A is the minimum variance portfolio. B is an efficient portfolio. Portfolios 1-5 are not efficient portfolios. AC represents the Efficient Frontier.



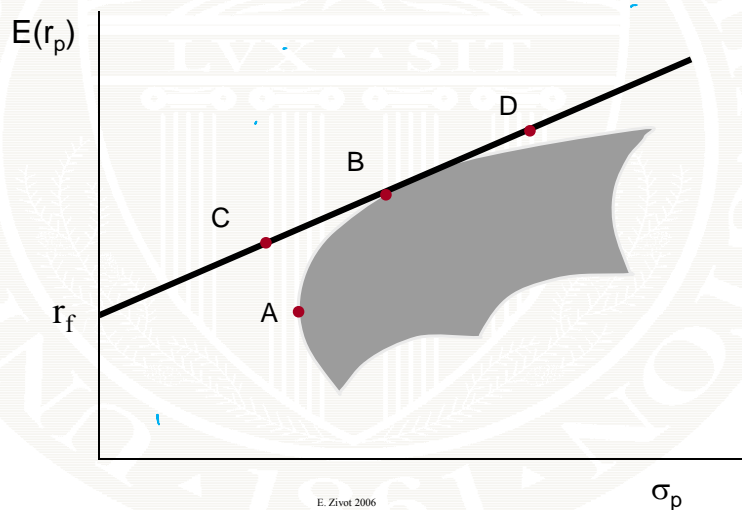
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Borrowing/Lending Expands the Investor's Opportunity Set

Portfolios C, B, and D are efficient. The ability to borrow or lend at r_f causes A to no longer be efficient, i.e., as there are higher return opportunities for the given standard deviation.

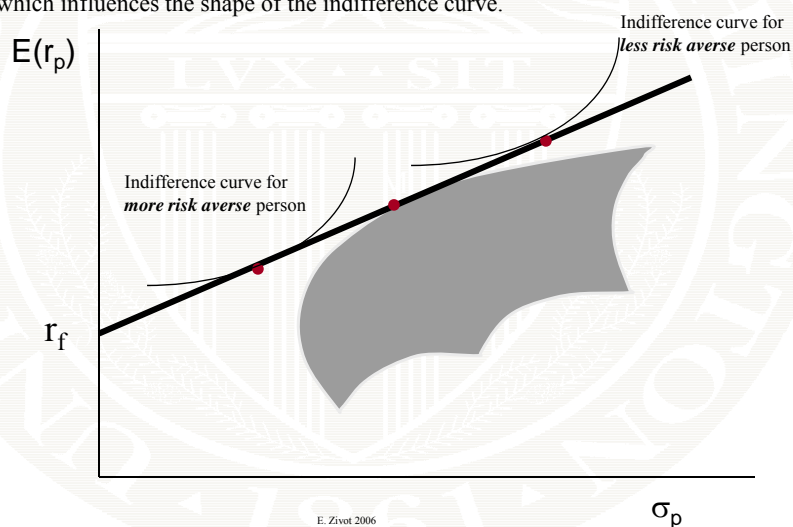


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The Investor's Optimal Portfolio

- Lies on the expanded efficient portfolio locus
- The position depends on the investor's attitude toward risk, i.e., degree of risk aversion which influences the shape of the indifference curve.



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Capital Market Equilibrium

- We have focused on individual consumer/investor's choice problem choosing optimal consumption stream of present and future consumption, and optimal investment or portfolio characteristics.
- We have derived properties regarding borrowing (leverage) and lending.
- What implications does Portfolio Theory (our mean-variance framework) have for the market in general or in aggregate?

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Understanding Capital Market Equilibrium

- What assumptions do we need to aggregate and what conditions must hold for the aggregate market to be in equilibrium?
- Investor's are rational and risk averse
 - » More wealth (higher portfolio expected return) is preferred to less wealth
 - » Less risk (lower portfolio variance) is preferred to more risk
- Investor's can borrow and lend at the riskless rate r_f .

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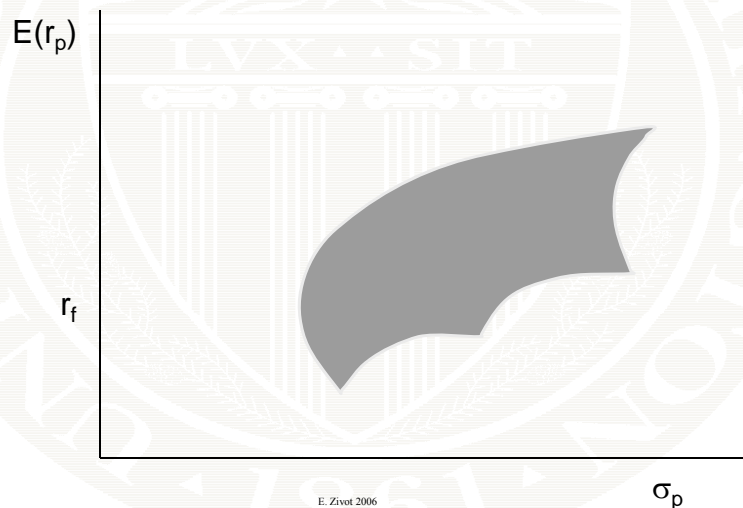
Understanding Capital Market Equilibrium

- Homogeneous expectations
 - » Investors have access to the same information and process it in the same way
- All investors use portfolio theory to determine the demand for risky assets
- Supply of assets (market) is all publicly traded assets
- Asset markets clear
 - » Asset prices are such that supply equals demand

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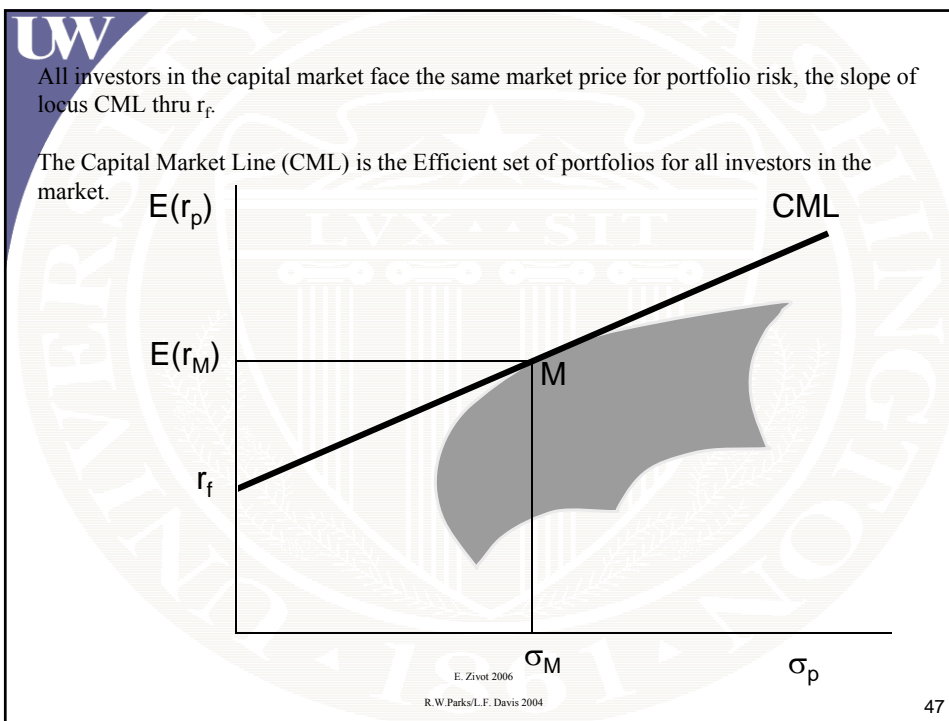
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Set of Capital Market Risky Portfolios



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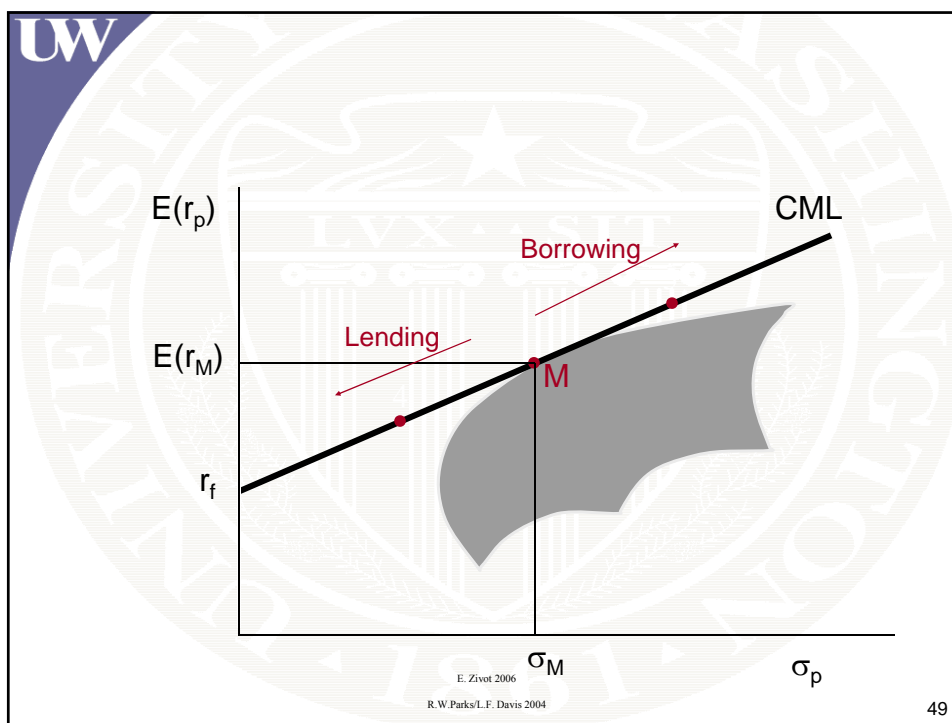
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Capital Market Line

- M must be the “market portfolio” of risky assets. It includes all risky assets, held in fractions that correspond with the shares of their market capitalization in total market value.
- Investors can hold less risky portfolios by combining M with risk-free lending.
- Investors can hold more risky portfolios, using leverage (borrowing) to expand their holdings of risky assets M.
- Sharpe’s Slope: $[E(r_m) - r_f] / \sigma_m = \text{market risk premium} / \text{market risk}$

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The Two-Fund Separation Result

Investors can create their optimal portfolio using a combination of two mutual funds:

1. The M fund, a mutual fund corresponding with the market portfolio of risky assets.
2. A money market fund giving the risk-free return or borrowing at the risk-free rate.

This result is the justification for “passive” investing

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