FINANCE THEORY: Inter-temporal Consumption-Saving and Optimal Firm Investment Decisions

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Econ 422

Summer 2010

Reading

- PCBR, Chapter 1 (general overview of financial decision making)
- Varian, *Intermediate Microeconomics: A Modern Approach*. Chapter 10, Intertemporal Choice.
- Hirshleifer and Hirshleifer, *Price Theory and Applications,* Chapter 14, The Economics of Time.

Goals of this Section

- Understand the economic principles behind inter-temporal consumption-savings decisions
- Introduce present value concepts
- Understand the roll of financial markets for the efficient allocation of savings and capital investment
- Understand what determines the level of interest rates

The Fisher Model

- Model of intertemporal choice involving consumption and investment decisions. (Named after Yale economist Irving Fisher)
- Key Assumptions:
  - » Two periods (generalizing to many future periods is straightforward).
  - » Perfect capital markets
  - » the absence of uncertainty

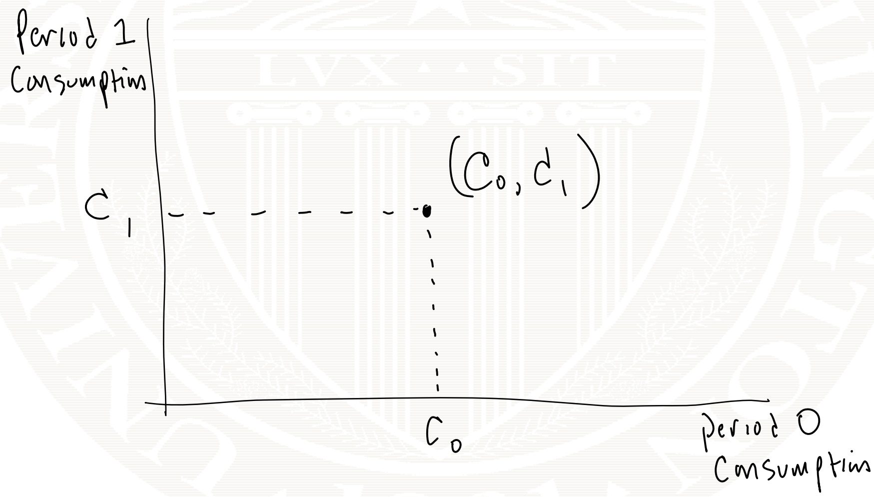
I. Intertemporal Exchange Model: Outline

A. Objects of choice, endowments and trade opportunities, preferences
B. Individual optima and comparative statics
C. Market exchange equilibrium and the determination of interest rates.

Objects of choice

- What is the consumer choosing?
- One of the many possible "Consumption Streams"
- A consumption stream is a sequence of time dated consumption, for the present and for the future; e.g. (C_0,C_1)
  - » C_0 is the standard of living or consumption level for period 0 (the present)
  - » C_1 is the standard of living or consumption level for period 1 (the future)

Representing a Consumption Stream



Consumer Preferences: Basic Assumptions

- Consumers are able to choose between alternative consumption streams.
- Choices are consistent (transitive)
- They prefer more consumption to less; i.e., they prefer higher standards of living to lower.
- Consumers choose the most preferred consumption stream among those attainable.

Ways to Represent Consumer Preferences

- Simple ranking of consumption choices
- Utility function, $U(C_0, C_1)$
- Indifference curves: level sets of utility function
  - » Combinations of C_0 and C_1 such that utility is constant (doesn't change)
  - » downward sloping, non intersecting, and convex shape

Utility Function, $U(C_0,C_1)$

- The utility function gives an <u>index value</u> for each consumption stream.
- The utility function value ranks consumption streams
- The marginal rate of substitution, MRS, gives:
  - » slope of an indifference curve at a point.
  - » the rate at which a consumer is willing to exchange future consumption for present consumption, (while maintaining the same level of satisfaction.)

Characteristics of preferences over consumption streams

- Present oriented
- Future oriented
- High degree of substitutability
- Low degree of substitutability

Examples of Utility Functions

$$U(C_0, C_1) = C_0^{1/2} C_1^{1/2}$$

$$U(C_0, C_1) = C_0^{2/3} C_1^{1/3}$$

$$U(C_0, C_1) = C_0^{1/3} C_1^{2/3}$$

From utility function to MRS

- Utility function or index $U=U(C_0,C_1)$
- Marginal utility tells us the rate at which utility changes when we change C_0, holding C_1 fixed or when we change C_1, holding C_0 fixed.
- $U_0=\partial U/\partial C_0$ = partial derivative (derivative holding C_1 fixed) of U with respect to C_0
- $U_1=\partial U/\partial C_1$ = partial derivative (derivative holding C_0 fixed) of U with respect to C_1

From utility function to MRS

- Total derivative of utility function:

$$dU = U_0 dC_0 + U_1 dC_1$$

- For changes along a given IC, utility stays constant ($dU = 0$):

$$0 = U_0 dC_0 + U_1 dC_1$$

- Solve for slope dC_1/dC_0:

$$\frac{dC_1}{dC_0} = -\frac{U_0}{U_1} = MRS$$

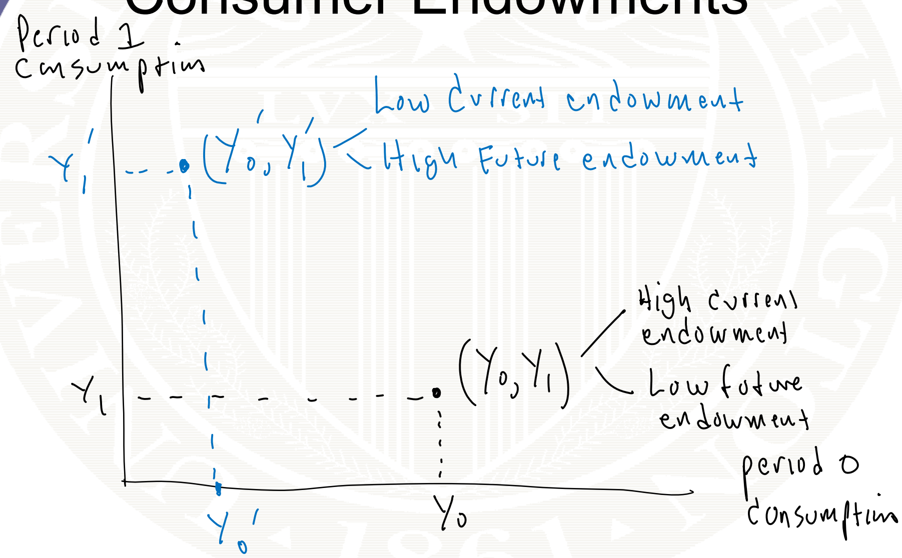
Consumer Endowments

- Consumer’s endowment is a claim to goods and services in the present and in the future.
- (Y_0, Y_1) represents the consumer’s endowment
  - » Y_i is the endowment in the ith period.

Consumer’s Endowment: Interpretation

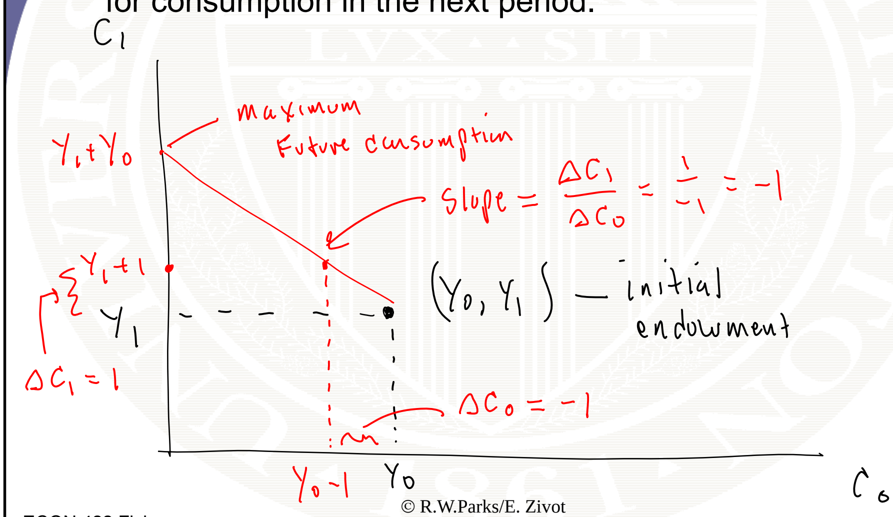
- The endowment might represent income that is expected in each of the two periods, from wages, from a pension trust, etc.
- The consumer can always choose a consumption stream equal to the endowment, but there may be other opportunities as well; e.g., through storage or by borrowing or lending.

Consumer Endowments



Storage

Some of the *present* endowment is saved and *stored* for consumption in the next period.



Market Exchange: Borrowing or Lending

- The consumer can borrow or lend consumption claims between periods
- Must be consistent with the endowment, i.e. you can't borrow more than you can repay. No uncertainty, lender knows your capacity.
- The real interest rate = r; e.g., r = 0.10 or 10%
- What consumption streams are possible?

Consumer's Budget Constraint: Lending

- If the consumer does not consume the entire present endowment, he or she can lend the amount $(Y_0 - C_0) = S_0$.
- This loan will be repaid with interest r
- Future consumption will be

$$C_1 = Y_1 + (1+r)(Y_0 - C_0).$$

Consumer's Budget Constraint: Borrowing

- To consume more than the present endowment the consumer must borrow $(C_0 - Y_0)$.
- The loan must be repaid with interest
- Future consumption will be
  $C_1 = Y_1 - (1+r)(C_0 - Y_0)$.
- Changing signs: $C_1 = Y_1 + (1+r)(Y_0 - C_0)$

Consumer's Budget Constraint

- $C_1 = Y_1 + (1+r)(Y_0 - C_0)$
  - » Covers both lending and borrowing because $(Y_0 - C_0)$ changes sign.
- Rewrite as:
  $C_1 = (1+r)Y_0 + Y_1 - (1+r)C_0$ or as

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}$$

Wealth

- What is the maximum present consumption that can be obtained with a given endowment, when we leave no resources for the future?
- Set the C_1 variable to zero in the budget constraint and solve for C_0:

  $C_0 = Y_0 + Y_1/(1+r) = W$

Interpreting the Budget Constraint

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r} = W$$

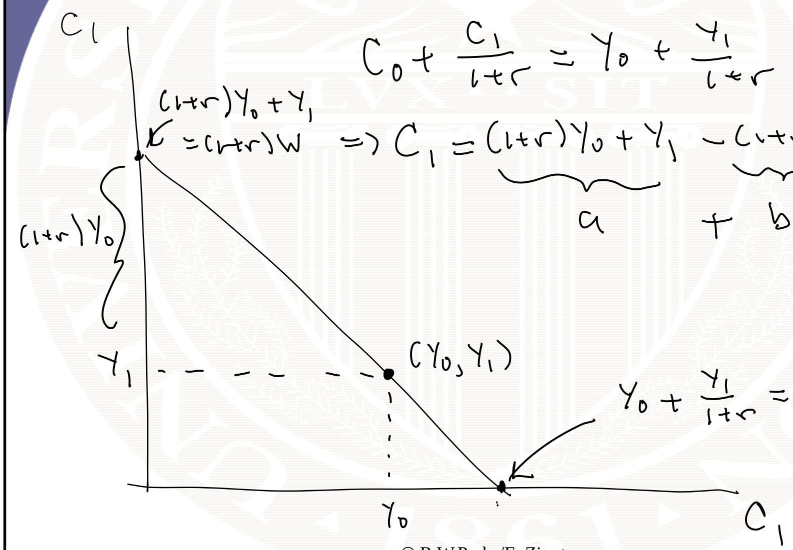
• Let C_0 denote consumption today in today's \$
=> $P_{0,0} = 1$
• Define $P_{0,1} = 1/(1 + r)$ = price today of \$1 to be received in period 1 = *present value* of \$1
• Re-interpretation of budget constraint in terms of present value:

$$P_{0,0}C_0 + P_{0,1}C_1 = P_{0,0}Y_0 + P_{0,1}Y_1 = W$$

Budget Constraint and Wealth

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r} = W$$

$$\Rightarrow C_1 = \underbrace{(1+r)Y_0 + Y_1}_{a} \; \underbrace{-(1+r)}_{+\, b} C_0$$



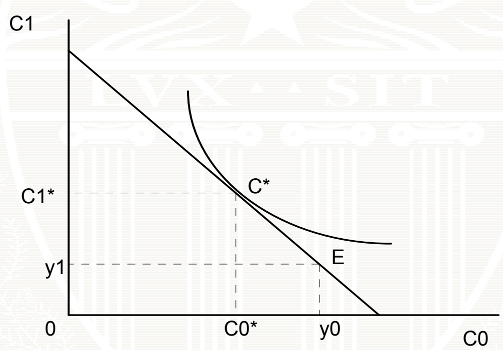
Budget Constraint and Wealth

- Consumers can attain (choose) any point on or inside the budget line.
- The line goes through the endowment point (Y_0,Y_1) ,has slope -(1+r).
- The horizontal intercept gives the consumer's wealth, W.

Class Exercise

- Person has endowment
  - » (y_0,y_1)=(10,000,11,000)
- Real interest rate is r=.10
- Find the person's wealth
- Find the future value of the endowment
- Write an equation for the budget constraint. Sketch it, i.e. indicate slope, intercepts.

The Consumer Optimum: Maximize Utility s.t. Intertemporal Budget Constraint



- This person saves $S_0 = Y_0 - C_0^*$ and lends it
- Repayment with interest is $(Y_0 – C_0^*)(1+r)$
- $Y_1 + (Y_0 – C_0^*)(1+r) = C_1^*$ the person's future consumption

Characteristics of the optimum

- The optimum consumption stream (C_0*,C_1*) must satisfy:
- The budget constraint $C_0 + C_1/(1+r) = W$ or $C_1 = (1+r)y_0 + y_1 - (1+r)C_0$
- Slope of IC = MRS = $-U_0/U_1$ = $-(1+r)$ = slope of BC

Example

$U = C_0^{0.5} C_1^{0.5}$ (a specific utility function)

$$MRS = -\frac{U_0}{U_1} = -\frac{C_1}{C_0}$$

$$(1)\ C_0 + \frac{C_1}{1+r} = W = Y_0 + \frac{Y_1}{1+r}$$

$$(2)\ MRS = -\frac{C_1}{C_0} = -(1+r) = \text{slope of B.C.}$$

Solve for C_0,C_1 in terms of r,W (or r, Y_0, and Y_1)

The solution:

$$C_1 = C_0(1+r) \quad \text{from (2)}$$

Substitute into (1):

$$C_0 + \frac{C_0(1+r)}{1+r} = W$$

$$2C_0 = W$$

$$C_0^* = \frac{1}{2}W \qquad C_1^* = \frac{1}{2}W(1+r)$$

See example Spreadsheet econ422Utility.xls on class notes page for numerical example using Excel

Formal Optimization Problem

$$\max_{C_0, C_1} U(C_0, C_1) \text{ subject to}$$

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r} = W$$

Solution Using Lagrange Multipliers

Step 1: Put constraint in homogeneous form

$$C_0 + \frac{C_1}{(1+r)} = W \Rightarrow C_0 + \frac{C_1}{(1+r)} - W = 0$$

Step 2: Form the Lagrangian function

$$L(C_0, C_1, \lambda) = U(C_0, C_1) + \lambda \left[C_0 + \frac{C_1}{1+r} - W \right]$$

Solution Using Lagrange Multipliers

Step 3: Maximize Lagrangian function

$$L(C_0, C_1, \lambda) = U(C_0, C_1) + \lambda\left[C_0 + \frac{C_1}{1+r} - W\right]$$

First order conditions

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial C_0} = U_0 + \lambda = 0$$

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial C_1} = U_1 + \frac{\lambda}{1+r} = 0$$

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial \lambda} = C_0 + \frac{C_1}{1+r} - W = 0$$

Use first to conditions to solve for λ:

$$-U_0 = \lambda$$

$$-U_1(1+r) = \lambda$$

$$\Rightarrow \frac{U_0}{U_1} = (1+r)$$

Hence, solving optimization problem gives solution that MRS = slope of budget constraint and that the budget constraint is satisfied.

Numerical Solution: Excel Solver

- Lagrangian maximization problem can be solved numerically using the Excel solver add-in
- See econ422Utility.xls for example

Borrowers vs. Lenders

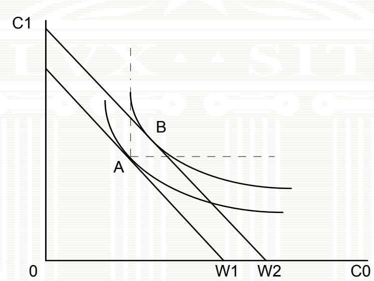
- Individuals with strong preferences toward future consumption and/or those with high initial endowments will be lenders
- Individuals with strong preferences toward current consumption and/or those with high future endowments will be borrowers

Comparative Statics for the Fisher Exchange Model

- What happens to the consumer optimum when the constraint changes?
  - » Start with an original optimum
  - » Change something
  - » Find the new optimum
  - » Compare it with the original
- In this model we can change:

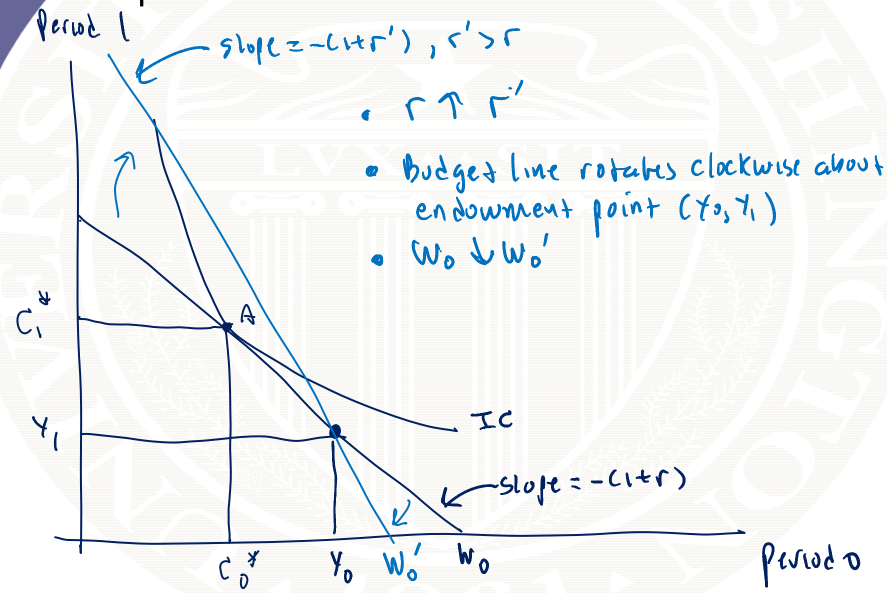
  (a)The endowment or (b) the interest rate

Effect of a Wealth Change with Fixed r



- With wealth W_1, the optimum is at A
- When the wealth increases to W_2, the new optimum is at B
- If both goods are "normal," B must be above and to the right of A.

Comparative statics: Increase in r for a lender



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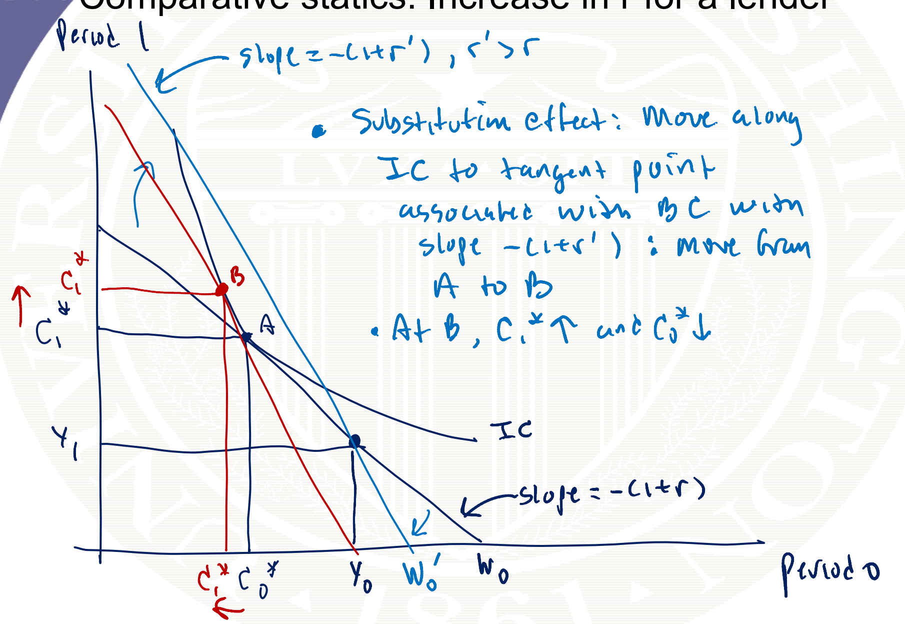
Comparative statics: Increase in r for a lender



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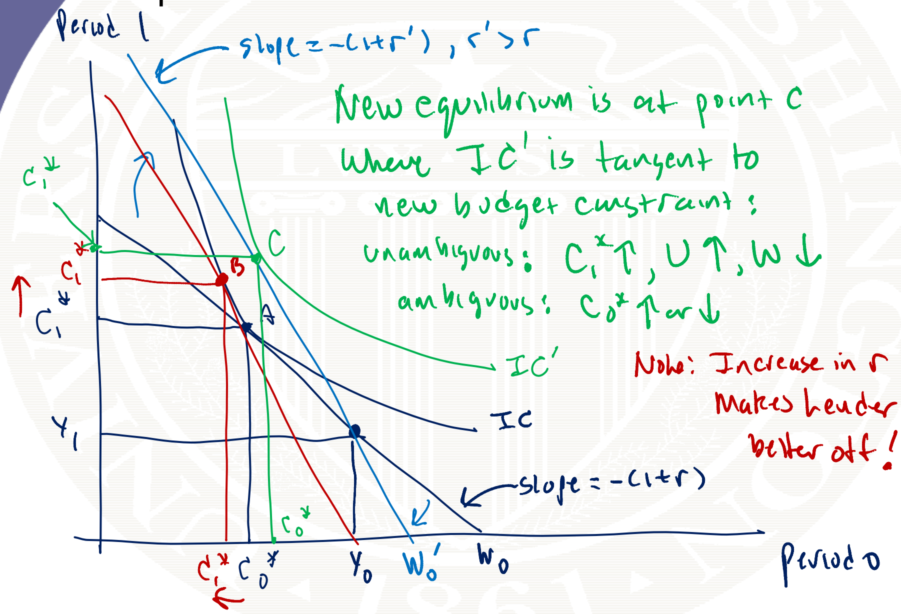
Comparative statics: Increase in r for a lender



Summary of Comparative Statics Results: Changes in the interest rate r

Results for a lender					
Variable	W	C_0	C_1	S_0	U
r ↑	↓	?	↑	?	↑
r ↓	↑	?	↓	?	↓
Results for a borrower					
r ↑	↓	↓	?	↑	↓
r ↓	↑	↑	?	↓	↑

The Market Equilibrium Interest Rate

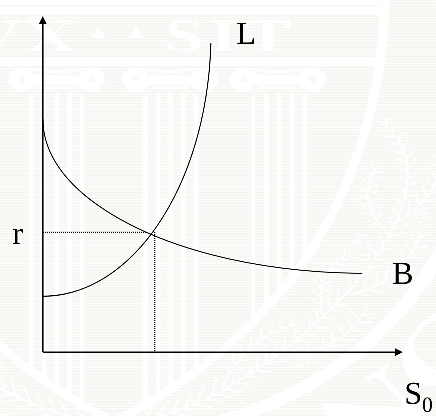
- Lenders need borrowers and vice versa
- Market clearing means that there is a match between the amount lenders want to lend with the amount borrowers want to borrow
- If dissaving is just negative saving, then market clearing means that aggregate saving is zero

The Market Real Interest Rate

Aggregating over the various consumers in the economy provides us with the Aggregate Supply (Lending) curve and Aggregate Demand (Borrowing) curve.

The intersection of these two curves illustrates the market clearing real rate of interest r.

Market Borrowing & Lending Equilibrium



Example

- 2 individuals A and B with utility $U(C_0,C_1)=(C_0C_1)^{1/2}$
- Endowments: A = (240,160); B = (320, 440)
- Determine equilibrium interest rate r such that aggregate saving is zero

Determinants of the Level of Real Interest Rates

- Societal Preferences
  - » The more present oriented are societal preferences, the higher the market r
    - – Shifts borrowing curve out
- Societal Endowments
  - » The more present oriented are societal endowments, the lower the market r
    - – Shifts lending curve out
- Productive Opportunities [see later]