FINANCE THEORY

THE FISHER MODEL

The Fisher Model

- Model of intertemporal choice involving consumption and investment decisions. (Named after Irving Fisher)
- Key Assumptions:
  - » Two periods (generalizing to many future periods is straightforward).
  - » Perfect capital markets
  - » the absence of uncertainty

I. Intertemporal Exchange Model: Outline

A. Objects of choice, endowments and trade opportunities, preferences

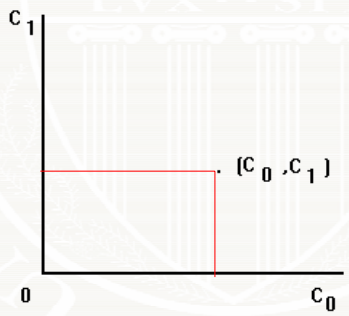
B. Individual optima and comparative statics

C. Market exchange equilibrium and the determination of interest rates.

Objects of choice

- What is the consumer choosing?
- One of the many possible “Consumption Streams”
- A consumption stream is a sequence of time dated consumption, for the present and and for the future. e.g. (C_0,C_1)
  - » C_0 is the standard of living or consumption level for period 0 (the present)
  - » C_1 is the standard of living or consumption level for period 1 (the future)

Representing a Consumption Stream



Consumer Preferences: Basic Assumptions

- Consumers are able to choose between alternative consumption streams.
- Choices are consistent (transitive)
- They prefer more consumption to less, i.e. they prefer higher standards of living to lower.
- Consumers choose the most preferred consumption stream among those attainable.

Ways to Represent Consumer Preferences

a. Simple ranking of consumption choices

b. Indifference curves
   - » downward sloping, non intersecting, and convex shape.

c. Utility function, $U(C_0,C_1)$

Utility Function, $U(C_0,C_1)$

- The utility function gives an index value for each consumption stream.
- The utility function value ranks consumption streams
- The marginal rate of substitution, MRS, gives:
  - » slope of an indifference curve at a point.
  - » the rate at which a consumer is willing to exchange future consumption for present consumption, (while maintaining the same level of satisfaction.)
  - » $MRS = -U_0/U_1$ where U_i is the marginal utility of consumption in the ith period.

Example: Preferences

- Sketch indifference curves for a person who has a high MRS (when measured at points of equal present and future consumption) and for a person who has a low MRS (at points of equal consumption through time).
- Sketch indifference curves for someone with a high but not perfect degree of substitutability between present and future consumption and for someone with low substitutability. (again measured at points of equal consumption through time.)

Characteristics of preferences

- Present oriented
- Future oriented
- High degree of substitutability
- Low degree of substitutability

From utility function to MRS

- Utility function or index $U=U(C_0,C_1)$
- Marginal utility tells us the rate at which utility changes when we change C_0, holding C_1 fixed.
- $U_0=\Delta U/\Delta C_0$ (holding C_1 fixed)

$$U_0 = \frac{\partial U}{\partial C_0}$$ the partial derivative of U with respect to C_0

The partial derivative holds constant other variables.

From utility function to MRS (continued)

- When C_0 and C_1 change, the change in utility is the sum of the changes in C_0 and C_1, each multiplied by their marginal utilities

$$dU = U_0 dC_0 + U_1 dC_1$$

But for changes along a given indifference curve, $dU = 0$

Solve to get $$\frac{dC_1}{dC_0} = -\frac{U_0}{U_1} = MRS$$

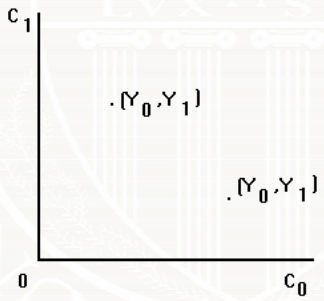
Consumer opportunities: Endowments

- Consumer's endowment is a claim to goods and services in the present and in the future.
- (Y_0,Y_1) represents the consumer's endowment
  - » Y_i is the endowment in the ith period.

Consumer's Endowment: Interpretation

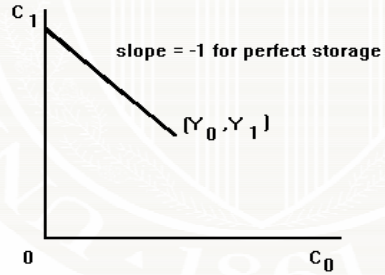
- The endowment might represent income that is expected in each of the two periods, from wages, from a pension trust, etc.
- The consumer can always choose a consumption stream equal to the endowment, but there may be other opportunities as well. e.g.through storage or by borrowing or lending.

Consumer Endowments



Storage

- Some of the *present* endowment is saved and *stored* for consumption in the next period.



Market Exchange: Borrowing or Lending

- The consumer can borrow or lend consumption claims between periods
- Must be consistent with the endowment, i.e. you can't borrow more than you can repay. No uncertainty, lender knows your capacity.
- The real interest rate = r.
- What consumption streams are possible?

Consumer's Budget Constraint: Lending

- If the consumer does not consume the entire present endowment, he or she can lend the amount $(Y_0 - C_0) = S_0$.
- This loan will be repaid with interest r
- Future consumption will be

  $C_1 = Y_1 + (1+r)(Y_0 - C_0)$.

Consumer's Budget Constraint: Borrowing

- To consume more than the present endowment the consumer must borrow $(C_0 - Y_0)$.
- The loan must be repaid with interest
- Future consumption will be

  $C_1 = Y_1 - (1+r)(C_0 - Y_0)$.
- Changing signs: $C_1 = Y_1 + (1+r)(Y_0 - C_0)$

Consumer's Budget Constraint

- $C_1 = Y_1 + (1+r)(Y_0 - C_0)$
  - » Covers both lending and borrowing because $(Y_0 - C_0)$ changes sign.
- Rewrite as:

  $C_1 = (1+r)Y_0 + Y_1 - (1+r)\,C_0$ or as

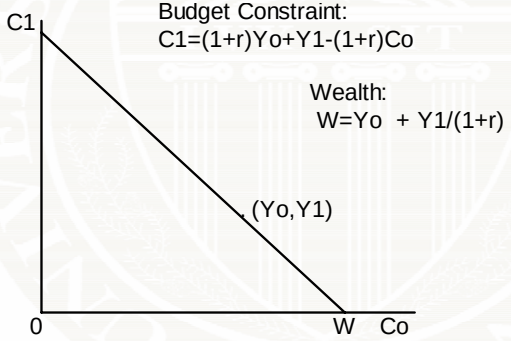
$$C_0 + \frac{C_1}{(1+r)} = Y_0 + \frac{Y_1}{(1+r)} = W$$

Wealth

- What is the maximum present consumption that can be obtained with a given endowment, when we leave no resources for the future?
- Set the C_1 variable to zero in the budget constraint and solve for C_0:

  $C_0 = Y_0 + Y_1/(1+r) = W$

Budget Constraint and Wealth



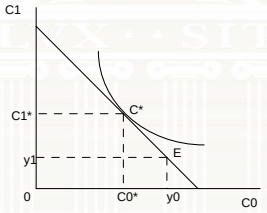
Budget Constraint and Wealth

- Consumers can attain (choose) any point on or inside the budget line.
- The line goes through the endowment point (Y_0,Y_1) ,has slope -(1+r).
- The horizontal intercept gives the consumer's wealth, W.

Class Exercise

- Person has endowment
  - » $(y_0,y_1)=(10000,11000)$
- Real interest rate is r=.10
- Find the person's wealth
- Find the future value of the endowment
- Write an equation for the budget constraint. Sketch it, i.e. indicate slope, intercepts.

The Consumer Optimum



- This person saves $S_0=y_0-C_0^*$ and lends it
- Repayment with interest is $C_1^*-y_1$
- $y_1+(C_1^*-y_1)=C_1^*$ the person's future consumption

Characteristics of the optimum

- The optimum consumption stream (C_0^*,C_1^*) must satisfy

1. the budget constraint
   $C_0+C_1/(1+r)=W$
   or $C_1=(1+r)y_0+y_1-(1+r)C_0$ and

2. slope of IC = MRS = $-U_0/U_1$ = $-(1+r)$ = slope of BC

Mathematical Example $U=U(C_0,C_1)$

$U = C_0^{0.5}C_1^{0.5}$ (a specific utility function)

$$MRS = -\frac{U_0}{U_1} = -\frac{C_1}{C_0}$$

$$1) C_0 + \frac{C_1}{1+r} = W = Y_0 + \frac{Y_1}{1+r}$$

$$2) MRS = -\frac{C_1}{C_0} = -(1+r) = \text{slope of B.C.}$$

Solve for C_0,C_1 in terms of r,W (or r, Y_0, and Y_1)

The solution:

$C_1 = C_0(1+r)$ from 2)

Substitute into 1):

$$C_0 + \frac{C_0(1+r)}{1+r} = W$$

$$2C_0 = W$$

$$C_0^* = \frac{1}{2}W \qquad C_1^* = \frac{1}{2}W(1+r)$$

See example Spreadsheet econ422Utility.xls on class notes page for numerical example using Excel

Formal Optimization Problem

$$\max_{C_0, C_1} U(C_0, C_1) \text{ subject to}$$

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}$$

Solution Using Lagrange Multipliers

Step 1: Put constraint in homogeneous form

$$C_0 + \frac{C_1}{(1+r)} = W \Rightarrow C_0 + \frac{C_1}{(1+r)} - W = 0$$

Step 2: Form the Lagrangian function

$$L(C_0, C_1, \lambda) = U(C_0, C_1) + \lambda\left[C_0 + \frac{C_1}{1+r} - W\right]$$

Solution Using Lagrange Multipliers

Step 3: Maximize Lagrangian function

$$\max_{C_0, C_1, \lambda} L(C_0, C_1, \lambda)$$

First order conditions

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial C_0} = 0$$

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial C_1} = 0$$

$$\frac{\partial L(C_0, C_1, \lambda)}{\partial \lambda} = 0$$

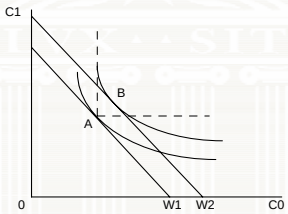
Borrowers vs. Lenders

- Individuals with strong preferences toward future consumption will be lenders
- Individuals with strong preferences toward current consumption will be borrowers

Comparative Statics for the Fisher Exchange Model

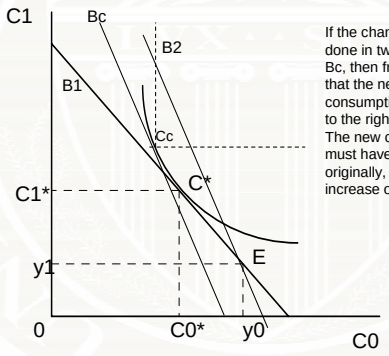
- What happens to the consumer optimum when the constraint changes?
  - » Start with an original optimum
  - » Change something
  - » Find the new optimum
  - » Compare it with the original
- In this model we can change:
  (a)The endowment or (b)the interest rate

Effect of a Wealth Change with Fixed r



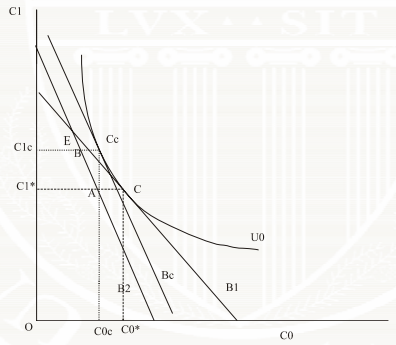
- With wealth W_1, the optimum is at A
- When the wealth increases to W_2, the new optimum is at B
- If both goods are “normal,” B must be above and to the right of A.

Comparative Statics: Increase in Interest Rate r for a Lender



If the change from B1 to B2 is done in two steps, first from B1 to Bc, then from Bc to B2, we see that the new optimal consumption must be above and to the right of Cc.
The new consumption stream must have more C1 than originally, but C0 may either increase or decrease.

Comparative Statics: Increase in Interest Rate r for a Borrower



Summary of Comparative Statics Results: Changes in the interest rate r

Results for a lender:

Variable:	W	C_0	C_1	S_0	U
$r\uparrow$	↓	?	↑	?	↑
$r\downarrow$	↑	?	↓	?	↓

Results for a borrower:

Variable:	W	C_0	C_1	S_0	U
$r\uparrow$	↓	↓	?	↑	↓
$r\downarrow$	↑	↑	?	↓	↑

The Market Equilibrium Interest Rate

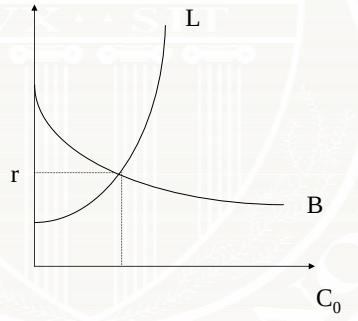
- Lenders need borrowers and vice versa
- Market clearing means that there is a match between the amount lenders want to lend with the amount borrowers want to borrow
- If dissaving is just negative saving, the market clearing means that aggregate saving is zero

The Market Real Interest Rate

Market Borrowing & Lending Equilibrium

Aggregating over the various consumers in the economy provides us with the Aggregate Supply (Lending) curve and Aggregate Demand (Borrowing) curve.

The intersection of these two curves illustrates the market clearing real rate of interest r.



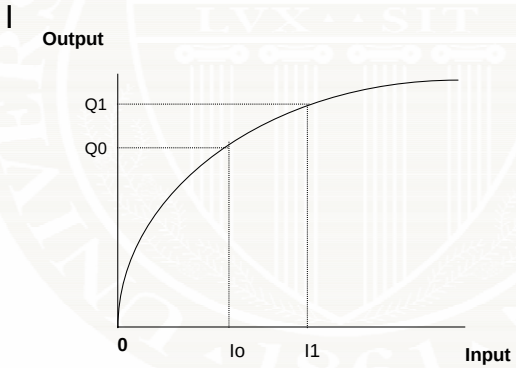
Determinants of the Level of Real Interest Rates

- Societal Preferences
  - » The more present oriented are societal preferences, the higher the market r
    - – Shifts borrowing curve out
- Societal Endowments
  - » The more present oriented are societal endowments, the lower the market r
- Productive Opportunities [see later]

II. Intertemporal Production (Real Investment) and Exchange: Outline

A. Individual optima
B. The Fisher Separation Theorem
C. Market equilibrium
D. Applications and examples

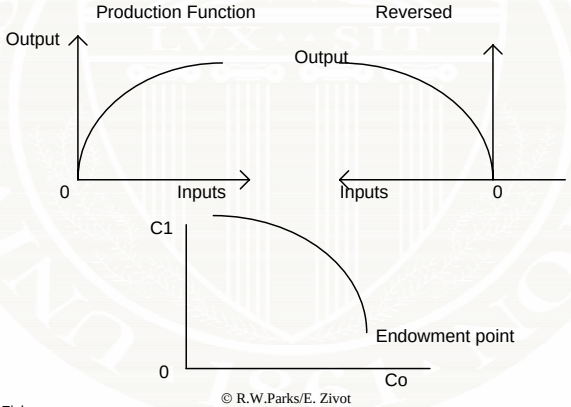
An Intertemporal Production Function Shows the Relationship Between Inputs Today and Outputs in the Future



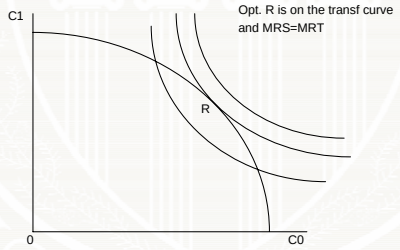
From Production Function to Fisher Diagram

- Reverse the production function around the vertical axis
- Drag the origin to the endowment point of the Fisher diagram

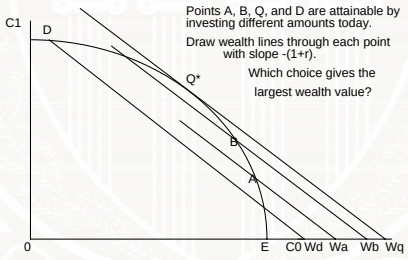
Production Function to Fisher Diagram



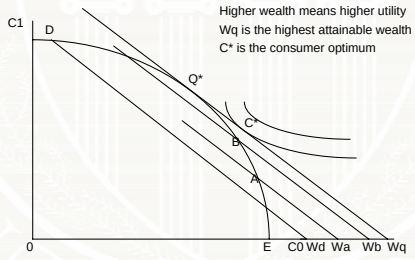
Consumer-Investor Optimum With No Financial Market



Investor Optimum with Financial Market



Consumer- Investor Optimum with Financial Market



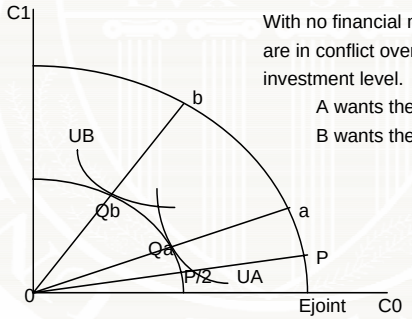
The Fisher Separation Theorem

- The consumer - investor's two-fold problem of determining the optimal level of investment and the optimal consumption stream can be separated into two steps:
  - » First choose the investment level that maximizes wealth. This choice does not depend on preferences.
  - » Next determine the optimum consumption stream, based on the maximized wealth.

The Fisher Separation Theorem (continued)

- The choice of optimal investment can be separated from the choice of optimal consumption. i.e it does not depend on investor preferences.
- A necessary condition for utility maximization is wealth maximization.
- Note: The separation result depends on the existence of a perfect capital market. Investors can borrow or lend at the market rate r.

Markets Resolve Differences in Preferences:Partners and Corporations

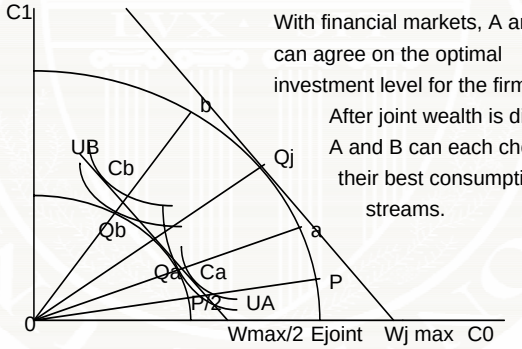


With no financial markets, A and B are in conflict over the optimal investment level.

A wants the company to be at a.

B wants the company to be at b.

Markets Resolve Differences in Preferences:Partners and Corporations

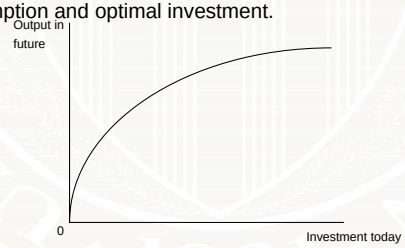


With financial markets, A and B can agree on the optimal investment level for the firm at Qj.

After joint wealth is divided A and B can each choose their best consumption streams.

Development or venture capital

- Endowment (0,0).
- Knowledge about a productive opportunity.(see below)
- Access to capital market at rate r.
- Apply the two period Fisher analysis and find the optimal consumption and optimal investment.



Determinants of the Level of Real Interest Rates Revisited

- Societal Preferences
  - » The more present oriented are societal preferences, the higher the market r
    - – Shifts Borrowing curve out
- Societal Endowments
  - » The more present oriented are societal endowments, the lower the market r
    - – Shifts lending curve out
- Productive Opportunities
  - » The more productive are the opportunities for converting present into future resources through real investment (i.e. production), the higher the market r.

Real versus Nominal Interest Rates

- Fisher model uses real interest rates.
- Real interest rates indicate the rate at which goods at one date exchange for goods at another.
- Nominal interest rates refer to the rate at which dollars at one date exchange for dollars at another

Relationship b/w Nominal & Real Interest Rates

Recall from Fisher Model:

MRS = $\Delta C_1/\Delta C_0$ = (1+r), r = real interest rate

Multiplying thru by nominal price ratio P_1/P_0:

$\Delta C_1/\Delta C_0 * (P_1/P_0)$ = (1+r)*(P_1/P_0) = $(1 + r_n)$

$P_1 = (P_0 + \Delta P_0)$ therefore: $P_1/P_0 = (P_0 + \Delta P_0)/P_0 = 1 +$ *inflation rate* $= 1 + \Pi$

$\Delta C_1/\Delta C_0 * (P_1/P_0)$ = $(1 + r_n)$= (1+r)*$(1 + \Pi)$ ***Fisher Equation***

$(1 + r_n)$= 1+r +r* Π+ $\Pi \approx$ 1+r + Π

$r_n \approx r + \Pi$

The nominal interest rate is approximately equal to the real interest rate plus the rate of inflation

Global Comparison of Nominal Interest Rates

Country	ST Interest Rates (% p.a.)	Country	ST Interest Rates (% p.a.)
Australia	5.58%	Taiwan	1.10%
Britain	4.13%	Thailand	1..34%
Canada	2.27%	Argentina	4.94%
Denmark	2.15%	Brazil	16.30%
Japan	0.03%	Chile	1.68%
Sweden	2.51%	Columbia	7.99%
Switzerland	0.24%	Mexico	7.56%
United States	1.02%	Peru	2.55%
Germany	2.09%	Venezuela	N/A
China	N/A	Egypt	6.89%
Hong Kong	0.13%	Israel	1.26%
India	4.24%	South Africa	8.10%
Indonesia	8.18%	Turkey	23.0%
Malaysia	3.03%	Czech Republic	2.07%
Philippines	7.63%	Hungary	12.68%
Singapore	0.75%	Poland	5.47%
South Korea	4.18%	Russia	14.0%

Source: The Economist, February 2004

Global Comparison of Inflation Rates (% change on year ago)

Country	Consumer Price Change	Country	Consumer Price Change
Australia	2.4%	Taiwan	0.6%
Britain	1.3%	Thailand	2.2%
Canada	1.2%	Argentina	2.3%
Denmark	0.9%	Brazil	6.7%
Japan	-0.3%	Chile	0.8%
Sweden	0.8%	Columbia	6.3%
Switzerland	0.1%	Mexico	4.2%
United States	1.7%	Peru	3.4%
Germany	0.9%	Venezuela	21.9%
China	2.1%	Egypt	5.5%
Hong Kong	-1.5%	Israel	-2.5%
India	4.3%	South Africa	0.2%
Indonesia	4.6%	Turkey	14.3%
Malaysia	0.9%	Czech Republic	2.3%
Philippines	3.4%	Hungary	7.1%
Singapore	1.3%	Poland	1.6%
South Korea	3.3%	Russia	10.8%

Source: The Economist, February 2004