Analysis of Structural Time Series Models Using SsfPack in S+FinMetrics 2.0

Eric Zivot

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Abstract

To be completed.

1 Structural Time Series Models

The basic univariate unobserved components *structural time series model* (STSM) for a time series y_t has the form

$$
\begin{aligned}
y_t &= \mu_t + \gamma_t + \psi_t + \xi_t \\
&\quad \xi_t \sim GWN(0, \sigma_\xi^2)
\end{aligned}
\tag{1}
$$

where μ_t represents the unobserved *trend* component, γ_t represents the unobserved *seasonal* component, ψ_t represents the unobserved *cycle* component, and ξ_t represents the unobserved *irregular* component.

1.1 Trend component specification

The nonstationary trend component μ_t has the form of a *local linear trend*:

$$
\begin{aligned}
\mu_{t+1} &= \mu_t + \beta_t + \eta_t, \ \eta_t \sim GWN(0, \sigma_\eta^2) \tag{2} \\
\beta_{t+1} &= \beta_t + \varsigma_t, \ \varsigma_t \sim GWN(0, \sigma_\varsigma^2) \tag{3}
\end{aligned}
$$

with $\mu_1 \sim N(0, \kappa)$ and $\beta_1 \sim N(0, \kappa)$ where κ is a large number, e.g. $\kappa = 10^6$. If $\sigma_\varsigma^2 = 0$ then μ_t follows a random walk with drift β_1. If both $\sigma_\varsigma^2 = 0$ and $\sigma_\eta^2 = 0$ then μ_t follows a linear deterministic trend. Some special trend specifications are listed in Table 1.

Level Specification	σ_ξ	σ_η	
constant term	*	0	
local level (LL)	*	*	
random walk (RW)	0	*	
Trend Specification	σ_ξ	σ_η	σ_ς
deterministic	*	0	0
LL with fixed slope	*	*	0
RW with fixed drift	0	*	0
local linear (LLT)	*	*	*
smooth trend	*	0	*
second differencing	0	0	*
Hodrick-Prescott	*	0	$0.025\sigma_\xi$

Asterix * indicates any positive value

Table 1: Some special level and trend specifications

1.2   Seasonal component specification

The seasonal component may have a dummy variable form, a trigonometric form, or a Harrison-Stevens form. The number of seasonal frequencies in a period of time, which is usually a year, is given by the integer s. When s is even, $[s/2] = s/2$; when s is odd, $[s/2] = (s-1)/2$. The dummy variable form of the stochastic seasonal component γ_t is

$$
\begin{aligned}
S(L)\gamma_t &= \omega_t, \ \omega_t \sim GWN(0, \sigma_\omega^2) \\
S(L) &= 1 + L + \cdots + L^{s-1}
\end{aligned}
\tag{4}
$$

When $\sigma_\omega^2 = 0$, the seasonal component becomes fixed. In this case, the seasonal component sums to zero over the year. This ensures that the seasonal effect is not confounded with the other components.

The trigonometric form of γ_t is

$$
\gamma_t = \sum_{j=1}^{[s/2]} \gamma_{j,t}
$$

where each $\gamma_{j,t}$ is generated by

$$
\begin{pmatrix} \gamma_{j,t+1} \\ \gamma_{j,t+1}^* \end{pmatrix} = \begin{pmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{pmatrix} \begin{pmatrix} \gamma_{j,t} \\ \gamma_{j,t}^* \end{pmatrix} + \begin{pmatrix} \omega_{j,t+1} \\ \omega_{j,t+1}^* \end{pmatrix}
$$

where $\lambda_j = 2\pi j/s$ is the jth seasonal frequency, in radians, and the seasonal disturbances $\omega_{j,t}$ and $\omega_{j,t}^*$ are mutually uncorrelated Gaussian white noise random variables with common variance σ_ω^2

$$
\begin{pmatrix} \omega_{j,t} \\ \omega_{j,t}^* \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_\omega^2 \mathbf{I}_2 \right), \ j = 1, \ldots, s/2
$$

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For simplicity, a common variance σ_ω^2 is assumed for each trigonometric term. However, this assumption can be relaxed and the state space representation may involve separate variances for each trigonometric term. For s even, the component at $j = s/2$ collapses to

$$\gamma_{j,t+1} = \gamma_{j,t} \cos(\lambda_j) + \omega_{j,t+1}$$

The dummy and trigonometric specifications for γ_j have different dynamic properties; see Harvey (1989, p. 56). For example, the trigonometric seasonal process evolves more smoothly; it can be shown that the sum of the seasonals over the past year follows an $\mathrm{MA}(s-2)$ rather than white noise.

In the Harrison-Stevens seasonal specification, (see Harrison and Stevens, 1976) all seasonal effects γ_j $(j = 1\ldots, s)$ are assumed to follow a random walk

$$\begin{pmatrix} \gamma_{1,t+1} \\ \vdots \\ \gamma_{s,t+1} \end{pmatrix} = \begin{pmatrix} \gamma_{1,t} \\ \vdots \\ \gamma_{s,t} \end{pmatrix} + \begin{pmatrix} \omega_{1,t+1} \\ \vdots \\ \omega_{s,t+1} \end{pmatrix}$$

or

$$\boldsymbol{\gamma}_{t+1} = \boldsymbol{\gamma}_t + \boldsymbol{\omega}_t$$

where

$$\boldsymbol{\omega}_t \sim N\left(0, \sigma_\omega^2 \left(\frac{s\mathbf{I}_s - \mathbf{1}_s \mathbf{1}_s'}{s-1} \right) \right)$$

The specific covariance structure between the s disturbance terms enforces the seasonal effects to sum to zero over the previous year. Also, the covariances between the s seasonal disturbances are equal.

1.3   Cycle component specification

The stochastic cycle component ψ_t is specified as

$$\begin{pmatrix} \psi_{t+1} \\ \psi_{t+1}^* \end{pmatrix} = \rho \begin{pmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{pmatrix} \begin{pmatrix} \psi_t \\ \psi_t^* \end{pmatrix} + \begin{pmatrix} \chi_t \\ \chi_t^* \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \chi_t \\ \chi_t^* \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_\psi^2(1-\rho^2)\mathbf{I}_2 \right)$$

where $\psi_0 \sim N(0, \sigma_\psi^2)$, $\psi_0^* \sim N(0, \sigma_\psi^2)$ and $cov(\psi_0, \psi_0^*) = 0$. The parameter $\rho \in (0,1]$ is interpreted as a damping factor. The frequency of the cycle is $\lambda_c = 2\pi/c$ and c is the period. When $\rho = 1$ the cycle reduces to a deterministic sine-cosine wave.

2 Specifying the State Space Form

The linear Gaussian *state space model* may be represented as the system of equations

$$\underset{m\times1}{\boldsymbol{\alpha}_{t+1}} = \underset{m\times1}{\mathbf{d}_t} + \underset{m\times m}{\mathbf{T}_t} \cdot \underset{m\times1}{\boldsymbol{\alpha}_t} + \underset{m\times r}{\mathbf{H}_t} \cdot \underset{r\times1}{\boldsymbol{\eta}_t} \tag{6}$$

$$\underset{N\times1}{\boldsymbol{\theta}_t} = \underset{N\times1}{\mathbf{c_t}} + \underset{N\times m}{\mathbf{Z_t}} \cdot \underset{m\times1}{\boldsymbol{\alpha_t}} \tag{7}$$

$$\underset{N\times1}{\mathbf{y}_t} = \underset{N\times1}{\boldsymbol{\theta}_t} + \underset{N\times N}{\mathbf{G}_t} \cdot \underset{N\times1}{\boldsymbol{\varepsilon}_t} \tag{8}$$

where $t = 1, \ldots, n$ and

$$\boldsymbol{\alpha}_1 \sim N(\mathbf{a}, \mathbf{P}), \tag{9}$$

$$\boldsymbol{\eta}_t \sim \text{iid } N(0, \mathbf{I}_r) \tag{10}$$

$$\boldsymbol{\varepsilon}_t \sim \text{iid } N(\mathbf{0}, \mathbf{I}_N) \tag{11}$$

and it is assumed that

$$E[\boldsymbol{\varepsilon}_t \boldsymbol{\eta}_t'] = \mathbf{0}$$

In (9), $\mathbf{a}$ and $\mathbf{P}$ are fixed and known but that can be generalized. The *state vector* $\boldsymbol{\alpha}_t$ contains unobserved stochastic processes and unknown fixed effects and the *transition equation* (6) describes the evolution of the state vector over time using a first order Markov structure. The *measurement equation* (8) describes the vector of observations $\mathbf{y}_t$ in terms of the state vector $\boldsymbol{\alpha}_t$ through the signal $\boldsymbol{\theta}_t$ and a vector of disturbances $\boldsymbol{\varepsilon}_t$. It is assumed that the innovations in the transition equation and the innovations in the measurement equation are independent, but this assumption can be relaxed. The deterministic matrices $\mathbf{T}_t, \mathbf{Z}_t, \mathbf{H}_t, \mathbf{G}_t$ are called *system matrices* and are usually sparse selection matrices. The vectors $\mathbf{d}_t$ and $\mathbf{c}_t$ contain fixed components and may be used to incorporate known effects or known patterns into the model; otherwise they are equal to zero.

The state space model (6)-(11) may be compactly expressed as

$$\begin{pmatrix} \boldsymbol{\alpha}_{t+1} \\ \mathbf{y}_t \end{pmatrix} = \underset{(m+N)\times1}{\boldsymbol{\delta}_t} + \underset{(m+N)\times m}{\boldsymbol{\Phi}_t} \cdot \underset{m\times1}{\boldsymbol{\alpha}_t} + \underset{(m+N)\times1}{\mathbf{u}_t}, \tag{12}$$

$$\boldsymbol{\alpha}_1 \sim N(\mathbf{a}, \mathbf{P}) \tag{13}$$

$$\mathbf{u}_t \sim \text{iid } N(\mathbf{0}, \boldsymbol{\Omega}_t) \tag{14}$$

where

$$\boldsymbol{\delta}_t = \begin{pmatrix} \mathbf{d}_t \\ \mathbf{c}_t \end{pmatrix}, \boldsymbol{\Phi}_t = \begin{pmatrix} \mathbf{T}_t \\ \mathbf{Z}_t \end{pmatrix}, \mathbf{u}_t = \begin{pmatrix} \mathbf{H}_t \boldsymbol{\eta}_t \\ \mathbf{G}_t \boldsymbol{\varepsilon}_t \end{pmatrix}, \boldsymbol{\Omega}_t = \begin{pmatrix} \mathbf{H}_t \mathbf{H}_t' & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_t \mathbf{G}_t' \end{pmatrix}$$

The initial value parameters are summarized in the $(m+1) \times m$ matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{P} \\ \mathbf{a}' \end{pmatrix} \tag{15}$$

State Space Parameter	List Component Name
$\boldsymbol{\delta}$	mDelta
$\boldsymbol{\Phi}$	mPhi
$\boldsymbol{\Omega}$	mOmega
$\boldsymbol{\Sigma}$	mSigma

Table 2: S+FinMetrics/SsfPack state space form list components

For multivariate models, i.e. $N > 1$, it is assumed that the $N \times N$ matrix $\mathbf{G}_t \mathbf{G}_t'$ is diagonal. In general, the system matrices in (12) are time varying.

The variance matrix $\mathbf{P}$ of the initial state vector $\boldsymbol{\alpha}_1$ is assumed to be of the form

$$\mathbf{P} = \mathbf{P}_* + \kappa \mathbf{P}_\infty \tag{16}$$

where $\mathbf{P}_\infty$ and $\mathbf{P}_*$ are symmetric $m \times m$ matrices with ranks r_∞ and r_*, respectively, and κ is a large scalar value, e.g. $\kappa = 10^6$. The matrix $\mathbf{P}_*$ captures the covariance structure of the stationary components in the initial state vector, and the matrix $\mathbf{P}_\infty$ is used to specify the initial variance matrix for nonstationary components. When the ith diagonal element of $\mathbf{P}_\infty$ is negative, the corresponding ith column and row of $\mathbf{P}_*$ are assumed to be zero, and the corresponding row and column of $\mathbf{P}_\infty$ will be taken into consideration. When some elements of state vector are nonstationary, the S+FinMetrics/SsfPack algorithms implement an "exact diffuse prior" approach as described in Durbin and Koopman (2001) and Koopman, Shephard and Doornik (2001).

2.1 State Space Representation in S+FinMetrics/SsfPack

State space models in S+FinMetrics/SsfPack utilize the compact representation (12) with initial value information (15). In S+FinMetrics/SsfPack, a state space model is specified by creating either a list variable with components giving the minimum components necessary for describing a particular state space form or by creating an "ssf" object. In the list variable representing a state space model, the component names must match the state space form parameters in (12) and (15) using the naming convention summarized in Table 2 The four components in Table must be specified for any state space model. The S+FinMetrics function CheckSsf takes a list variable with a minimum state space form, coerces the components to matrix objects and returns an "ssf" object giving the full parameterization of a state space model used in many of the S+FinMetrics/SsfPack state space modeling functions.

The S+FinMetrics/SsfPack function GetSsfStsm creates the state space system matrices for the univariate STSM (1). The arguments expected by GetSsfStsm are

```
> args(GetSsfStsm)
function(irregular = 1, level = 0.1, slope = NULL,
```

Argument	STSM parameter
`irregular`	σ_η
`level`	σ_ξ
`slope`	σ_ς
`seasonalDummy`	σ_ω, s
`seasonalTrig`	σ_ω, s
`seasonalHS`	σ_ω, s
`cycle0`	σ_ψ, λ_c, ρ
$\vdots$	$\vdots$
`cycle9`	σ_ψ, λ_c, ρ

Table 3: Arguments to the `S+FinMetrics` function `GetSsfStsm`

```
seasonalDummy = NULL, seasonalTrig = NULL, seasonalHS
= NULL, cycle0 = NULL, cycle1 = NULL, cycle2 = NULL,
cycle3 = NULL, cycle4 = NULL, cycle5 = NULL, cycle6 =
NULL, cycle7 = NULL, cycle8 = NULL, cycle9 = NULL)
```

These arguments and the relationships to the component parameters of the STSM are explained in Table 3.

If a component argument has a `NULL` value, then that component is excluded from the STSM. If the standard deviation of a component is set equal to zero, then the component becomes deterministic. The seasonal components are specified by a two element vector, with the first element giving the standard deviation of the stochastic seasonal and the second element giving the number of seasons. The cycle components are specified by a three element vector, with the first element giving the standard deviation of the stochastic cycle, the second element giving the frequency of the cycle, and the third element giving the damping factor. The function `GetSsfStsm` creates a list variable with the appropriate system matrices for the specified STSM.

2.2   Simulating Observations

Once a STSM has been specified, simulated observations from the model can be generated using the S+FinMetrics function SsfSim. The arguments expected by `SsfSim` are:

```
> args(SsfSim)
function(ssf, n = 100, mRan = NULL, seed = NULL, a1 = NULL)
```

where `ssf` represents either a list with components giving a minimal state space form or a valid "ssf" object, `n` is the number of simulated observations, `mRan` is user-specified matrix of disturbances, and `a1` is the initial state vector.

The following sub-sections describe the specification of some common STSMs using the S+FinMetrics/SsfPack functions.

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2.3 Local Level Model

Consider the following simple model for the stochastic evolution of the level of a variable y_t

$$\mu_{t+1} = \mu_t + \eta_t, \ \eta_t \sim \text{iid } N(0, \sigma_\eta^2) \tag{17}$$

$$y_t = \mu_t + \varepsilon_t, \ \varepsilon_t \sim \text{iid } N(0, \sigma_\varepsilon^2) \tag{18}$$

$$\alpha_1 \sim N(a, P) \tag{19}$$

where it is assumed that $E[\varepsilon_t \eta_t] = 0$. In the above model, the observed variable y_t is the sum of two unobserved components, μ_t and ε_t^*. The component μ_t is the state variable and represents the fundamental value (signal) of the variable. The transition equation (17) shows that the fundamental values evolve according to a random walk. The component ε_t^* represents random deviations (noise) from the fundamental value that are assumed to be independent from the innovations to μ_t. The strength of the signal in the fundamental value relative to the random deviation is measured by the signal-to-noise ratio of variances $q = \sigma_\eta^2 / \sigma_\varepsilon^2$. The model (17)-(19) is called the *random walk plus noise model, signal plus noise model* or the *local level model*.[1]

The state space form (12) of the local level model has time invariant parameters

$$\boldsymbol{\delta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Phi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Omega = \begin{pmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix} \tag{20}$$

Since the state variable μ_t is $I(1)$, the unconditional distribution of the initial state μ_1 doesn't have finite variance. In this case, it is customary to set $\mu = E[\mu_1] = 0$ and $P = \text{var}(\mu_1)$ to some large positive number, e.g. $P = 10^7$, in (19) to reflect that no prior information is available. Using (16), the initial variance is specified with $P_* = 0$ and $P_\infty = 1$. Therefore, the initial state matrix (15) for the local level model has the form

$$\Sigma = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{21}$$

where -1 implies that $P_\infty = 1$.

The local level model may also be parameterized in terms of the signal-to-noise ratio $q = \sigma_\eta^2 / \sigma_\varepsilon^2$ and the noise variance σ_ε^2. This parameterization is often preferred for estimation using the concentrated log-likelihood.

Example 1 *State space form of local level model*

The state space for the local level model (17)-(19) with $\sigma_\xi = 1$ and $\sigma_\eta = 0.5$ may be constructed using

---

[1]A detailed technical analysis of this model is given in Durbin and Koopman (2001), chapter 2.

```
> ssf.stsm = GetSsfStsm(irregular=1, level=0.5)
> class(ssf.stsm)
[1] "list"
> ssf.stsm
$mPhi:
      [,1]
[1,]     1
[2,]     1

$mOmega:
      [,1] [,2]
[1,] 0.25    0
[2,] 0.00    1

$mSigma:
      [,1]
[1,]    -1
[2,]     0
```

The arguments `irregular=1` and `level=0.5` specify $\sigma_\varepsilon = 1$ and $\sigma_\eta = 0.5$ in (18) and (19), respectively.

Simulated data from the local level model may be constructed using the S+FinMetrics function `SsfSim`.

2.4   Local Linear Trend Model

2.5   Seasonal Models

Example 2 *Seasonal dummy model*

Consider the following local-level model with a quarterly stochastic dummy variable specification

$$
\begin{aligned}
y_t &= \mu_t + \gamma_t + \xi_t \\
\mu_{t+1} &= \mu_t + \eta_t \\
\gamma_{t+1} &= -\gamma_t - \gamma_{t-1} - \gamma_{t-2} + \omega_t \\
&\begin{pmatrix} \xi_t \\ \eta_t \\ \omega_t \end{pmatrix} \sim N \begin{pmatrix} \sigma_\xi^2 & 0 & 0 \\ 0 & \sigma_\eta^2 & 0 \\ 0 & 0 & \sigma_\omega^2 \end{pmatrix}
\end{aligned}
$$

The state space form for this model with $\sigma_\xi = 1$, $\sigma_\eta = 0.2$, $\sigma_\omega = 0.2$ and initial seasonal values $\gamma_1 = 2.5, \gamma_0 = 5.0, \gamma_{-1} = -2.5$ is created using

```
> ssf.lltdum = GetSsfStsm(irregular=1,
+                         level=0.5,
+                         slope=0.1,
+                         seasonalDummy = c(0.2, 4))
> ssf.lltdum$mSigma[6,3:5] = c(2.5, 5.0, -2.5)
> ssf.lltdum
$mPhi:
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    1    0    0    0
[2,]    0    1    0    0    0
[3,]    0    0   -1   -1   -1
[4,]    0    0    1    0    0
[5,]    0    0    0    1    0
[6,]    1    0    1    0    0


$mOmega:
     [,1] [,2] [,3] [,4] [,5] [,6]
[1,] 0.25 0.00 0.00    0    0    0
[2,] 0.00 0.01 0.00    0    0    0
[3,] 0.00 0.00 0.04    0    0    0
[4,] 0.00 0.00 0.00    0    0    0
[5,] 0.00 0.00 0.00    0    0    0
[6,] 0.00 0.00 0.00    0    0    1


$mSigma:
     [,1] [,2] [,3] [,4] [,5]
[1,]   -1    0  0.0    0  0.0
[2,]    0   -1  0.0    0  0.0
[3,]    0    0 -1.0    0  0.0
[4,]    0    0  0.0   -1  0.0
[5,]    0    0  0.0    0 -1.0
[6,]    0    0  2.5    5 -2.5
```

Eighty simulated observations from the model are computed and plotted using

```
> lltdum.sim = SsfSim(ssf.lltdum, n=80, seed=10)
> seriesPlot(lltdum.sim, one.plot=F)
```

Example 3 *Trigonometric seasonal model*

Example 4 *Harrison-Stevens seasonal model*

3 ML Estimation

3.1 Concentrated Likelihood Function

3.2 Examples

Example 5 *Local level model*

References

[1] Durbin, J., and Koopman, S.J. (2003). State Space, Cambridge University Press.

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[3] Harvey, A. (1989). *Forecasting, Structural Time Series and the Kalman Filter*. MIT Press, Cambridge, MA.

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