EVANESCE Implementation in S-PLUS FinMetrics Module

July 2, 2002

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The Extreme Value Analysis Employing Statistical Copula Estimation (EVANESCE) library for S-PLUS FinMetrics module provides a set of functions for bivariate extreme value analyses using parametric and non-parametric copula estimation methods. It is contributed by Rene Carmona and described in Carmona (2001) and Carmona and Morrison (2001). Some of the original functions are renamed and model objects restructured when the library is incorporated into FinMetrics to be consistent with the other extreme value theory library in the package (EVIS by Alexander McNeil). This document gives an overview of the copula concept and their implementation. The detailed function documentation is available both in S-PLUS FinMetrics Reference Manual and the product online help.

1. About Copulas

Suppose we are interested in modeling the stochastic behavior of two random variables X and Y based on a set of n independent observations of the couple (X, Y), say $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$. Furthermore, suppose we have estimated their respective marginal distributions, $G_1(x)$ and $G_2(y)$, using standard statistical techniques. We are then interested to know how we test to know whether X and Y are independent, and how we describe their dependence structure if they are not independent.

One way to describe the dependence structure between two random variables is to estimate and use their joint distribution function

$$F(x, y) = I\!P\{X \le x, Y \le y\}$$

A test of independence between X and Y is simply to test whether F(x, y) is a product of the respective marginal distributions.

To construct a measure of dependence structure we introduce two uniform distribution random variables $G_1(X)$ and $G_2(Y)$, and their joint distribution is

$$C(u, v) = I\!P\{G_1(X) \le u, G_2(Y) \le v\} = F(G_1^{-1}(u), G_2^{-1}(v)),$$

where G_1^{-1} and G_2^{-1} are the quantile functions of X and Y respectively

$(G_1^{-1}(q) = \inf\{x \in I\!R : G(x) > q\})$.

If we can estimate the function C, we have F, i.e.

$$F(x, y) = C(G_1(x), G_2(y))$$

C is a two-dimensional copula, i.e. a cumulative distribution function of two random variables with uniform (0, 1) marginals. Nelsen [1999] and Joe [1997] are the two recently published textbooks on the subject.

Note that if X and Y are continuous random variables, the function C satisfying the above definition is unique. If X and Y are not continuous random variables, C is uniquely determined on RangeG1 × RangeG2 (this result is known as Sklar's theorem [Sklar, 1959], see also Nelsen [1999], p. 15). In order to estimate F, we can transform the observations of X and Y , $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, into observations of $U = G_1(X)$ and $V = G_2(Y)$, $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$, where $u_i = G_1(xi)$ and $v_i = G_2(yi)$, $i = 1, 2, \ldots, n$, and C may then be estimated as the joint distribution of U and V . Then we have an estimate of F. There are a few non-parametric methods developed to estimate the copula [Deheuvels, 1978; Genest and Rivest, 1993; Genest et al., 1995; Cap´era`a et al., 1997]. One can also assume that the copula has a particular parametric form and estimate its parameters using, for example, the method of maximum likelihood.

C is a two-dimensional copula if and only if it is a function

$C(u, v) : \mathbf{I}^2 \mapsto \mathbf{I} \;\; (\mathbf{I} = [0, 1])$

that satisfies the following two conditions:

- For every u and v in $\mathbf{I}$:

  $$C(u, 0) = C(0, v) = 0, \qquad C(u, 1) = u, \qquad \text{and} \qquad C(1, v) = v;$$

- For every $u_1 \leq u_2$, and $v_1 \leq v_2$, and $u_1, u_2, v_1, v_2 \in \mathbf{I}$, the following inequality holds:

  $$C(u_1, v_1) - C(u_2, v_1) - C(u_1, v_2) + C(u_2, v_2) \geq 0$$

If C is considered to be a distribution function of two random variables U and V, the first condition ensures that U and V have uniform marginal distributions. The second condition, often referred to as the rectangular inequality, simply requires that C is a valid distribution function, i.e. $\mathbb{P}\{u_1 \leq U \leq u_2, v_1 \leq V \leq v_2\} \geq 0$

A variety of parametric copula families are supported in EVANESCE. The most popular measures of dependence structure of copulas, for example Kendall’s tau and Spearman’s rho, and tail dependence index are implemented in EVANESCE. Algorithms suggested by Genest and Mackay [1986] and Cap´era`a et al. [2000] for the generation of random pairs from certain parametric copulas are implemented in EVANESCE too.

2. Parametric Copula Classes and Empirical Copulas

There are 16 parametric copula families and an empirical copula class implemented in EVANESCE library. Users can use these to construct joint cumulative and probability density functions, generate random variables, compute Kendall's tau, Spearman's rho, and the tail index parameter, or use the maximum likelihood method to estimate parameters of any of these copulas.

1) normal copula (normal.copula)

One of the most frequently used copulas (especially for financial modeling) is the copula of a bivariate Gaussian distribution with correlation δ. It is defined by

$$\begin{aligned} C(u,v) &= \int_{-\infty}^{\Phi^{-1}(u)} dx \int_{-\infty}^{\Phi^{-1}(v)} dy \, \frac{1}{2\pi\sqrt{1-\delta^2}} \exp\left\{-\frac{x^2-2\delta xy+y^2}{2(1-\delta^2)}\right\} \\ &= \Phi_\delta\left(\Phi^{-1}(u), \Phi^{-1}(v)\right), \end{aligned}$$

where Φ^{-1} is the quantile function of the standard univariate Gaussian distribution, and Φ_δ is the joint cumulative distribution function of a standard bivariate Gaussian distribution with correlation coefficient δ ($0<\delta<1$). Since this copula is a very familiar object to most researchers (especially when used with Gaussian marginals), it has been incorporated in a number of applications simply because it was the only tool available for quite some time [Embrechts et al., 2000b]. In fact, J. P. Morgan's RiskMetrics [1995] has been using this copula for portfolio risk management by Monte Carlo simulations long before it was related that one were dealing with copulas.

2) normal mixture copula (normal.mix.copula)

Suppose two pairs of random variables (U_1, V_1) and (U_2, V_2) independent of each other. The joint distribution of the two pairs are given by normal copulas with parameter δ1 and δ2 respectively. Let (X,Y) be a random pair, such that it is equal to (U_1, V_1) with probability P, and it is equal to (U_2, V_2) with probability (1-P). Note that since marginal distributions of U_1, V_1, U_2, V_2 are uniform, so are the marginals of X, and Y. So the joint distribution of (X,Y) is given by the following copula

$$C(x,y) = p\, C_{\delta_1}^{(\mathrm{B1})}(x,y) + (1-p)\, C_{\delta_2}^{(\mathrm{B1})}(x,y),$$

where $0 \le p, \delta_1, \delta_2 \le 1$, $C_\delta^{(\mathrm{B1})}(x,y)$ is a normal copula with parameter δ

3) The Extreme Value copula class

An important class of copulas is the Extreme Value class (ev.copula). A copula is said to be an EV copula if for all t>0 the scaling property

$$C(u^t, v^t) = (C(u,v))^t$$

holds for all $(u,v) \in \mathbf{I}^2$.

EV copulas are max-stable, meaning that, if (X_1, Y_1), (X_2, Y_2), . . . , (Xn, Yn) are i.i.d. random pairs from an EV copula C and Mn =max{X_1,X_2, . . . ,Xn}, Nn = max{Y_1, Y_2, . . . , Yn}, a copula associated with the random pair (Mn,Nn) is also C. It can be shown [Joe, 1997, p. 175] that EV copulas can be represented in the form:

$$C(u,v) = \exp\left\{\log(uv)\, A\left(\frac{\log(u)}{\log(uv)}\right)\right\}$$

where A: [0,1] → [1/2,1] is a convex function such that max(t,1-t) < A(t) < 1 for all $t \in [0,1]$. The function A(t) is called the dependence function. As in the case of univariate random variables, it can be shown that the limiting copula for the sequence $\{(a_n + b_n M_n, c_n + d_n N_n)\}$ is an EV copula, if the sequence converges weakly in distribution for some sequence of numbers a_n , b_n , c_n and d_n under certain regularity conditions [Galambos, 1987].

4) Gumbel copula (gumbel.copula)

Well-known Gumbel copula [Gumbel, 1960] (an EV copula as well as an Archimedean copula class) has the following form:

$$C(u,v) = \exp\left\{-\left[(-\log u)^\delta + (-\log v)^\delta\right]^{1/\delta}\right\},$$

with $\delta \geq 1$ and the dependence function of the form:

$$A(t) = (t^\delta + (1-t)^\delta)^{1/\delta}.$$

5) Galambos copula (galambos.copula)

Galambos copula [Galambos, 1975] (an EV copula) has the following form:

$$C(u,v) = uv \exp\left\{\left[(-\log u)^{-\delta} + (-\log v)^{-\delta}\right]^{-1/\delta}\right\}, 0 \leq \delta < \infty.$$

and the dependence function is:

$$A(t) = 1 - (t^{-\delta} + (1-t)^{-\delta})^{-1/\delta}$$

6) Husler and Reiss copula (husler.reiss.copula)

Husler and Reisss copula [Husler and Reiss, 1989] (an EV copula) has the following form:

$$C(u,v) = \exp\left\{-\tilde{u}\ \Phi\left[\frac{1}{\delta}+\frac{1}{2}\delta\log\left(\frac{\tilde{u}}{\tilde{v}}\right)\right]-\tilde{v}\ \Phi\left[\frac{1}{\delta}+\frac{1}{2}\delta\log\left(\frac{\tilde{v}}{\tilde{u}}\right)\right]\right\}$$

where $0 \le \delta < \infty,\ \tilde{u} = -\log u,\ \tilde{v} = -\log v,\ \text{and}\ \Phi$ is a cdf of a standard Gaussian. The dependence function is

$$A(t) = t\ \Phi\left[\delta^{-1}+\frac{1}{2}\delta\log\left(\frac{t}{1-t}\right)\right]+(1-t)\ \Phi\left[\delta^{-1}-\frac{1}{2}\delta\log\left(\frac{t}{1-t}\right)\right]$$

7) Tawn copula (tawn.copula)

Tawn copula [Tawn, 1988, 1997] (an EV copula) is an asymmetric extension of the Gumbel copula with the dependence function of the form:

$$A(t) = 1-\beta+(\beta-\alpha)t+\{\alpha^r t^r+\beta^r(1-t)^r\}^{1/r}$$

where $0 \le \alpha,\ \beta \le 1,\ \ 1 \le r < \infty$

8) BB5 copula (bb5.copula)

BB5 copula [Joe, 1997] (an EV copula) is a two-parameter extension of the Gumbel copula and has the form of:

$$C(u,v) = \exp\left\{-\left[\tilde{u}^\theta+\tilde{v}^\theta-\left(\tilde{u}^{-\theta\delta}+\tilde{v}^{-\theta\delta}\right)^{-1/\delta}\right]^{1/\theta}\right\}$$

where $\delta > 0,\ \theta \ge 1,\ \tilde{u} = -\log u,\ \tilde{v} = -\log v$

The dependence function is:

$$A(t) = \left[t^{\theta} + (1-t)^{\theta} - \left(t^{-\delta\theta} + (1-t)^{-\delta\theta}\right)^{-1/\delta}\right]^{1/\theta}$$

9) The Archimedean copula class

Another general class of copulas is the Archimedean copula (archm.copula). A copula is said to be an Archimedean copula if its distribution function can be written in the following form:

$$C(u, v) = \phi^{-1}[\phi(u) + \phi(v)]$$

with some function $\phi(t) : \mathbf{I} \mapsto I\!R^{+}$ ϕ which is continuous, strictly decreasing, convex, and satisfying $\phi(1) = 0$. This function ϕ is called the Archimedean generator [Nelsen, 1999, chapter 4]. $\phi(0)$ is defined as $\lim_{t \to 0^{+}} \phi(t)$, and

$\phi^{-1}(z) = 0 \text{ for all } z > \phi(0), \text{ if } \phi(0) < \infty$

Notice that Gumbel copula is an EV copula as well as Archimedean copula because it can be written in the above standard form of an Archimedean copula with a generator function of $\phi(t) = (-\log t)^{\delta}$

10) Frank copula (frank.copula)

Frank copula [Frank, 1979] (an Archimedean copula) has the following distribution function:

$C(u, v) = -\delta^{-1} \log\left([\eta - (1 - e^{-\delta u})(1 - e^{-\delta v})]/\eta\right)$ where

$0 < \delta < \infty$, and $\eta = 1 - e^{-\delta}$

and the generator function is given by:

$$\phi(t) = -\log \frac{e^{-\delta t}-1}{e^{-\delta}-1}$$

11) Kimeldorf and Sampson copula (kimeldorf.sampson.copula)

Kimeldorf and Sampson copula [Kimeldorf and Sampson, 1975] (an Archimedean copula) has the following form:

$C(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$ where $0 < \delta < \infty$ and its generator function is

$$\phi(t) = t^{-\delta} - 1$$

12) Joe copula (joe.copula)

Joe copula [Joe, 1993] (an Archimedean copula) has the form of

$C(u,v) = 1 - \left((1-u)^{\delta} + (1-v)^{\delta} - (1-u)^{\delta}(1-v)^{\delta}\right)^{1/\delta}$ where $\delta \geq 1$ and the generator function is

$$\phi(t) = -\log\left(1-(1-t)^{\delta}\right)$$

13) BB1 copula (bb1.copula)

BB1 copula [Joe, 1997] (an Archimedean copula) is given by

$C(u,v) = \left\{1 + \left[(u^{-\theta}-1)^{\delta} + (v^{-\theta}-1)^{\delta}\right]^{1/\delta}\right\}^{-1/\theta}$ with $\theta > 0, \delta \geq 1$ and the generator function is

$$\phi(t) = \left(t^{-\theta} - 1\right)^{\delta}$$

14) BB2 copula (bb2.copula)

BB2 copula [Joe, 1997] (an Archimedean copula) has the form of

$$C(u,v) = \left[1 + \delta^{-1}\log\left(e^{\delta(u^{-\theta})} + e^{\delta(v^{-\theta})} - 1\right)\right]^{1/\theta}$$ where $\theta > 0, \delta > 0$

The generator function has the form of $\phi(t) = e^{\delta(t^{-\theta}-1)} - 1$

15) BB3 copula (bb3.copula)

BB3 copula [Joe, 1997] (an Archimedean copula) has the form of

$$C(u,v) = \exp\left\{-\left[\delta^{-1}\log\left(e^{\delta\tilde{u}^{\theta}} + e^{\delta\tilde{v}^{\theta}} - 1\right)\right]^{1/\theta}\right\}$$ with $\theta \geq 1, \delta > 0,$ and

$\tilde{u} = -\log(u)$, $\tilde{v} = -\log(v)$

The generator function

$$\phi(t) = \exp\left\{\delta\ (-\log t)^{\theta}\right\} - 1$$

16) BB6 copula (bb6.copula)

BB6 copula [Joe, 1997] (an Archimedean copula) has the form of

$$C(u,v) = 1 - \left(1 - \exp\left\{-\left[\left(-\log(1-(1-u)^{\theta})\right)^{\delta} + \left(-\log(1-(1-v)^{\theta})\right)^{\delta}\right]^{\frac{1}{\delta}}\right\}\right)^{\frac{1}{\theta}}$$

where $\theta \geq 1, \delta \geq 1$ and the generator function is

$$\phi(t) = \left[-\log\left(1-(1-t)^{\theta}\right)\right]^{\delta}$$

17) BB7 copula (bb7.copula)

BB7 copula [Joe, 1997] (an Archimedean copula) has the form of

$$C(u,v) = 1 - \left(1 - \left[\left(1-(1-u)^{\theta}\right)^{-\delta} + \left(1-(1-v)^{\theta}\right)^{-\delta} - 1\right]^{-1/\delta}\right)^{\frac{1}{\theta}}$$

where $\theta \geq 1, \delta > 0$.

The generator function is given by

$$\phi(t) = \left(1-(1-t)^{\theta}\right)^{-\delta} - 1$$

18) The Archimax copula class – BB4 copula (bb4.copula)

Capéraà et al. [2000] combined the EV and Archimedean copula classes into a single class called Archimax copulas. The Archimax copulas are copulas which can be represented in the following form:

$$C(u,v) = \phi^{-1}\left[\left(\phi(u)+\phi(v)\right)A\left(\frac{\phi(u)}{\phi(u)+\phi(v)}\right)\right]$$

where A(t) is a valid dependence function and ϕ a valid Archimedean generator. Archimax copulas reduce to Archimedean copulas for A(t) =1 and to EV copulas for $\phi(t) = -\log(t)$. Capéraà et al. [2000] proved that it is a valid copula for any combination of valid function $\phi(t)$ and A(t)

BB4 copula [Joe, 1997] is an example of this class of copula with

$$\phi(t) = t^{-\theta} - 1 \qquad \text{and} \qquad A(t) = 1 - (t^{-\delta} + (1-t)^{-\delta})^{-1/\delta}$$

The distribution function is given by

$$C(u,v) = \left(x^{-\theta} + u^{-\theta} - 1 - \left[\left(u^{-\theta} - 1\right)^{-\delta} + \left(v^{-\theta} - 1\right)^{-\delta}\right]^{-\frac{1}{\delta}}\right)^{-\frac{1}{\theta}}$$

where $\theta \geq 0, \delta > 0$

19) Empirical copula (empirical.copula)

If $u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n)}$ and $v_{(1)} \leq v_{(2)} \leq \ldots \leq v_{(n)}$ are the order statistics of the univariate samples, the empirical copula $\hat{C}_{\text{emp}}$ is defined at the point $\left(\frac{i}{n}, \frac{j}{n}\right)$ by the formula:

$$\hat{C}_{\text{emp}}\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} 1_{\{u_k \leq u_{(i)}, v_k \leq v_{(j)}\}}, \qquad i,j = 1,2,\ldots,n$$

An empirical copula can be created by calling the following function

empirical.copula(x,y)

where x and y are data points that are assumed to have a uniform (0,1) marginal distribution [Nelson, 1999]. They are either a vector, a list, or a matrix.

3. A List of Major Functions in EVANSCE and Brief Description by Category

1) Parameter estimation for GEV and GPD distribution using the method of L-Moments and method of maximum likelihood

`sample.LMOM`:
compute unbiased estimates of mean, second L-moment, L-skewness, and L-kurtosis.

`PlotPos.LMOM`:
compute plotting position estimates of sample L-moments

`gev.lmom`:
compute L-moment parameter estimates for GEV.

`gev.mix1`:
compute MIX1 parameter estimates for GEV.

`gev.mix2`:
compute MIX2 parameter estimates for GEV.

`gpd.lmom`:
compute L-moment parameter estimates for GPD.

`gpd.ml`:
compute MLE parameter estimate for GPD.

2) Peak Over Threshold (POT) estimation and CDF & quantile functions of a random variable with power decaying tails

`gpd.tail`:
fit a GPD to excesses on two tails (standard POT analysis).

`gpd.1p`:
semi-parametric estimation of CDF based on a GPD model (one tail)

`gpd.2p`:
semi-parametric estimation of CDF based on a GPD model (two tails)

`gpd.1q`:
semi-parametric estimation of the quantile based on a GPD model (one tail)

`gpd.2q`:
semi-parametric estimation of the quantile based on a GPD model (two tails)

3) CDF, PDF, random variable generation from copula objects and exploratory plots

dcopula, pcopula, rcopula:
density, CDF and random number generation of two r.v.s with uniform marginals and joint CDF given by a "copula" object respectively.

contour.dcopula, contour.pcopula, persp.dcopula, persp.pcopula:
generate 2-D contour plots and 3-D perspective plots of the pdf and cdf of a "copula" object

4) Dependence structure of copula objects

tail.index:
compute the tail dependence index for a parametric or empirical copula.
(no method function implemented on Frank copula, Kimeldorf and Sampson copula, Joe copula, normal copula, and normal mixture copula)

Spearmans.rho:
compute Spearman's rho for a copula.

Kendalls.tau:
compute Kendall's tau for a copula.

5) Estimation of copula parameters

fit.copula:
MLE parameter estimates for copulas.

6) Functions Specific to Extreme Value copulas and Archimax copulas

Afunc:
calculate the dependence function for an extreme value copula.

7) Functions Specific to Archimedean copulas and Archimax copulas

PHI:
calculate the Archimedean generator function for an Archimedean copula.

8) Functions related to creating and calculating bivariate distributions

bivd:
create a bivd object representing a child class object of a particular parametric copula bivd and a bivariate distribution with 2 arbitrary marginals

The syntax of the function:
bivd(cop, marginX = "unif", marginY = marginX, param.marginX = c(0, 1), param.marginY = param.marginX)

`pbivd`:
CDF of arbitrary bivariate distribution.

`dbivd`:
density of arbitrary bivariate distribution.

`rbivd`:
random variable generation of arbitrary bivariate distribution.

9) Estimation of a Bivariate Joint CDF

`gpdjoint.1p`:
an empirical and semi-parametric estimate of bivariate joint CDF (one tail).

`gpdjoint.2p`:
an empirical and semi-parametric estimate of bivariate joint CDF (two tails).

10) Value-at-Risk (VaR) calculation

`VaR.exp.portf`:
calculate the Value-at-Risk of a two asset portfolio based on the copula parameters and fitted GPD models

`VaR.exp.sim`:
calculate the Value-at-Risk and expected shortfall of a two asset portfolio by simulation methods.

11) Creating copula objects

`archm.copula`:
create an Archimedean copula object

`ev.copula`:
create an Extreme Value copula object

`empirical.copula`:
create an "`empirical.copula`" object.

12) Others

`shape.plot`:
calculate and plot how shape parameter of a GPD varies with thresholds
(equivalent to shape function in EVIS library)

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