Juggling With Numbers

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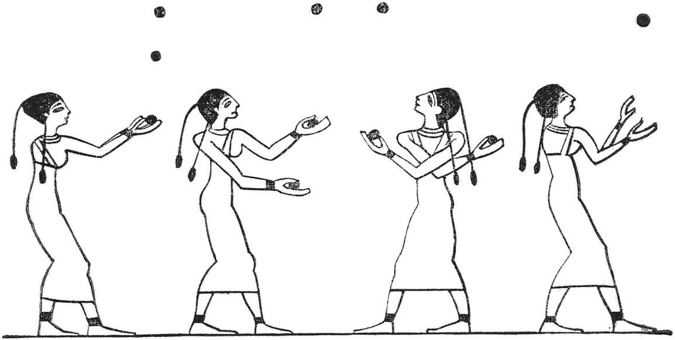
Juggling Is Old!

Oldest known depictions appear in an Egyptian temple at Beni Hasan (c. 1994-1781 BCE).



Everybody Juggles!

Mughal Emperor Babur wrote of jugglers and acrobats in India (c. 1528):

> *... the Hindustani bazigers [jugglers] were brought in and performed their tricks, and the lulis [tumblers] and rope-dancers exhibited their feats... One of these is the following—they take seven rings, one of which they suspend over their forehead, and two on their thighs; the other four they place, two on two of their fingers, and the other two on two of their toes, and then whirl them all around with a quick uninterrupted motion.*

Georg Forster wrote of jugglers in Tonga (1773):

> *A girl of 10-12 years... had five ball shaped fruits, which she threw permanently high and caught by an admirable skill and quickness.*

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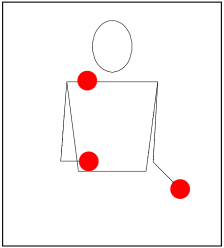
- Idea: use a numerical code to describe the throws.
- Measure height of throw according to number of "beats" until it comes back down (usually, "beats" = "thuds")



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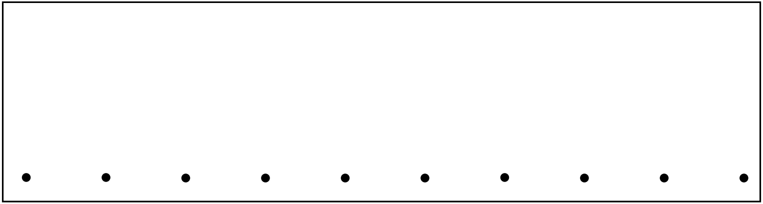

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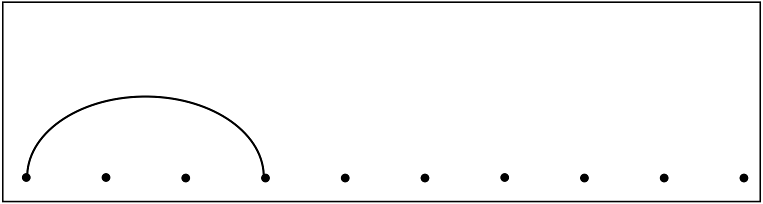

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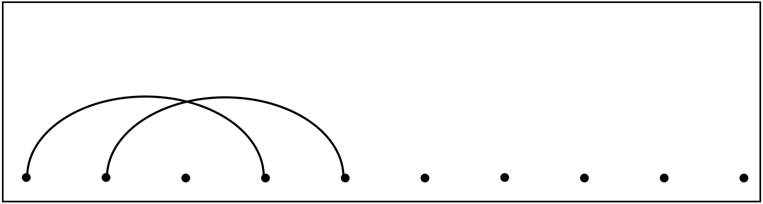

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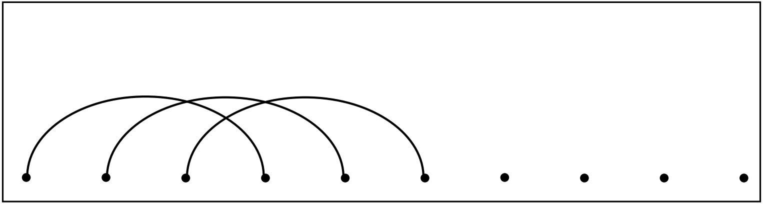

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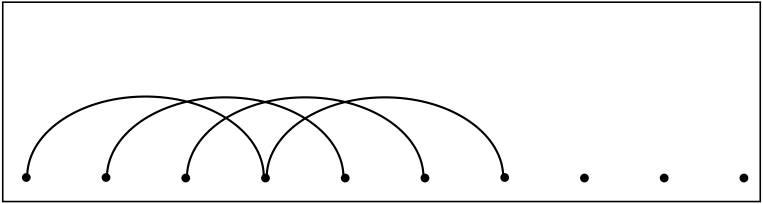

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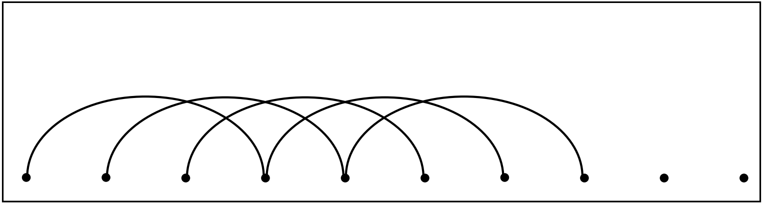

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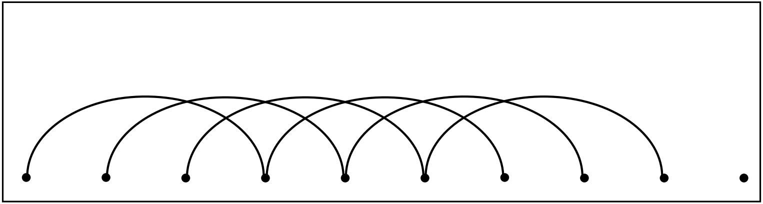

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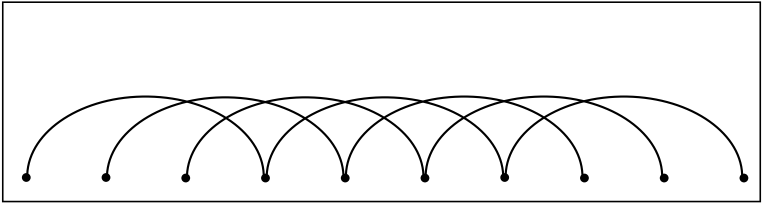

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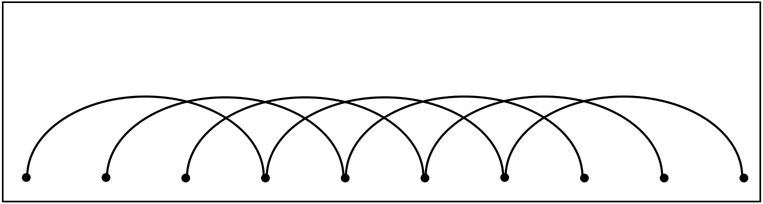

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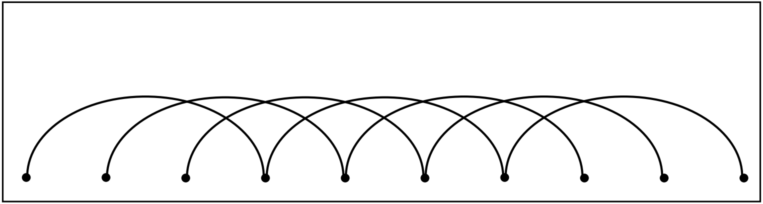
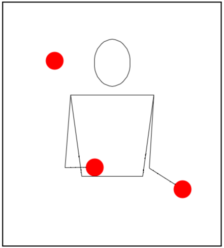
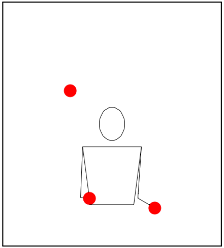


- Repeated throws of height 3.
- This pattern can be represented by $(\ldots 3333 \ldots)$, or just (3).
- This is the *siteswap* for the juggling pattern.

Introduction to Siteswap Notation



(441)



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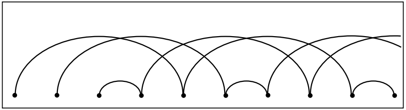

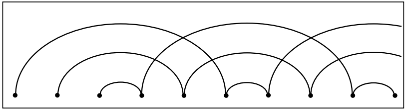

Properly Defining A Siteswap

Some remarks:

- The beats always alternate between left and right hands.
- The *length* (or, *period*) of a siteswap is the number of beats that occur before it repeats.
- We are only interested in *monoplex* juggling: at most one ball caught/thrown at once.
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More Precisely: If two balls are thrown at times i and j, and remain in the air for t_i beats and t_j beats, respectively, it cannot be the case that $t_i + i = t_j + j$ (since this would create a collision).

Defining A Sitewap

Definition

A *siteswap* is a finite sequence of nonnegative integers. A *valid* siteswap is one with no collisions, i.e., the quantities $t_i + i \pmod{n}$ are distinct for $1 \leq i \leq n$.

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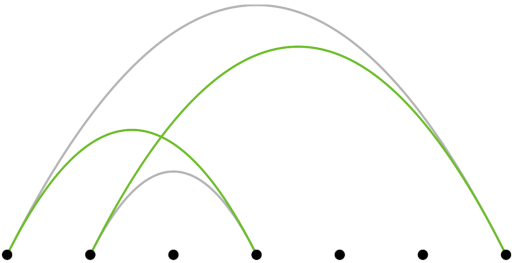
Lemma

Let s be a valid siteswap with successive throws a and b, so that $s = (\ldots, a, b, \ldots)$. *Then the siteswap* $s' = (\ldots, b+1, a-1, \ldots)$ *is also valid and requires the same number of balls as s.*

The “Height Swap”

Example: $(\ldots, 6, 2, \ldots) \longrightarrow (\ldots, 3, 5, \ldots)$ switches the landing times of successive throws.



So, no new collisions can be introduced from a height swap (and no old collisions can be removed).

A Game: Number Morphs

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Try it! Play number morphs for $5, 1, 6, 3, 5$.

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My version of the game:

- Starting siteswap: (51635)

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Final score: $4 - 4 = 0$ — *win!*

Theorem

A sequence s of nonnegative integers is a valid siteswap if, and only if, the number morphs game results in a constant sequence $(b, b, \ldots, b)$.

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- Every height swap reduces the spread by either 1 or 2.
- Unless you get stuck, the spread will eventually equal 0.

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Theorem

The number of balls required to juggle a valid siteswap s is equal to the average of the nonnegative integers appearing in s.

The Reverse Question

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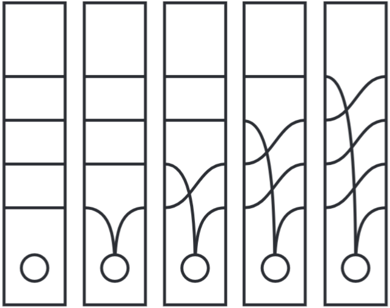
Examples:

- For $b = 2$ and $n = 2$, there are five: $(22), (40), (04), (31), (13)$.
- For $b = 3$ and $n = 3$, there are 37:

$$(900), (090), (009), (630), (603), (063), (360), (036), (306), (333),$$
$$(711), (171), (117), (441), (414), (144), (522), (252), (225), (720),$$
$$(180), (126), (450), (423), (153), (027), (018), (612), (045), (342),$$
$$(351), (702), (801), (261), (504), (234), (135)!$$

Juggling Cards

Take $b = 4$, and consider this set of five "juggling cards."

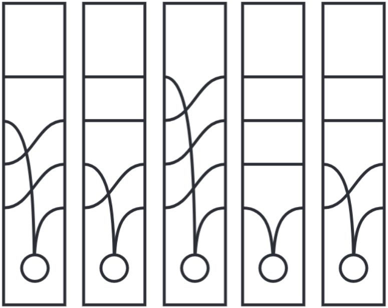


C_0 C_1 C_2 C_3 C_4

You can build any 4-ball juggling diagram from these cards.

Juggling Cards

Example: What siteswap corresponds to this sequence of cards?



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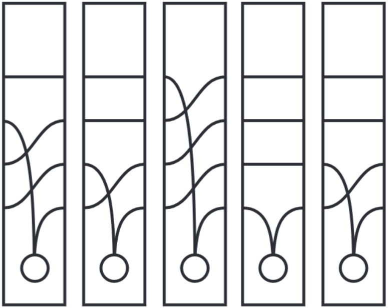

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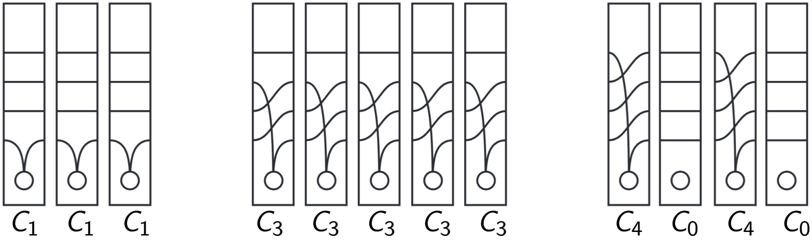
Answer: (53192).

Playing Around With Juggling Cards



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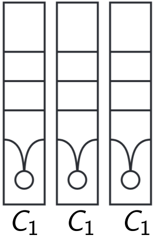

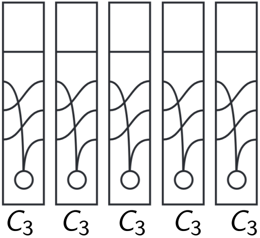

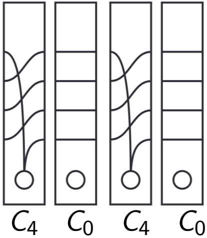

Answers: (1), (3), (80)

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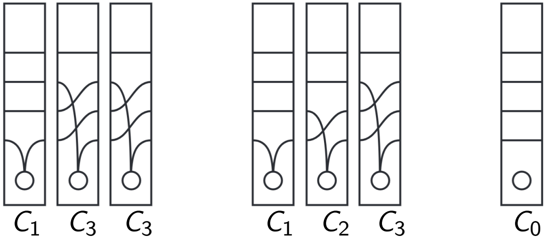

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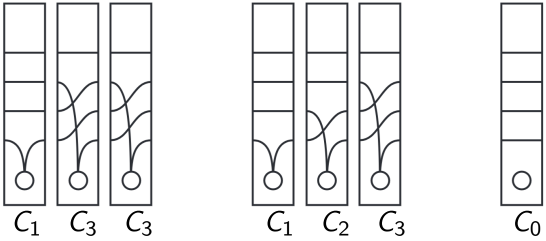

Answers: (441), (531), ()

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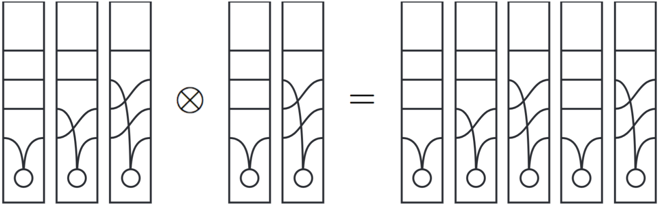
Corollary

Given an integer $n \geq 1$, there exist $(b+1)^n - b^n$ valid siteswaps with b balls and length n, counting repetitions and cyclic permutations separately.

Examples: For $b = 2$ and $n = 2$, there are $3^2 - 2^2 = 5$ valid siteswaps. For $b = 3$ and $n = 3$, there are $4^3 - 3^3 = 37$ valid siteswaps.

How to Multiply Juggling Patterns?

Juggling cards: it's easy to "concatenate" old patterns to get new ones.



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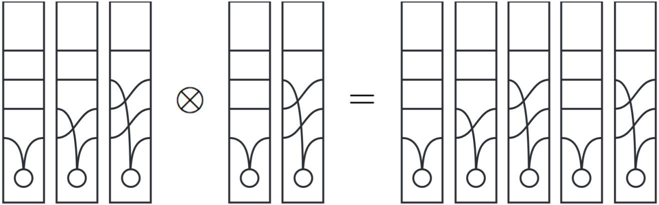

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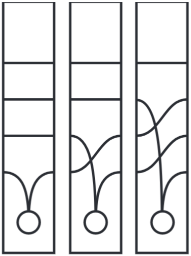
Siteswaps: direct concatenation won't always work: $(531)(51) = (53151)$, but (53151) is not a valid siteswap.

How to Multiply Juggling Patterns

Solution: Restrict to sets of "compatible" patterns.

Definition

A juggling pattern is a *ground state* pattern if there is a moment when the juggler can stop juggling, after which b "thuds" are heard as the balls hit the ground on each of the next b beats.



Ground State Siteswaps

Facts about ground state siteswaps:

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- Any two ground state siteswaps (with same b) can be "multiplied" via concatenation: $(441)(6231) = (4416231)$.
- Multiplication isn't always commutative: $(3)(42) = (42)(3)$, but $(3)(42)(522) \neq (42)(3)(522)$.
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- Most ground state siteswaps can be "factored" into shorter ones: $(53403426231) = (5340)(3)(42)(6231)$.
- If a siteswap can't be factored, it is "primitive."
- The "identity" siteswap is ().

Counting Ground State Siteswaps

Question: Given b, how many ground state siteswaps are there with length $n \geq 0$?

Theorem (Chung & Graham, 2008)

Given $b, n \geq 0$, the number of ground state juggling patterns with b balls and length n is given by

$$J_b(n) = \begin{cases} n! & \text{if } n \leq b \\ b! \cdot (b+1)^{n-b} & \text{if } n > b. \end{cases}$$

Counting Ground State Siteswaps

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Examples:

- $b = 3,\ n = 0$: $0! = 1$ — ()
- $b = 3,\ n = 3$: $3! = 6$ — (333), (342), (423), (441), (531), (522).
- $b = 4,\ n = 7$: $4! \cdot 5^3 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5 \cdot 5 = 3000$ siteswaps!

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Theorem (τ, 2019)

Given $b \geq 4$, the number of primitive, ground state juggling patterns with b balls and length n is approximated by

$$P_b(n) \sim \frac{b+1-\rho}{|s_b'(1/\rho)|} \cdot \rho^n,$$

where $s_b(z)$ is a b-degree polynomial and ρ is a constant satisfying $0.73 \cdot \frac{1}{e^b\sqrt{b}} < 1 - \frac{\rho}{b+1} < 6.04 \cdot \frac{\sqrt{b}}{e^b}$.

An Analogy Instead of a Proof

The Classic Question: Given a positive integer n, what proportion of the numbers from 1 to n are prime?

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Our Question: Given b, what proportion of ground state siteswaps of length n are primitive?

The Answer (2019): The proportion is approximately $C_b \cdot \left(\frac{\rho}{b+1}\right)^n$, i.e., the primitive siteswaps are sparse since $\frac{\rho}{b+1} < 1 - \frac{0.73}{e^b\sqrt{b}} < 0.994$:

$$\lim_{n\to\infty} C_b \cdot \left(\frac{\rho}{b+1}\right)^n \; < \; \lim_{n\to\infty} C_b \cdot \left(0.994\right)^n \; = \; 0.$$

Some Open Questions

- What other arithmetic properties does the set of ground state juggling sequences have?
- What happens when you allow for a ball to be added or dropped (i.e., what if b can change)?

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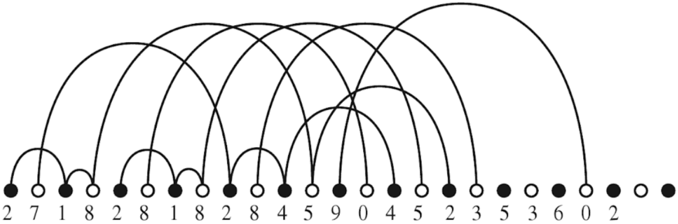
- What happens when you allow for a ball to be added or dropped (i.e., what if b can change)?
- What irrational numbers are the most "jugglable"?



Slides online: `https://tinyurl.com/JugglingWithNumbers`
Also see: *The Mathematics of Juggling* by Burkard Polster (Springer, 2003).