Leonhard Euler and the Invention of Modern Math

Erik R. Tou

University of Washington, Tacoma

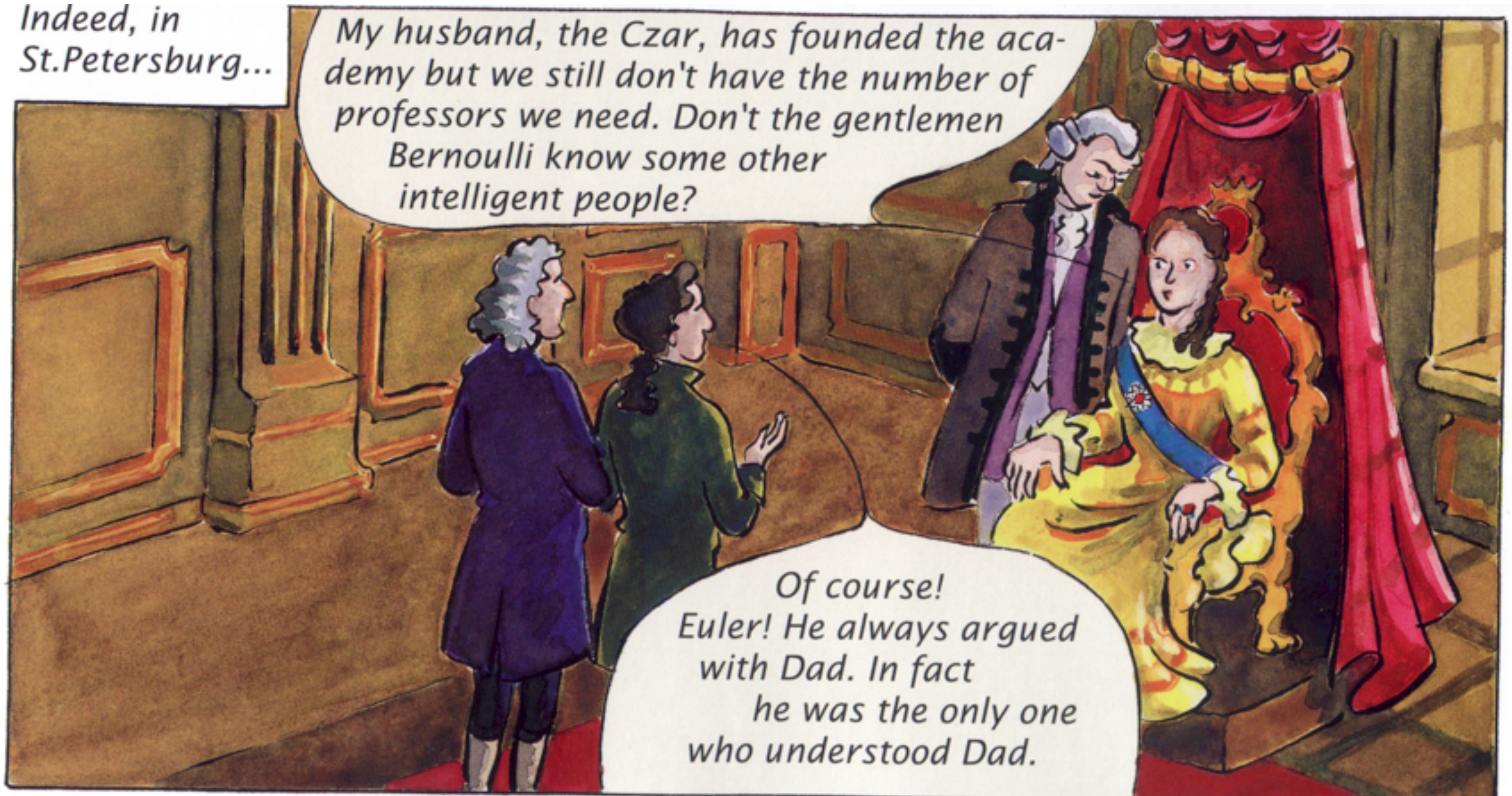
23 February 2018

University of the Pacific

WHO WAS LEONHARD EULER?

- **1707**: Born in Riehen, Switzerland, near Basel
- As a youth, was tutored by Johann Bernoulli
- **1726**: tied for second place in the Paris Academy's annual prize competition (on the masting of ships)
- **1727**: Graduated from University of Basel
- **1727**: Applied for a Physics professorship at University of Basel, but was rejected.
- **Later in 1727**: offered a position at the Czar Peter's St. Petersburg Academy (with help of Bernoulli's sons…)

Euler and the Hiring Process



Euler and the Hiring Process

Euler and the Hiring Process

Euler and the Hiring Process

Images courtesy *Euler: A Man to be Reckoned With*, by Heyne, Heyne, and Pini

Euler's Early Career

- **1730**: Appointed Professor of Physics

  "…I made a new contract for four years, granting me 400 rubles for each of the first two and 600 for the next two, along with 60 rubles for lodging, wood, and light." [*]

[*] Bradley, R. and Sandifer, C. E., eds. *Leonhard Euler: Life, Work and Legacy*, Elsevier Science, 2007.

Euler's Early Career

- **1730**: Appointed Professor of Physics

  "…I made a new contract for four years, granting me 400 rubles for each of the first two and 600 for the next two, along with 60 rubles for lodging, wood, and light." [*]

- **1735**: Euler becomes famous—solves the "Basel problem":

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$$

[*] Bradley, R. and Sandifer, C. E., eds. *Leonhard Euler: Life, Work and Legacy*, Elsevier Science, 2007.

Euler's Early Career

- **1730**: Appointed Professor of Physics

  "…I made a new contract for four years, granting me 400 rubles for each of the first two and 600 for the next two, along with 60 rubles for lodging, wood, and light." [*]

- **1735**: Euler becomes famous—solves the "Basel problem":

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$$

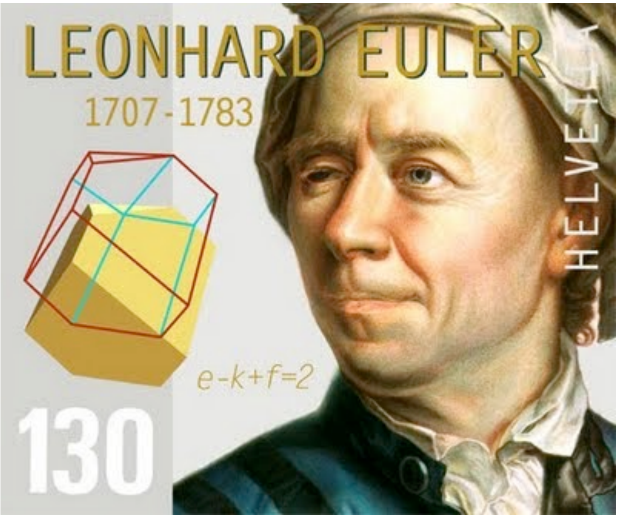
- **1738 and 1740**: Euler becomes rich—wins two Paris Academy prizes (the first of 12)

- **By 1739**: published 69 papers and four books

[*] Bradley, R. and Sandifer, C. E., eds. *Leonhard Euler: Life, Work and Legacy*, Elsevier Science, 2007.

Euler's Later Career

- Went completely blind later in life—apparently, this increased his productivity (30 papers in 1772 alone!)
- By the time he died at age 76, Euler had published over 500 papers and wrote more than 20 books.
- But that's not all…



Euler's Posthumous Career

- Euler died on 18 September 1783.

Euler's Posthumous Career

- Euler died on 18 September 1783.
- This did not stop him from publishing…
  - Euler's unpublished papers continued to appear in Academy journals until 1830
  - Eight more were found in 1849
  - 48 more were found in 1860

Euler's Posthumous Career

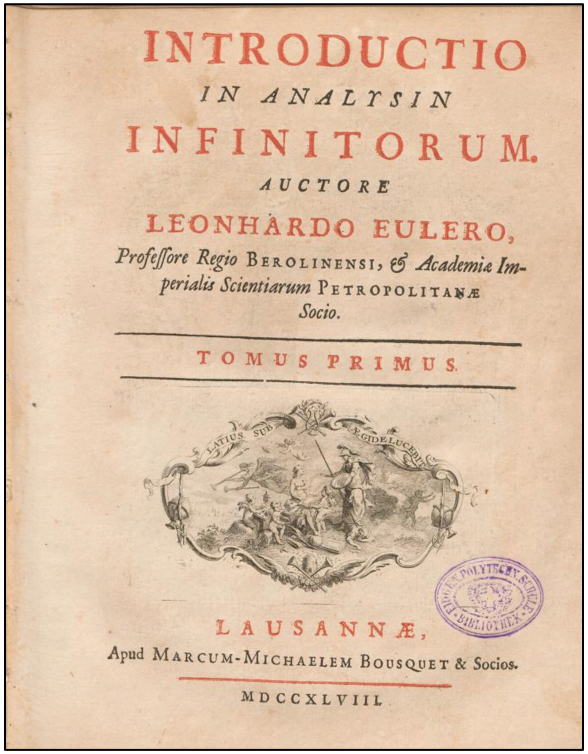
- Euler died on 18 September 1783.
- This did not stop him from publishing…
  - Euler's unpublished papers continued to appear in Academy journals until 1830
  - Eight more were found in 1849
  - 48 more were found in 1860
- To date: more than 870 papers and books are known (over 30,000 pages)

- **In 2002**, an idea: Make the original, unedited sources available online ***for free***.
- **By 2006:** 90% of Euler's numbered works online, with 7% of these translated into English.
- **Today:** 97% of numbered works available, with 23% of these translated into English, along with a compilation of his letters.
- Hosted by the MAA since 2011:

http://eulerarchive.maa.org

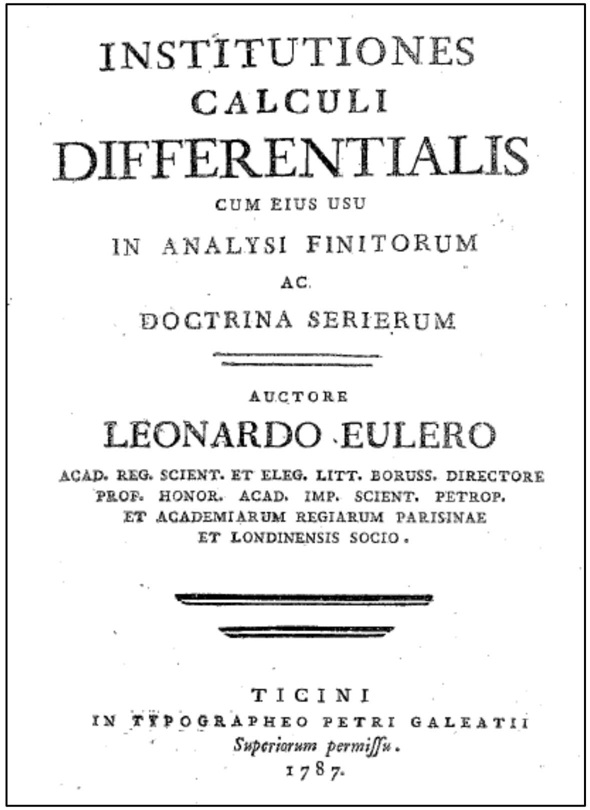
Things You Can Find on the Euler Archive

The origins of the modern Calculus sequence

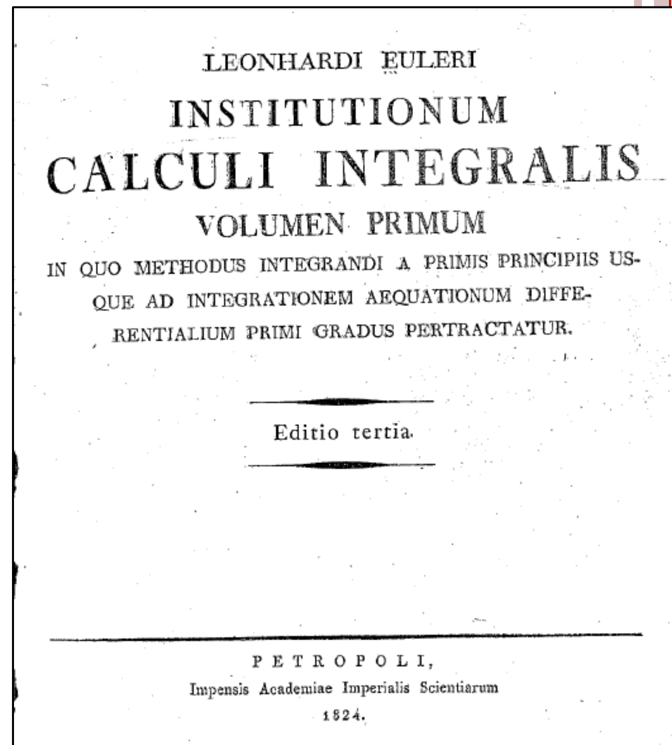

INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*
TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

E101 & E102


INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS USU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM
AUCTORE
LEONARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP.
ET ACADEMIARUM REGIARUM PARISINAE
ET LONDINENSIS SOCIO.
TICINI
IN TYPOGRAPHEO PETRI GALEATII
Superiorum permissu.
1787.

E212


LEONHARDI EULERI
INSTITUTIONUM
CALCULI INTEGRALIS
VOLUMEN PRIMUM
IN QUO METHODUS INTEGRANDI A PRIMIS PRINCIPIIS USQUE AD INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM PRIMI GRADUS PERTRACTATUR.
Editio tertia.
PETROPOLI,
Impensis Academiae Imperialis Scientiarum
1824.

E342, E366, E385

Things You Can Find on the Euler Archive

Euler's calculation of the derivative for the logarithm function (from *Institutiones Calculi Differentialis* [E212])

181. Si igitur cuiusque ipsius x functionis p logarithmus lp proponatur, eodem ratiocinio reperietur eius differentiale esse $= \frac{dp}{p}$, unde ad logarithmorum differentialia invenienda haec habetur regula. *Quantitatis* p, *cuius logarithmus proponitur, sumatur differentiale, hocque per ipsam quantitatem* p *divisum dabit differentiale logarithmi quaesitum.* Sequitur haec eadem regula quoque ex forma $\frac{p^0 - 1^0}{0}$, ad quam superiori libro logarithmum ipsius p reduximus. Sit $\omega = 0$, & cum sit

$$lp = \frac{p^\omega - 1}{\omega} : \quad \text{erit} \quad d.lp = d.\frac{1}{\omega}p^\omega = p^{\omega-1}dp = \frac{dp}{p} \quad \text{ob } \omega = 0.$$

Notandum autem est $\frac{dp}{p}$ esse differentiale logarithmi hyperbolici ipsius p; ita ut, si logarithmus vulgaris ipsius p proponeretur, differentiale illud $\frac{dp}{p}$ multiplicari deberet per hunc numerum 0,43429448 &c.

Things You Can Find on the Euler Archive

Euler's calculation of the derivative for the logarithm function

in English

For any quantity p whose logarithm is proposed, we take the differential of that quantity p and divide by the quantity p itself in order to obtain the desired differential of the logarithm.

This same rule follows from the form

$$\frac{p^0 - 1^0}{0},$$

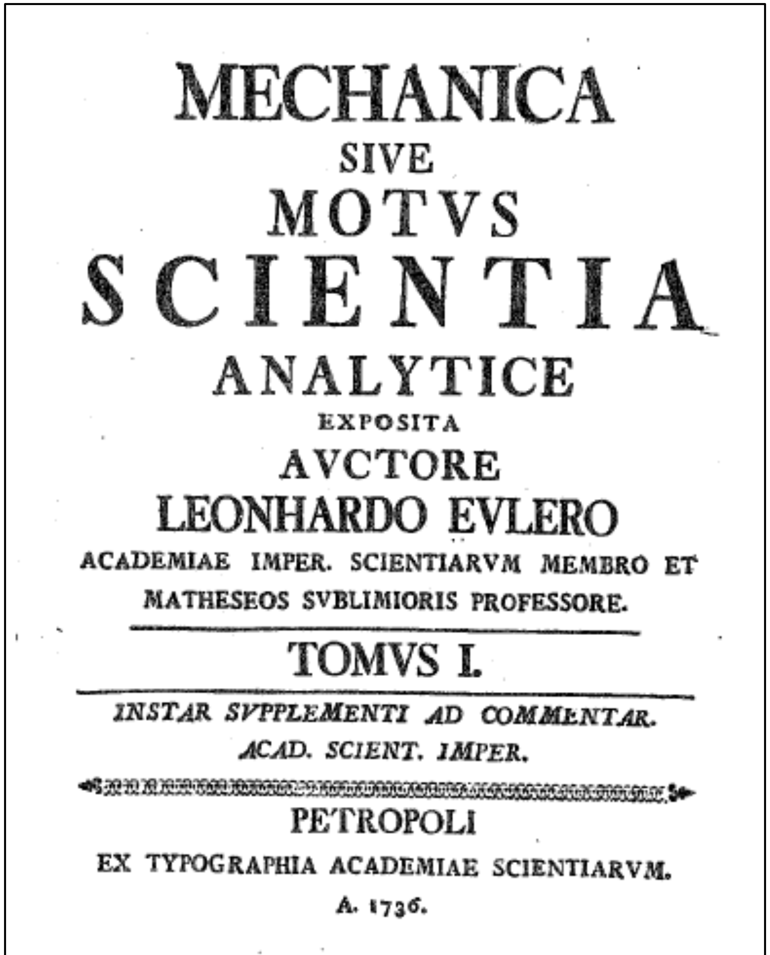
to which we reduced the logarithm of p in the previous book.[2] Let $\omega = 0$, and since $\ln p = (p^\omega - 1)/\omega$, we have

$$d \ln p = d\frac{1}{\omega}p^\omega = p^{\omega-1}dp = \frac{dp}{p},$$

since $\omega = 0$. It is to be noted, however, that dp/p is the differential of the hyperbolic logarithm of p, so that if the common logarithm of p is desired, this differential dp/p must be multiplied by the number $0.43429448\ldots$.

THINGS YOU CAN FIND ON THE EULER ARCHIVE

The first modern physics textbook

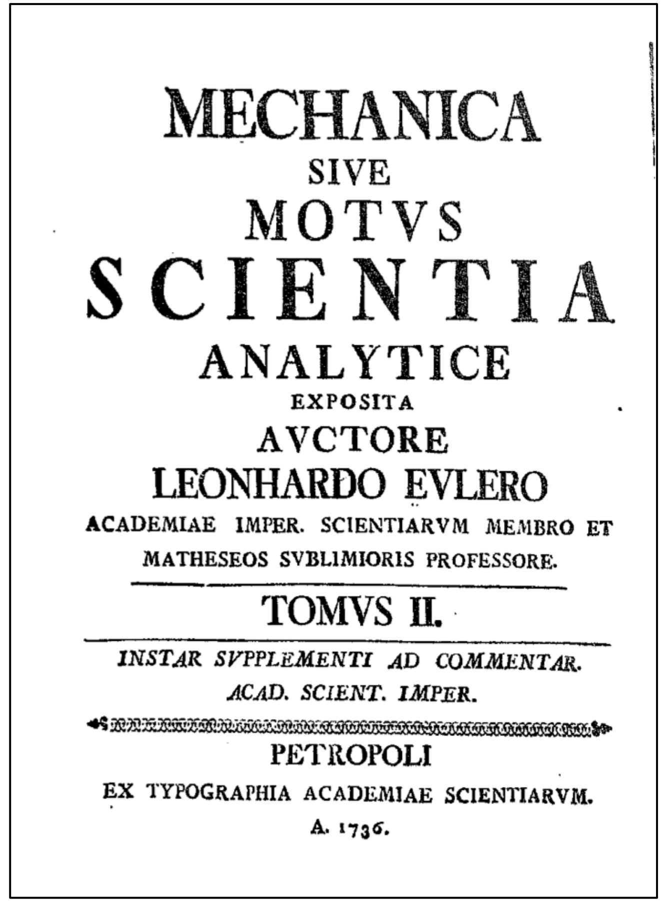


MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS I.

INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.



MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS II.

INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

E15 & E16

Things You Can Find on the Euler Archive

English translations of the personal correspondence between Euler and his childhood friend Caspar Wettstein:

- 19 Nov 1746: "…you have easily figured out that you have incited a great appetite in me for the excellent English tobacco that you sent, that I have yet to taste and really enjoy."
- 21 Nov 1750: "…I find myself obliged to request of you, Sir, to send me some tobacco, not doubting that this one will arrive before the one that I am expecting from Hamburg."
- 27 April 1751: "I am very impatient to receive the tobacco soon, that you have had the kindness to send to me, since I am forced to smoke filler and as to what I see, the tobacco will be paid for by the books that I sent to you on two separate occasions…"

Things You Can Find on the Euler Archive

The emergence of the "Goldbach conjecture":

numeros unico modo in duo quadrata divisibiles gäbe. Auf solche Weise will ich auch eine conjecture hazardiren: dass jede Zahl, welche aus zweyen numeris primis zusammengesetzt ist, ein aggregatum so vieler numerorum primorum sey, als man will (die unitatem mit dazu gerechnet), bis auf die congeriem omnium unitatum *); zum Exempel

$$4 = \begin{cases} 1+3 \\ 1+1+2 \\ 1+1+1+1 \end{cases} \qquad 5 = \begin{cases} 2+3 \\ 1+1+3 \\ 1+1+1+2 \\ 1+1+1+1+1 \end{cases}$$

Goldbach to Euler, 7 June 1742

Things You Can Find on the Euler Archive

The emergence of the "Goldbach conjecture":

Dass eine jegliche Zahl, welche in zwey numeros primos resolubilis ist, zugleich in quot, quis voluerit, numeros primos zertheilt werden könne, kann aus einer Observation, so Ew. vormals mit mir communicirt haben, dass nehmlich ein jeder numerus par eine summa duorum numerorum primorum sey, illustrirt und confirmirt werden. Denn, ist der numerus propositus n par, so ist er eine summa duorum numerorum primorum, und da $n-2$ auch eine summa duorum numerorum primorum ist, so ist n auch eine summa

Euler claims: If *n* is an even number,
it is the sum of two prime numbers.

Euler to Goldbach, 30 June 1742

Number Theory and Euler

Euclid (c. 300 BCE) developed a comprehensive theory for geometry, also wrote on primes and perfect numbers (*Elements*, Book IX):

> "If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect."

Translation: If $1 + 2 + 2^2 + 2^3 + \ldots + 2^{p-1} = 2^p - 1$ is prime, then $n = (2^p - 1)2^{p-1}$ is perfect.

Euler proved the converse in 1747, so we know these are the *only* even perfect numbers.

Number Theory and Euler

Diophantus of Alexandria (c. 250 CE) developed a set of techniques to solve linear and quadratic equations, now called "Diophantine analysis."

In *Arithmetica*, he was concerned with rational solutions x to the equations $ax^2 + bx = c$, $ax^2 = bx + c$ and $ax^2 + c = bx$.

NUMBER THEORY AND EULER

Diophantus of Alexandria (c. 250 CE) developed a set of techniques to solve linear and quadratic equations, now called "Diophantine analysis."

In *Arithmetica*, he was concerned with rational solutions x to the equations $ax^2 + bx = c$, $ax^2 = bx + c$ and $ax^2 + c = bx$.

For example, Book II Problem 10 goes as follows:

> *To find two square numbers having a given difference.*
>
> Given difference 60. Side of one [square is the] number x, side of the other [square is] x plus any number, the square of which is not greater than 60, say 3. Therefore, $(x+3)^2 - x^2 = 60$, so $x = 8\frac{1}{2}$ and the required squares are $72\frac{1}{4}$ and $132\frac{1}{4}$.

Number Theory and Euler

Three claims by Pierre de Fermat (1607-1665):

1. All numbers in the following sequence are prime:

$$2^{2^1}+1, 2^{2^2}+1, 2^{2^3}+1, 2^{2^4}+1, \ldots$$

2. For any integer a and prime p, $a^p - a$ is divisible by p.
3. For any $n \geq 3$, there are no nontrivial integer solutions to the equation $x^n + y^n = z^n$.

Number Theory and Euler

Three claims by Pierre de Fermat (1607-1665):

1. All numbers in the following sequence are prime: $$2^{2^1}+1, 2^{2^2}+1, 2^{2^3}+1, 2^{2^4}+1, \ldots$$
2. For any integer a and prime p, $a^p - a$ is divisible by p.
3. For any $n \geq 3$, there are no nontrivial integer solutions to the equation $x^n + y^n = z^n$.

Three proofs by Euler:

1. 1732: Shows that $2^{2^5}+1$ is not prime.
2. 1736: Proves it with mathematical induction, later generalizes to non-primes (1775).
3. 1738/1770: Proves it for $n = 3$. Complete proof by Wiles et al. in 1994.

Euler and the Pell Equation

- The *Pell* equation is the Diophantine equation

$$x^2 = dy^2 + 1.$$

- First studied by Indian mathematicians Brahmagupta (c. 650 CE) and Bhaskara II (c. 1150 CE).

Euler and the Pell Equation

- The *Pell* equation is the Diophantine equation

$$x^2 = dy^2 + 1.$$

- First studied by Indian mathematicians Brahmagupta (c. 650 CE) and Bhaskara II (c. 1150 CE).
- Easy factorization: $(x + y\sqrt{d})(x - y\sqrt{d}) = 1$
- If $d = n^2$ is a square, then this factorization implies that $x + yn$, $x - yn = \pm 1$, so that $x = \pm 1$ and $y = 0$.
- Typically, assume d is positive and non-square.

Euler and the Pell Equation

- The *Pell* equation is the Diophantine equation

$$x^2 = dy^2 + 1.$$

- First studied by Indian mathematicians Brahmagupta (c. 650 CE) and Bhaskara II (c. 1150 CE).
- Easy factorization: $(x + y\sqrt{d})(x - y\sqrt{d}) = 1$
- If $d = n^2$ is a square, then this factorization implies that $x + yn$, $x - yn = \pm 1$, so that $x = \pm 1$ and $y = 0$.
- Typically, assume d is positive and non-square.
- Euler (1729): Shows that solutions to $x^2 = dy^2 + 1$ can be used to find infinitely many solutions to a general equation $p^2 = aq^2 + bq + c$.

Euler and the Pell Equation

1767: Euler's paper "On the use of a new algorithm in solving the Pell problem" appears in print.

Euler's algorithm* for $x^2 = dy^2 + 1$:

1. Set $z_0 = \sqrt{d}$, $a_0 = [\sqrt{d}]$ = greatest integer $\leq d$.
2. For each n, define $z_n = (z_{n\text{-}1} - a_{n\text{-}1})^{-1}$ and $a_n = [z_n]$.
3. Eventually, the values of z_n and a_n will repeat.

Example: $d = 14$ (so $\sqrt{d} \approx 3.742$).

n	0	1	2	3	4	5
z_n	3.742	1.348	2.871	1.148	6.742	1.348
a_n	3	1	2	1	6	1

The World of Continued Fractions

- If we write it out as a single computation,

$$\sqrt{14} = 3 + \frac{1}{z_1} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{z_1}}}}} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \sqrt{14}}}}}$$

The World of Continued Fractions

- If we write it out as a single computation,

$$\sqrt{14} = 3 + \frac{1}{z_1} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{z_1}}}}} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \sqrt{14}}}}}$$

- Delete $1/(3+\sqrt{14})$ and simplify to get $\sqrt{14} \approx 15/4$.
- This gives a solution to the Pell equation for $d = 14$, $15^2 = 225 = 14 \cdot 4^2 + 1$.

New Solutions From Old

If x_1 and y_1 are solutions to the Pell equation, then we can generate infinitely many new solutions by taking powers: $(x_1+y_1\sqrt{d})^n = x_n + y_n\sqrt{d}$.

Example: $d = 14$, $x_1 = 15$, $y_1 = 4$.

- $(15 + 4\sqrt{14})^2 = 225 + 120\sqrt{14} + 224 = 449 + 120\sqrt{14}$.
- So $x_2 = 449$, $y_2 = 120$: $449^2 = 201{,}601 = 14 \cdot 120^2 + 1$.

n	1	2	3	4	5	6
x_n	15	449	13455	403201	12082575	362074049
y_n	4	120	3596	107760	3229204	96768360

Archimedes' Cattle Problem

Attributed to Archimedes (c. 287-212 BCE), the cattle problem is a poem inviting the reader to calculate the size of the Sun god's cattle herd:

> The Sun god's cattle, friend, apply thy care
> to count their number, hast thou wisdom's share.
> They grazed of old on the Thrinacian floor
> of Sic'ly's island, herded into four,
> colour by colour: one herd white as cream,
> the next in coats glowing with ebon gleam,
> brown-skinned the third, and stained with spots the last.
> Each herd saw bulls in power unsurpassed,
> in ratios these: count half the ebon-hued,
> add one third more, then all the brown include;
> thus, friend, canst thou the white bulls' number tell.
>
> …

Archimedes' Cattle Problem

- Four colors: white, black, brown, spotted.
- Males (bulls) and females (cows) of each color.
- Eight variables:
  - Bulls: x (white), y (black), z (spotted), t (brown)
  - Cows: x' (white), y' (black), z' (spotted), t' (brown)
- Translating the first condition (bulls):

  "count half the ebon-hued, add one third more, then all the brown include; thus, friend, canst thou the white bulls' number tell."

$$x = \left(\frac{1}{2} + \frac{1}{3}\right) y + t$$

Archimedes' Cattle Problem

- To be considered *skillful*, solve this system:

$$x = \left(\tfrac{1}{2} + \tfrac{1}{3}\right)y + t \qquad x' = \left(\tfrac{1}{3} + \tfrac{1}{4}\right)\left(y + y'\right) \qquad t' = \left(\tfrac{1}{6} + \tfrac{1}{7}\right)\left(x + x'\right)$$

$$y = \left(\tfrac{1}{4} + \tfrac{1}{5}\right)z + t \qquad y' = \left(\tfrac{1}{4} + \tfrac{1}{5}\right)\left(z + z'\right)$$

$$z = \left(\tfrac{1}{6} + \tfrac{1}{7}\right)x + t \qquad z' = \left(\tfrac{1}{6} + \tfrac{1}{7}\right)\left(t + t'\right)$$

- Any multiple *k* of this set of numbers is a solution:

$x =$	10366482	$x' =$	7206360
$y =$	7460514	$y' =$	4893246
$z =$	7358060	$z' =$	3515820
$t =$	4149387	$t' =$	5439213

Archimedes' Cattle Problem

Any multiple k of this set of numbers is a solution:

$x =$	10366482	$x' =$	7206360
$y =$	7460514	$y' =$	4893246
$z =$	7358060	$z' =$	3515820
$t =$	4149387	$t' =$	5439213

Tell'st thou unfailingly how many head
the Sun possessed, o friend, both bulls well-fed
and cows of ev'ry colour—no-one will
deny that thou hast numbers' art and skill,
though *not yet dost thou rank among the wise*.

Archimedes' Cattle Problem

- To be considered *wise*, solve this additional problem:

  Whene'er the Sun god's white bulls joined the black,
  their multitude would gather in a pack
  of equal length and breadth, and squarely throng
  Thrinacia's territory broad and long.
  But when the brown bulls mingled with the flecked,
  in rows growing from one would they collect,
  forming a perfect triangle, with ne'er
  a diff'rent-coloured bull, and none to spare.

  Friend, canst thou analyse this in thy mind,
  and of these masses all the measures find,
  go forth in glory! be assured all deem
  thy wisdom in this discipline supreme!

- So: $x + y$ is a square, $z + t$ is a triangular number.

Archimedes' Cattle Problem

- $x + y = 17826482 \cdot k = 2^2 \cdot 4456749 \cdot k = n^2$,
  - So $k = (n/2)^2 = 4456749 \cdot Y^2$.
- $z + t = m(m+1)/2$, so $8(z + t)+1 = (2m+1)^2 = X^2$.
  - So $X^2 = 8(11507447 \cdot k) + 1 = 92059576 \cdot k + 1$.

Archimedes' Cattle Problem

- $x + y = 17826482 \cdot k = 2^2 \cdot 4456749 \cdot k = n^2$,
  - So $k = (n/2)^2 = 4456749 \cdot Y^2$.
- $z + t = m(m+1)/2$, so $8(z + t)+1 = (2m+1)^2 = X^2$.
  - So $X^2 = 8(11507447 \cdot k) + 1 = 92059576 \cdot k + 1$.
- Putting these together, we get

$$X^2 = (4456749 \cdot 92059576)Y^2 + 1,$$

or:

$$X^2 = 410{,}286{,}423{,}278{,}424 \cdot Y^2 + 1.$$

Archimedes' Cattle Problem

- $x + y = 17826482 \cdot k = 2^2 \cdot 4456749 \cdot k = n^2$,
  - So $k = (n/2)^2 = 4456749 \cdot Y^2$.
- $z + t = m(m+1)/2$, so $8(z + t)+1 = (2m+1)^2 = X^2$.
  - So $X^2 = 8(11507447 \cdot k) + 1 = 92059576 \cdot k + 1$.
- Putting these together, we get

$$X^2 = (4456749 \cdot 92059576)Y^2 + 1,$$

or:

$$X^2 = 410{,}286{,}423{,}278{,}424 \cdot Y^2 + 1.$$

- Euler's method of solution requires 203,254 steps, and arrives at a solution for the size of the herd: $\approx 7.76 \cdot 10^{206544}$. (Euler did not attempt this!)

REFERENCES

- Euler, Leonhard, "De solutione problematum Diophanteorum per numeros integros," *Commentarii academiae scientiarum Petropolitanae* **6** (1732/3) 1738, pp. 175-188. Available online at the Euler Archive.
- Euler, Leonhard, "De solutione problematum Diophanteorum per numeros integros," *Novi commentarii academiae scientiarum Petropolitanae* **11** (1765) 1767, pp. 28-66. Available online at the Euler Archive.
- Lenstra, H. W. "Solving the Pell Equation," *Notices of the AMS* **49** (2) 2002, pp. 182-192.
- Nelson, H. L. "A solution to Archimedes' cattle problem," *J. Recreational Math.* **13** (3) 1980–81, pp. 162-176.
- Sandifer, C. Edward. "How Euler Did It: Euler and Pell," *MAA Online*, April 2005. Available online at the Euler Archive.

Slides available online at: https://tinyurl.com/EulerPell

Email: etou@uw.edu

Thank you!