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Optimal output-transitions for linear systems☆

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Abstract

This article addresses the optimal (minimum-input-energy) output-transition problem for linear systems. The goal is to transfer the output from an initial value $y(t)=\underline{y}$ (for all time $t \leqslant t_i$) to a final output value $y(t)=\bar{y}$ (for all time $t \geqslant t_f$). Previous methods solve this output-transition problem by transforming it into a state-transition problem; the initial and final states ($x(t_i), x(t_f)$, respectively) are chosen and a minimum-energy state-to-state transition problem is solved. However, the choice of the initial and final states can be ad hoc and the resulting output-transition cost (input energy) may not be minimal. The contribution of this article is the solution of the optimal output-transition problem. An example system with elastic dynamics is studied to illustrate the proposed method. Simulation results are presented that show substantial reduction of transition costs with the use of the proposed method when compared to the use of minimum-energy state-to-state transitions.

Keywords: Optimal control; Output transition; Minimum energy; Inversion; Precision positioning; Vibration compensation

1. Introduction

Changing the output of a system from one value to another (i.e. output transition) is a fundamental control problem. Formally, the problem studied in this article is to transfer the system output, with minimum input energy, from an initial value $y(t)=\underline{y}$ (for all time $t \leqslant t_i$) to a final output value $y(t)=\bar{y}$ (for all time $t \geqslant t_f$) as shown in Fig. 1. Such output-transition problems arise in a wide range of applications, for example, in the positioning of flexible structures which include: (I) large-scale light-weight (and therefore, flexible) space manipulators and antennae (Farrenkopf, 1979; Singhose, Banerjee, & Seering, 1997; Wie, Sinha, & Liu, 1993); (II) medium-scale read-write heads of disk drives (Ho, 1997; Miu & Bhat, 1991); and (III) relatively small-scale piezo-based nano-positioners (Bleuler, Clavel, Breguet, Langeu, & Peanette, 2000; Croft, Shedd, & Devasia, 2001). During output transitions, the elastic dynamics of these structures can lead to residual vibrations after the completion of a positioning maneuver, which results in loss of positioning precision. Such vibrations in the output could take a prohibitively long time to reduce to an acceptable level. Therefore, procedures that minimize (or remove) residual output-vibrations are needed to achieve acceptable transitions in these systems; such an approach to output transitions is studied in this article.

The contribution of this article is the solution of the minimal-input-energy, output-transition problem. Previous methods solve the output-transition problem by transforming it into a state-transition problem. In such methods, the initial and final states ($x(t_i)=\underline{x}$, $x(t_f)=\bar{x}$, respectively) are chosen, and an optimal state-transition problem (from $x(t_i)$ to $x(t_f)$) is solved (Lewis & Syrmos, 1995). For example, the initial and final states can be chosen as equilibrium states of the system at the initiation ($t=t_i$) and completion ($t=t_f$) of the output transition. This choice, of initial and final states, enables the output to be maintained at the desired value $y(t)=\bar{y}$ without residual vibrations (for all time $t \geqslant t_f$). In flexible structures such equilibrium states correspond to rigid-body configurations of the structure. However, the choice of equilibrium states as the initial and final states may not be optimal. On the other hand, an arbitrary choice of the initial and final states is also not acceptable because it can lead to transient errors, for example, after the

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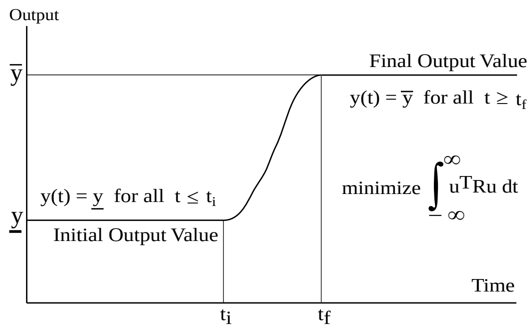

Fig. 1. The output-transition problem.

completion of the output transition. The novelty of the proposed approach is that it quantifies the possible choices in the initial and final states and then optimally chooses them to minimize the output-transition cost.

The proposed approach uses pre- and post-actuation inputs to reduce the transition cost (i.e., input energy) without changing the output-transition time ($T_{tran} = t_f - t_i$). For example, consider the transfer of a disk-drive read-write head from one track of a disk to another track. Pre-actuation can be used before the initiation of the output transition to set up the *optimal* initial condition without changing the output position, i.e., maintain the read-write head over the initial track ($y = \underline{y}$ for time $t \leqslant t_i$). Similarly, post-actuation can be used after the completion of the output transition to maintain the read-write head over the final track ($y = \bar{y}$ for time $t \geqslant t_f$). Because the output is precisely controlled during the pre- and post-actuation phase, read and write operations can still be performed during the pre- and post-actuation. Thus, the effective seek time (i.e., the time interval (t_i, t_f) during which read and write operations cannot be performed) is not effected by the use of the pre- and post-actuation.

In contrast to the output-transition problem, the state-transition problem is well understood. Note that a controllable linear system can be transferred from any initial state to any final state; there are multiple input trajectories that could achieve such a state transition. For example, if the boundary conditions (initial state $x(t_i) = \underline{x}$ and final state $x(t_f) = \bar{x}$) are prescribed, then a control input can be chosen using standard optimal control approaches (see, e.g., Lewis & Syrmos, 1995) to minimize (I) the time taken to achieve the state transition (i.e., minimum time problems), or (II) the input energy needed to achieve the transition. Such state-transition approaches have been used to achieve zero-residual-vibration output transitions in flexible structures. For example, the input-shaping scheme (e.g., in Singhose et al., 1997; Pao & Lau, 2000) finds an input such that the output and a suitable number of its time derivatives are zero after the transition. Necessary and sufficient conditions for an input to achieve such zero-residual-vibration state-transitions were characterized in the Laplace domain in Bhat and Miu (1990). The central idea in these techniques is to avoid vibrations by completing the maneuver with a final state ($x(t_f) = \bar{x}$ at time t_f) that is a rigid-body configuration with zero elastic-deformations. Thus, these techniques place a constraint on the state at the completion of the output transition; in contrast, the proposed approach does not constrain the final state. Rather the proposed output-transition approach only requires that the final output $\bar{y}$ be achieved at time t_f—the output is then maintained at the final value $y = \bar{y}$ using inversion-based post-actuation input for time $t < t_f$. Similarly, the proposed approach does not constrain the state at the initiation of the output transition ($x(t_i)$), and a pre-actuation input for time $t < t_i$ is used to maintain the output value $y = \underline{y}$ before the initiation of the output transition (Devasia, Chen, & Paden, 1996). The additional freedom in the choice of the boundary states ($x(t_i)$ and $x(t_f)$) is then exploited to optimally reduce the output-transition cost. It is noted that the use of pre-actuation requires that the output-transition's start time t_i be known in advance. If the start time t_i is not known in advance then the proposed approach can be modified to only use post-actuation without pre-actuation (see Remark 5).

The optimal output-transition problem was posed previously in Piazzi & Visioli (2000, 2001), in which an inversion-based approach was used to plan an output transition. However, these results require the user to specify (a priory) the set of acceptable output trajectories during the transition (using polynomials); the method to choose the output trajectory is ad hoc. Similar pre-specification of a desired output trajectory was also used by Dowd and Thanos (2000) to achieve smooth transitions between output-trajectory segments in industrial positioning systems. In contrast, we do not require the pre-specification of the output trajectory; rather, the best output trajectory is obtained as the result of the proposed optimization procedure. We do however, use the stable inversion approach (as in Piazzi & Visioli (2000, 2001)) to find the pre-actuation ($t < t_i$) inputs that maintain output tracking $y = \underline{y}$ before the initiation of the output transition, and similarly to find post-actuation ($t > t_f$) inputs to maintain the output at $\bar{y}$. The inversion-based approach, used to find pre- and post-actuation inputs, is then integrated with standard optimal control approaches during the output transition (between times t_i and t_f) to solve the optimal output-transition problem.

The paper is organized as follows. In Section 2, the point-to-point output-transition (PPT) problem is formulated. Two approaches are proposed: (I) using standard state-to-state transition approach; and (II) integrating the state-to-state transition approach with an inversion-based pre- and post-actuation approach, which yields the optimal solution to the PPT problem (presented in Section 3). The proposed method is illustrated using an example system with elastic dynamics in Section 4. Simulation results show substantial reduction of transition costs with the use of the proposed method when compared to the use of previous techniques that are based on state-to-state transitions. Conclusions are in Section 5.

2. Problem formulation

We begin by formulating the PPT problem.

2.1. PPT problem

Consider a square (same number of inputs as outputs), linear, time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^p$ is the input and $y(t) \in \Re^p$ is the output. A direct feedforward term $Du(t)$ can be included in the output equation; however, it is omitted to simplify the presentation. Furthermore, let $\underline{x}$ and $\bar{x}$ be equilibrium points of the system (also referred to as delimiting states), and let the corresponding outputs be $\underline{y}$ and $\bar{y}$ respectively, i.e.,

Definition 1. *Delimiting states for transition.*

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} = 0, \quad \underline{y} = C\underline{x}, \\ \dot{\bar{x}} &= A\bar{x} = 0, \quad \bar{y} = C\bar{x}. \end{aligned} \tag{2}$$

The output-transition problem is formally stated next.

Definition 2. *PPT problem.* Given the delimiting states and the transition time interval $[t_i, t_f]$, find a bounded input-state trajectory $[u_{ff}(\cdot), x_{ref}(\cdot)]$ such that the following three requirements are met:

(I) The system equations are satisfied for all time $(-\infty < t < \infty)$

$$\dot{x}_{ref}(t) = Ax_{ref}(t) + Bu_{ff}(t); \qquad y_{ref}(t) = Cx_{ref}(t).$$

(II) The output transitions from $\underline{y}$ to $\bar{y}$ in the time interval $[t_i, t_f]$. The output is maintained at the desired value before and after the output transition

$$y_{ref}(t) = \underline{y} \quad \text{for all } t \leqslant t_i;$$

$$y_{ref}(t) = \bar{y} \quad \text{for all } t \geqslant t_f.$$

(III) The system state approaches the delimiting states as time goes to (plus or minus) infinity,

$$x_{ref}(t) \to \underline{x} \quad \text{as } t \to -\infty;$$

$$x_{ref}(t) \to \bar{x} \quad \text{as } t \to \infty.$$

Remark 1. Although the goal is to transfer the system between the delimiting states $\underline{x}$ and $\bar{x}$, the critical issue is to change the output from $\underline{y}$ to $\bar{y}$ in a specified transition time, $T_{tran} = t_f - t_i$. Therefore, the states at the beginning and end of output transitions, $x_{ref}(t_i)$ and $x_{ref}(t_f)$, need not be the delimiting states, $\underline{x}$ and $\bar{x}$. Pre- and post-actuation inputs, applied outside of the transition interval, $[t_i, t_f]$, could be used to transfer the state x_{ref} to the delimiting states. Such changes in state x_{ref} outside of the transition interval are acceptable as long as the output is maintained at the desired value.

2.2. Approach 1: state-to-state transition

One approach to achieve the output transition is to transfer the system between the delimiting states, i.e., from $x(t_i) = \underline{x}$ to $x(t_f) = \bar{x}$ within the transition time. It is noted that the final delimiting state is an equilibrium point; therefore, the system state can be maintained at that value without changing the output. (For flexible structures, these delimiting states could correspond to rigid body configurations; therefore, there are no residual vibrations.) If the system is controllable, then there is an input that can achieve the desired state-to-state output transition; however, this input is not unique. In the following, we require that the input chosen for the state-to-state transition (SST) should minimize the input energy required for the transition. This minimal-energy SST problem can be posed and solved using standard linear quadratic optimal (LQ) control theory (see, e.g., Lewis & Syrmos, 1995).

Assumption 1. In the following, we assume that System (1) is controllable.

Definition 3. *Minimal-energy SST problem.* Find a bounded input-state trajectory $[u_{ff}(\cdot), x_{ref}(\cdot)]$ that transfers System (1) from an initial state $x(t_i)$ to a final state $x(t_f)$ and minimizes the following cost function

$$J_{sst}(t_i, t_f, x(t_i), x(t_f), u) = \int_{t_i}^{t_f} u^T R u \, dt, \tag{3}$$

where R is a positive-definite symmetric matrix.

Definition 4. *Final-state difference.* If no input is applied during the time interval $[t_i, t_f]$, then the state of the system at time t_f will be $e^{A(t_f - t_i)} x(t_i)$. The difference between this no-applied-input state and the desired state $x(t_f)$ at time t_f, is referred to as the final-state difference, and can be written as

$$d(t_i, t_f) = x(t_f) - e^{A(t_f - t_i)} x(t_i). \tag{4}$$

Lemma 1. *Solution to minimal-energy SST problem. Given the boundary states $x(t_i)$ and $x(t_f)$, the input $u_{sst}(\cdot)$ that minimizes the cost function J_{sst} is given by*

$$u_{sst}(t) = R^{-1} B^T e^{A^T(t_f - t)} G^{-1}_{(t_i, t_f)} d(t_i, t_f), \tag{5}$$

where $G_{(t_i, t_f)}$ is the invertible controllability grammian given by

$$G_{(t_i, t_f)} = \int_{t_i}^{t_f} e^{A(t_f - \tau)} B R^{-1} B^T e^{A^T(t_f - \tau)} \, d\tau. \tag{6}$$

The transition cost when using the minimum-energy SST input (5) is given by

$$J^*_{sst}(t_i, t_f, x(t_i), x(t_f)) = d(t_i, t_f)^T G^{-1}_{(t_i, t_f)} d(t_i, t_f). \tag{7}$$

Proof. See, for example, (Lewis & Syrmos, 1995, p. 165). □

An approach to achieve point-to-point output-transition (PPT) is to use the minimal energy solution to the SST problem, where the initial and final states for this SST problem are the initial and final delimiting states, respectively. This is stated formally in the next lemma.

Lemma 2. *Standard PPT. A solution to the PPT problem is to use the minimal energy solution to the SST problem (Lemma 1) with the state at the initiation of the output transition chosen as the initial delimiting state ($x(t_i) = \underline{x}$), and similarly with the state at the completion of the output transition chosen as the final delimiting state ($x(t_f) = \bar{x}$). The input $u_{ff} = u_{sppt}$ is given by*

$$u_{sppt}(t) = R^{-1}B^{T}e^{A^{T}(t_f - t)}G^{-1}_{(t_i,t_f)}\hat{d}(t_i,t_f) \quad if\ t_i \leqslant t \leqslant t_f$$
$$= 0\ otherwise, \tag{8}$$

where $\hat{d}(t_i,t_f) = [\bar{x} - e^{A(t_f - t_i)}\underline{x}]$. The reference state trajectory $x_{ref} = x_{sppt}$ is given by

$$x_{sppt}(t) = \underline{x} \quad if\ t < t_i$$
$$= e^{A(t-t_i)}\underline{x} + \int_{t_i}^{t}\{e^{A(t-\tau)}Bu_{sppt}(\tau)\}\,d\tau \quad if\ t_i \leqslant t \leqslant t_f$$
$$= \bar{x} \quad if\ t > t_f \tag{9}$$

and the cost J_{sppt} of using input u_{sppt} is given by

$$J_{sppt}(t_i,t_f) = \hat{d}(t_i,t_f)^{T}G^{-1}_{(t_i,t_f)}\hat{d}(t_i,t_f). \tag{10}$$

Proof. From Lemma 1, the state is transferred from $\underline{x}$ to $\bar{x}$. Conditions 2 and 3 for PPT (see Definition 2) are satisfied since $\underline{x}$ and $\bar{x}$ are delimiting states (see Definition 1). □

The solution to the PPT problem achieved using the minimum-energy solution to the SST problem is referred to as the standard approach to the PPT problem (or standard PPT, in short). It is noted that the standard PPT does not use pre- or post-actuation. In the following, we explore the use of pre- and post-actuation to further reduce the transition cost.

2.3. Approach 2: optimal output transition

The choice of the states $[x(t_i), x(t_f)]$, at the initiation $t = t_i$ and completion $t = t_f$ of the output transition, need not be fixed a priori as the delimiting states $(\underline{x}, \bar{x})$ (as was done in the standard PPT approach). In the following, the states at the initiation and completion of the output transition are considered as variables in the control-design, and are optimized. This optimal point-to-point output-transition (optimal PPT) problem, is stated next.

Definition 5. *Optimal PPT problem.* Find a bounded input-state trajectory $[u_{ff}(\cdot), x_{ref}(\cdot)]$ that solves the output-transition problem (see Definition 2), and minimizes the following cost function

$$J_{ppt}(t_i,t_f,u) = \int_{-\infty}^{\infty} u^{T}Ru\,dt, \tag{11}$$

where R is a positive-definite symmetric matrix (as in Eq. (3)).

3. Solution to the optimal PPT problem

The optimal point-to-point output-transition (optimal PPT) problem is solved in this section. This is done in three steps: (I) transforming the system into output-tracking form; (II) finding the input needed before the initiation of output transition (pre-actuation) and after the completion of the output transition (post-actuation); and (III) integrating the pre- and post-actuation with the minimal energy SST approach to find the solution to the optimal PPT problem.

3.1. Output-tracking form

In the PPT problem, the output has to be maintained constant outside the transition interval—this can be done using inversion-based approaches. We begin by rewriting the system equations in the output-tracking form (see, e.g., Isidori, 1989, Chapter 4; Sastry, 1999, Chapter 9), under the following assumption.

Assumption 2. System (1) has a well-defined relative degree ρ (e.g., see Sastry, 1999), where $\rho := [\rho_1, \rho_2, \ldots, \rho_p]$.

Output-tracking form. Under Assumption 2, there exists (I) a coordinate transformation, Φ, and (II) an input law that transforms the system into the output-tracking form (see, e.g., Isidori, 1989; Sastry, 1999).

Coordinate transformation: The system state x is transformed into the following new coordinates:

$$\Phi[\ \xi^{T}\quad \eta_s^{T}\quad \eta_u^{T}\quad \eta_c^{T}\ ]^{T} = x, \tag{12}$$

where the output and its time-derivatives upto order $\rho - 1 := [\rho_1 - 1, \rho_2 - 1, \ldots, \rho_p - 1]$ are denoted by

$$\xi = [y_1, \dot{y}_1, \ldots, d^{\rho_1 - 1}y_1/dt^{\rho_1 - 1}, y_2, \dot{y}_2, \ldots, d^{\rho_2 - 1}y_2/dt^{\rho_2 - 1}, \ldots, y_p, \dot{y}_p, \ldots, d^{\rho_p - 1}y_p/dt^{\rho_p - 1}]^{T} \tag{13}$$

and η_s, η_u, η_c represent the stable, unstable, and center subspaces of the internal dynamics η, respectively. Furthermore, the output and its time-derivatives upto order $\rho - 1$ are known if the desired output is specified; this known term is defined as

$$\xi_d(t) = \xi(t)|_{y(\cdot) = y_d(\cdot)}. \tag{14}$$

Input law: The input law needed for the transformation has the following general form

$$u_{ff}(t) = B_s\eta_s(t) + B_u\eta_u(t) + B_c\eta_c(t) + B_{\xi}\mathbb{Y}(t), \tag{15}$$

where the output and its time-derivatives upto order $\rho = [\rho_1, \rho_2, \ldots, \rho_p]$ are denoted by

$$\mathbb{Y} = [\xi^T, d^{\rho_1} y_1/dt^{\rho_1}, d^{\rho_2} y_2/dt^{\rho_2}, \ldots, d^{\rho_p} y_p/dt^{\rho_p}]^T. \tag{16}$$

Again, the output and its time-derivatives upto order ρ are known in terms of the desired output as

$$\mathbb{Y}_d(t) = \mathbb{Y}(t)|_{y(t)=y_d}. \tag{17}$$

Output-tracking form: System (1) can be transformed into the following output-tracking form using the coordinate transformation in Eq. (12) and input law in Eq. (15):

$$\dot{\xi}(t) = \dot{\xi}_d(t), \tag{18}$$

$$\begin{bmatrix} \dot{\eta}_s(t) \\ \dot{\eta}_u(t) \\ \dot{\eta}_c(t) \end{bmatrix} = [A_{int}] \begin{bmatrix} \eta_s(t) \\ \eta_u(t) \\ \eta_c(t) \end{bmatrix} + B_{int}\mathbb{Y}_d(t), \tag{19}$$

where

$$A_{int} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{bmatrix}, \quad B_{int} = \begin{bmatrix} B_{int,s} \\ B_{int,u} \\ B_{int,c} \end{bmatrix}$$

and the internal dynamics of the system, represented by Eq. (19), is assumed to be decoupled (without loss of generality); the stable, unstable and center subspaces of the internal dynamics are η_s, η_u, and η_c, respectively. The corresponding submatrices A_s, and $-A_u$ are Hurwitz, i.e., their eigenvalues lie on the open left half of the complex plane. The eigenvalues of submatrix A_c lie on the imaginary axis of the complex plane.

Remark 2. The well-defined relative degree in Assumption 2 can be replaced by general invertibility conditions with appropriate changes in the procedures to compute the inverse system and to obtain the internal dynamics (Silverman, 1969; Sain, 1969). The approach can be extended to non-square systems provided the system is invertible. However, the approach cannot be used if the number of inputs is less than the number of independent outputs.

3.2. Use of pre- and post-actuation in optimal PPT

Outside the transition interval $[t_i, t_f]$, the output is constant, and the input that maintains the output at a constant value can be obtained using inversion approaches; this is discussed next.

3.2.1. Maintaining output constant ($y = \underline{y}$) before output transition ($t \leqslant t_i$)

The pre-actuation input before the initiation of the output transition aims to maintain the output constant ($y = \underline{y}$ for all time $t \leqslant t_i$). To find the pre-actuation input, we begin with two coordinate transformations: (I) to rewrite the system equations in the output-tracking form; and (II) to shift the origin of the system to the initial delimiting state $\underline{x}$.

Coordinate transformation into output-tracking form: Consider the initial delimiting state $\underline{x}$ in the output-tracking coordinates (using the state transformation in Eq. (12))

$$\Phi[\,\underline{\xi}^T \quad \underline{\eta}^T\,]^T := \Phi[\,\underline{\xi}^T \quad \underline{\eta_s}^T \quad \underline{\eta_u}^T \quad \underline{\eta_c}^T\,]^T = \underline{x}. \tag{20}$$

Since $\underline{x}$ is an equilibrium point of the system (with input $u = 0$), we have the following relationships for the internal dynamics states (from Eq. (15) and Eq. (19))

$$0 = A_{int}\underline{\eta} + B_{int}\underline{\mathbb{Y}} \tag{21}$$

$$0 = B_s\underline{\eta_s} + B_u\underline{\eta_u} + B_c\underline{\eta_c} + B_\xi\underline{\mathbb{Y}}, \tag{22}$$

where, as in Eq. (17), the output and its time-derivatives upto order ρ computed at the constant output ($y = \underline{y}$) are denoted by $\underline{\mathbb{Y}}(t) = \mathbb{Y}(t)|_{y(t)=\underline{y}}$.

Coordinate transformation to shift the origin: Next, we move the origin of the system to the initial delimiting state with the following change of coordinates:

$$\hat{\underline{\xi}}(t) := \xi(t) - \underline{\xi},$$

$$\hat{\underline{\eta}}(t) := \begin{bmatrix} \hat{\underline{\eta}}_s(t) \\ \hat{\underline{\eta}}_u(t) \\ \hat{\underline{\eta}}_c(t) \end{bmatrix} := \begin{bmatrix} \eta_s(t) - \underline{\eta_s} \\ \eta_u(t) - \underline{\eta_u} \\ \eta_c(t) - \underline{\eta_c} \end{bmatrix}. \tag{23}$$

The system dynamics in the output-tracking form can be rewritten for all $t < t_i$, by subtracting Eq. (21) from Eq. (19) and using Eq. (23), as

$$\frac{d}{dt}\hat{\underline{\xi}}(t) = 0, \qquad \frac{d}{dt}\hat{\underline{\eta}}(t) = A_{int}\hat{\underline{\eta}}(t). \tag{24}$$

The exact output tracking input can be written for all $t < t_i$, by subtracting Eq. (22) from Eq. (15) and by using Eq. (23), as

$$u_{ff}(t) = B_s\hat{\underline{\eta}}_s(t) + B_u\hat{\underline{\eta}}_u(t) + B_c\hat{\underline{\eta}}_c(t). \tag{25}$$

Pre-actuation: The next lemma states that the internal state-components, related to the stable and center subspaces of the internal dynamics (η_s and η_c, respectively), must remain constant during pre-actuation. It also finds the pre-actuation input in terms of the state-component $\eta_u(t_i)$ (related to the unstable subspace of the internal dynamics) at the initiation of output transition.

Lemma 3. *Pre-actuation. Let system* (1) *be controllable and have a well defined relative degree* (*Assumptions* 1 *and* 2). *Then the pre-actuation input for point-to-point output transition is uniquely specified in terms of the state component $\eta_u(t_i)$ (related to the unstable subspace of the internal dynamics) at the initiation of the output transition (time t_i).*

(1) *The state-components related to the stable and center subspaces of the internal dynamics must remain constant during pre-actuation, i.e., for time $t \leqslant t_i$*

$$\eta_s(t) = \underline{\eta_s}, \qquad \eta_c(t) = \underline{\eta_c}. \tag{26}$$

(2) *The exact-output maintaining, pre-actuation input for time $t < t_i$ is given by*

$$u_{pre}(t) = B_u[e^{A_u(t-t_i)})\underline{\hat{\eta}_u}(t_i)] = B_u[e^{A_u(t-t_i)}(\eta_u(t_i) - \underline{\eta_u})]. \quad (27)$$

Proof. The internal dynamics Eq. (24) is autonomous, and an explicit solution to the internal dynamics (for time $t < t_i$) can be found as $\underline{\hat{\eta}}(t) = e^{-A_{int}(t_i-t)}\underline{\hat{\eta}}(t_i)$. Then, the requirement that the system state should tend to the delimiting state $\underline{x}$ as time t goes to $-\infty$ (see requirement 3, in Definition 2) can be expressed in terms of the above solution to the internal dynamics as

$$\begin{aligned} 0 &= \lim_{t\to-\infty} \underline{\hat{\eta}_s}(t) = \lim_{t\to-\infty} e^{-A_s(t_i-t)}\underline{\hat{\eta}_s}(t_i),\\ 0 &= \lim_{t\to-\infty} \underline{\hat{\eta}_u}(t) = \lim_{t\to-\infty} e^{-A_u(t_i-t)}\underline{\hat{\eta}_u}(t_i), \quad (28)\\ 0 &= \lim_{t\to-\infty} \underline{\hat{\eta}_c}(t) = \lim_{t\to-\infty} e^{-A_c(t_i-t)}\underline{\hat{\eta}_c}(t_i). \end{aligned}$$

From this equation, we obtain the following constraint on the state at the initiation of the output transition at time $t = t_i$

$$\underline{\hat{\eta}_s}(t_i) = 0, \quad \underline{\hat{\eta}_c}(t_i) = 0 \quad (29)$$

because submatrices $-A_s$ and $-A_c$ are not Hurwitz. For any other values of the stable subspace of the internal dynamics $\underline{\hat{\eta}_s}(t_i)$ and the center subspace of the internal dynamics $\underline{\hat{\eta}_c}(t_i)$, the system will not go to the delimiting state $\underline{x}$ as $t \to -\infty$. The first statement of the lemma (see Eq. (26)) follows from Eqs. (23) and (29). While the state components of the stable and center subspaces of the internal dynamics are constrained, there are no such restrictions on the choice of the state component of the unstable subspace, $\underline{\hat{\eta}_u}(t_i)$. The delimiting state can be achieved for any choice of the unstable subspace of the internal dynamics $\underline{\hat{\eta}_u}(t_i)$ because submatrix $-A_u$ is Hurwitz. The second statement of the Lemma (Eq. (27)) follows by substituting Eq. (29) into Eq. (25) and then using the definition of $\underline{\hat{\eta}_u}$ in Eq. (23). □

Note that the pre-actuation input is completely specified in terms of the unstable component $\eta_u(t_i)$ of the internal dynamics at the initiation of the desired output transition. Therefore, the cost of the pre-actuation input can be quantified in terms of this component as shown in the next Lemma.

Lemma 4. *Pre-actuation cost. For a specified unstable subspace of the internal dynamics $\eta_u(t_i)$ the cost of the pre-actuation input is equal to*

$$J_{pre}(\eta_u(t_i)) := \int_{-\infty}^{t_i} u_{pre}^T(t) R u_{pre}(t)\, dt = \eta_u^T(t_i) W_{pre}\eta_u(t_i) - 2\eta_u^T(t_i) W_{pre}\underline{\eta_u} + \underline{\eta_u}^T W_{pre}\underline{\eta_u}, \quad (30)$$

where

$$W_{pre} = \int_0^{\infty} e^{-A_u^T\tau} B_u^T R B_u e^{-A_u\tau}\, d\tau \quad (31)$$

can be found by solving the Lyapunov equation

$$W_{pre}A_u + A_u^T W_{pre} = B_u^T R B_u. \quad (32)$$

Proof. From Eqs. (27) and (30)

$$\begin{aligned} J_{pre}(\eta_u(t_i)) &= \underline{\hat{\eta}_u}^T \left[\int_{-\infty}^{t_i} e^{A_u^T(t-t_i)} B_u^T R B_u e^{A_u(t-t_i)}\, dt \right] \underline{\hat{\eta}_u} \\ &= \underline{\hat{\eta}_u}^T \left[\int_0^{\infty} e^{-A_u^T\tau} B_u^T R B_u e^{-A_u\tau} d\tau \right] \underline{\hat{\eta}_u} \\ &\quad \text{by setting } -\tau = (t - t_i) \\ &= [\eta_u(t_i) - \underline{\eta_u}]^T W_{pre}[\eta_u(t_i) - \underline{\eta_u}]. \end{aligned}$$

Since submatrix $-A_u$ is Hurwitz, W_{pre} in the above equation can be found as the unique, symmetric solution to the following Lyapunov equation (see, e.g., Chen, 1999, Theorem 5.6, Chapter 6), $W_{pre}(-A_u) + (-A_u)^T W_{pre} = -B_u^T R B_u$, which completes the proof. □

3.2.2. Maintaining output constant ($y = \bar{y}$) after output-transition ($t \geqslant t_f$)

The post-actuation after the completion of the output transition aims to maintain the output constant ($y(t) = \bar{y}$ for all time $t \geqslant t_f$). As in the pre-actuation case, we begin with two coordinate transformations: (I) to rewrite the system equations in the output-tracking form; and (II) to shift the origin of the system to the final delimiting state $\bar{x}$.

Coordinate transformation and shift of origin to $\bar{x}$; Consider the final delimiting state $\bar{x}$ in the output-tracking coordinates (using the state transformation in Eq. (12))

$$\Phi[\,\bar{\xi}^T \quad \overline{\eta_s}^T \quad \overline{\eta_u}^T \quad \overline{\eta_c}^T\,]^T = \bar{x}. \quad (33)$$

Furthermore, let the origin of the system be moved to the final delimiting state $\bar{x}$

$$\begin{aligned} \hat{\bar{\xi}}(t) &:= \xi(t) - \bar{\xi}, \quad \hat{\overline{\eta_s}}(t) := \eta_s(t) - \overline{\eta_s},\\ \hat{\overline{\eta_u}}(t) &:= \eta_u(t) - \overline{\eta_u}, \quad \hat{\overline{\eta_c}}(t) := \eta_c(t) - \overline{\eta_c}, \end{aligned} \quad (34)$$

where a bar above the variables indicates that the variables are related to the post-actuation phase of the PPT.

Post-actuation input and cost: The next lemma states that the state-components related to the unstable and center subspaces of the internal dynamics (η_u and η_c, respectively) must remain constant during post-actuation. It also obtains the post-actuation input u_{post} in terms of the state-component $\eta_s(t_f)$, which is related to the stable subspace of the internal dynamics at the completion of output transition. The results of this next lemma are then used to quantify the cost of using the post-actuation input.

Lemma 5. *Post-actuation. Let system* (1) *be controllable and have a well defined relative degree* (*Assumptions* 1 *and* 2). *Then the post-actuation input for point-to-point output transition is uniquely specified in terms of the state component* $\eta_s(t_f)$, *which is related to the stable subspace of the internal dynamics at the completion of output transition* (*time* t_f).

(1) *The state-components related to the unstable and center subspaces of the internal dynamics must remain constant during post-actuation, i.e., for all time* $t \geqslant t_f$ $\eta_u(t) = \overline{\eta_u}$ *and* $\eta_c(t) = \overline{\eta_c}$,

(2) *the exact-output maintaining, post-actuation input for time* $t > t_f$ *is given by*

$$u_{post}(t) = u_{ff}(t) = B_s[e^{A_s(t-t_f)}\hat{\overline{\eta_s}}(t_f)] = B_s[e^{A_s(t-t_f)}(\eta_s(t_f) - \overline{\eta_s})]. \quad (35)$$

Proof. This proof is similar to the proof of Lemma 3, and so this proof is omitted. □

Lemma 6. *Post-actuation cost. For a specified stable subspace of the internal dynamics* $\eta_s(t_f)$, *the cost of the post-actuation input is equal to*

$$J_{post}(\eta_s(t_f)) := \int_{t_f}^{\infty} u_{post}^T(t) R u_{post}(t)\,dt = \eta_s^T(t_f) W_{post} \eta_s(t_f) - 2\eta_s^T(t_f) W_{post}\overline{\eta_s} + \overline{\eta_s}^T W_{post}\overline{\eta_s} \quad (36)$$

where

$$W_{post} = \int_0^{\infty} e^{A_s^T \tau} B_s^T R B_s e^{A_s \tau}\,d\tau \quad (37)$$

is the solution to the Lyapunov equation

$$W_{post} A_s + A_s^T W_{post} = -B_s^T R B_s. \quad (38)$$

Proof. This proof is similar to the proof of Lemma 4, and is omitted. □

3.3. Optimal PPT

Lemmas 3 and 5 imply that the only freedom in the state $x(t_i)$, at the initiation of PPT, is in the choice of the unstable component $\eta_u(t_i)$ of the internal dynamics. Similarly, the only freedom in the state $x(t_f)$, at the completion of PPT, is in the choice of the stable component $\eta_s(t_f)$ of the internal dynamics. Therefore, a particular choice of these state component variables, $\eta_u(t_i)$ and $\eta_s(t_f)$, completely specifies the boundary states $[x(t_i), x(t_f)]$ at the initiation $t = t_i$ and completion $t = t_f$ of the PPT problem.

Definition 6. *Boundary conditions* Ψ. Components of the state, at the initiation and completion of output transition, that can be varied while achieving the desired output transition (i.e., solving the PPT problem) are

$$\Psi := [\eta_s^T(t_f) \quad \eta_u^T(t_i)]^T. \quad (39)$$

Remark 3. *Standard PPT boundary conditions*. The boundary conditions Ψ are constrained to be the delimiting states when using the standard PPT solution to the output-transition problem, i.e. $\Psi = \tilde{\Psi} := [\bar{\eta}_s^T \ \underline{\eta}_u^T]^T$.

Remark 4. *Optimal PPT boundary conditions*. The boundary conditions Ψ are not constrained to be $\tilde{\Psi}$ (as in Remark 3) in the optimal PPT approach; they are chosen to optimize the energy used for the output transition.

3.3.1. Minimum-energy PPT for a specified Ψ

Once this minimum-energy PPT is found for given boundary conditions Ψ, then the optimal PPT can be found by optimizing the boundary conditions. For a set of prescribed boundary conditions Ψ, the energy needed for the PPT can be found by adding (I) the minimum energy needed during the output-transition time-period $[t_i, t_f]$ and (II) the energy needed outside the output-transition time-period (i.e., $t < t_i$ and $t > t_f$). We begin by finding the minimum energy needed during the output-transition time-period.

Minimum energy during output transition: Lemmas 3 and 5 imply that the only freedom in the choice of state $x(t_i)$ at the initiation of output transition and in the choice of state $x(t_f)$ at the completion of the output transition are in the choice of boundary conditions Ψ. The choice of boundary conditions Ψ specifies the pre- and post-actuation inputs (and associated costs); however, it does not specify the input during the output-transition interval $[t_i, t_f]$. The input energy needed during the output-transition interval can be optimized by using minimum-energy SST technique (see Lemma 1) with the initial and final states chosen as

$$x(t_i) = \Phi[\underline{\xi}^T \quad \underline{\eta_s}^T \quad \eta_u(t_i)^T \quad \underline{\eta_c}^T]^T, := [\Phi_{\xi} \mid \Phi_{\eta_s} \mid \Phi_{\eta_u} \mid \Phi_{\eta_c}][\underline{\xi}^T \quad \underline{\eta_s}^T \quad \eta_u(t_i)^T \quad \underline{\eta_c}^T]^T \quad (40)$$

$$x(t_f) = \Phi[\bar{\xi}^T \quad \eta_s(t_f)^T \quad \overline{\eta_u}^T \quad \overline{\eta_c}^T]^T.$$

The associated minimum input-energy $J_{sst}^*(t_i, t_f, \Psi)$ during the output transition time-period can be written in terms of the boundary condition Ψ as (using Eqs. (4) and (7))

$$J_{sst}^*(t_i, t_f, \Psi) = d_{\Psi}(t_i, t_f)^T G_{(t_i,t_f)}^{-1} d_{\Psi}(t_i, t_f), \quad (41)$$

where $d_{\Psi}(t_i, t_f)$ is the final-state difference $d(t_i, t_f)$ rewritten in terms of the boundary conditions Ψ as (using Definition 4 and Eq. (40))

$$d_{\Psi}(t_i, t_f) = H_1 \hat{f} + H_2 \Psi, \quad (42)$$

where

$$H_1 := [\Phi_{\xi} \mid \Phi_{\eta_u} \mid \Phi_{\eta_c} \mid -W_{\xi} \mid -W_{\eta_s} \mid -W_{\eta_c}]$$

$$H_2 := [\Phi_{\eta_s} \mid -W_{\eta_u}]$$

$$[W_{\xi} \mid W_{\eta_s} \mid W_{\eta_u} \mid W_{\eta_c}] := e^{A(t_f-t_i)}[\Phi_{\xi} \mid \Phi_{\eta_s} \mid \Phi_{\eta_u} \mid \Phi_{\eta_c}]$$

$$\hat{f} := [\,\bar{\xi} \quad \bar{\eta}_u \quad \bar{\eta}_c \quad \underline{\xi} \quad \underline{\eta}_s \quad \underline{\eta}_c\,]^T.$$

The minimum input-energy $J^*_{sst}(t_i, t_f, \Psi)$ during the output transition time-period can be rewritten as (using Eqs. (41) and (42))

$$\begin{aligned} J^*_{sst}&(t_i, t_f, \Psi) \\ &= (H_1\hat{f} + H_2\Psi)^T G^{-1}_{(t_i,t_f)}(H_1\hat{f} + H_2\Psi) \\ &= \Psi^T H_2^T G^{-1}_{(t_i,t_f)} H_2 \Psi + 2\Psi^T H_2^T G^{-1}_{(t_i,t_f)} H_1 \hat{f} \\ &\quad + \hat{f}^T H_1^T G^{-1}_{(t_i,t_f)} H_1 \hat{f} \end{aligned} \tag{43}$$

Lemma 7. *Minimum-energy PPT for prescribed boundary conditions Ψ. For a given choice of the boundary conditions Ψ (as defined in Eq. (39)), the minimum energy needed to achieve the point-to-point output transition is quadratic in Ψ and has the form*

$$J_{ppt}(t_i, t_f, \Psi) = \Psi^T \Lambda \Psi - 2\Psi^T b + c. \tag{44}$$

Proof. The total cost $J_{ppt}(t_i, t_f, \Psi)$ over all time $t \in (-\infty, \infty)$, for minimal energy PPT with a specified choice of boundary conditions Ψ, can then be obtained by adding the cost $J^*_{sst}(t_i, t_f, \Psi)$ during output transition to the pre-actuation cost $J_{pre}(\eta_u(t_i))$ and post-actuation cost $J_{post}(\eta_s(t_f))$. The lemma follows by substituting for $J_{pre}(\eta_u(t_i))$, $J_{post}(\eta_s(t_f))$, and $J^*_{sst}(t_i, t_f, \Psi)$, from Eqs. (30), (36), and (43), respectively, and setting

$$\Lambda := \begin{bmatrix} W_{post} & 0 \\ 0 & W_{pre} \end{bmatrix} + H_2^T G^{-1}_{(t_i,t_f)} H_2, \tag{45}$$

$$b := \begin{bmatrix} W_{post}\overline{\eta_s} \\ W_{pre}\underline{\eta_u} \end{bmatrix} - H_2^T G^{-1}_{(t_i,t_f)} H_1 \hat{f},$$

$$c := \overline{\eta_s}^T W_{post} \overline{\eta_s} + \underline{\eta_u}^T W_{pre} \underline{\eta_u} + \hat{f}^T H_1^T G^{-1}_{(t_i,t_f)} H_1 \hat{f}. \qquad \square$$

3.3.2. Solution to the optimal PPT problem

Optimal values for the boundary conditions Ψ are found in the following theorem, which also obtains the input that solves the optimal PPT problem.

Theorem 1. *Optimal PPT. Let system* (1) *be controllable and have a well-defined relative degree (Assumptions* 1 *and* 2*). Then the optimal PPT problem always has a solution as described in the following three statements.*

(I) *The PPT cost function (Eq. (11)) is minimized by the following choice of boundary conditions $\Psi = \Psi^* = [(\eta_s^*)^T \ (\eta_u^*)^T]^T$ given by*

$$\Psi^* = \Lambda^{-1} b \quad \textit{if } \Lambda \textit{ is invertible},$$

$$\Psi^* = \Lambda^{\dagger} b \quad \textit{otherwise}, \tag{46}$$

where $\Lambda^{\dagger}$ is the pseudo (generalized) inverse of Λ (e.g., see Ortega, 1987 *for the definition of the generalized inverse).*

(II) *The optimal input $u_{ff} = u^*_{ppt}$ can include pre-actuation ($t < t_i$) and post-actuation ($t > t_f$), and is given by*

$$u^*_{ppt}(t) = B_u[e^{A_u(t-t_i)}(\eta_u^* - \underline{\eta_u})] \quad \textit{if } t < t_i,$$

$$= R^{-1}B^T e^{A^T(t_f-t)} G^{-1}_{(t_i,t_f)} d_{\Psi^*}(t_i, t_f)$$

if $t_i \leqslant t \leqslant t_f$, where $d_{\Psi^*}(t_i, t_f)$ is obtained by setting $\Psi = \Psi^*$ in Eq. (42) and

$$u^*_{ppt}(t) = B_s[e^{A_s(t-t_f)}(\eta_s^* - \overline{\eta_s})] \quad \textit{if } t > t_f.$$

(III) *The corresponding reference state trajectory $x_{ref} = x_{ppt}$ is given by*

$$\begin{aligned} x_{ppt}&(t) \\ &= \Phi[\underline{\xi}^T \quad \underline{\eta_s}^T \ [e^{-A_u(t_i-t)}(\eta_u^* - \underline{\eta_u}) + \underline{\eta_u}]^T \quad \underline{\eta_c}^T]^T \\ &\quad \textit{if } t < t_i \\ &= e^{A(t-t_i)}x(ti) + \int_{t_i}^{t} \{e^{A(t-\tau)}Bu^*_{ppt}(\tau)\}\, d\tau \\ &\quad \textit{if } t_i \leqslant t \leqslant t_f \\ &= \Phi[\bar{\xi}^T \ [e^{-A_s(t_f-t)}(\eta_s^* - \overline{\eta_s}) + \overline{\eta_s}]^T \ \overline{\eta_u}^T \ \overline{\eta_c}^T]^T \\ &\quad \textit{if } t_f < t. \end{aligned}$$

Proof. From Lemma 7, the cost function (Eq. (11)) can be written as a quadratic form in terms of boundary conditions Ψ as shown in Eq. (44). Since the cost function (Eq. (11)) is quadratic in the input, it has a lower bound (zero!). The existence of the lower bound implies that the optimization problem always has at least one solution (e.g., see Ortega, 1987, Theorem 4.2.1, in Chapter 4). Similarly, the first part of the Theorem follows from optimization of quadratic forms (e.g., see Ortega, 1987, Theorem 4.2.1, Chapter 4). The second part of the theorem follows from Lemmas 1, 3 and 5 (Eqs. (5), (27) and (35)). The third part of the Theorem follows by integrating system equation (1) with the optimal PPT input. $\square$

Thus, the optimal PPT integrates inversion-based pre- and post-actuation with minimum energy SST to solve the optimal output-transition problem—this is represented in Fig. 2.

Corollary 1. *The cost of the optimal PPT is less than or equal to the cost of the standard PPT (defined in Lemma* 2*).*

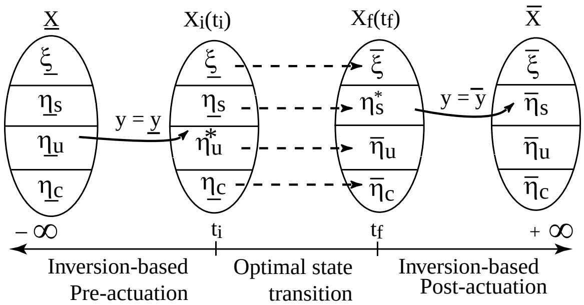

Fig. 2. Integration of inversion-based approach and minimum-energy SST approach to solve the optimal PPT problem.

Proof. This follows because the standard PPT scheme of using no pre- or post-actuation is a possible choice in the optimization scheme used to find the optimal PPT (in Theorem 1). □

Remark 5. If pre-actuation is not allowable, then the transition can be minimized by using post-actuation, i.e., by optimally choosing the stable component of the internal dynamics $\eta_s(t_f)$ at the completion of the output transition. In this case the state at the initiation of output transition is chosen as the initial delimiting state, i.e., $x(t_i) = \underline{x}$

Remark 6. Other cost functions could be used in the above approach. For example, minimum-time approaches could be used to choose the inputs during the output-transition time-interval, and then pre- and post-actuation inputs can be used to maintain constant output outside the output-transition time-interval. Similarly, the non-zero states of the system could also be penalized in the cost function as in standard optimal control techniques (e.g., to reduce vibrations in a flexible structure). The optimal input law and cost would be modified accordingly, however, the proposed approach would remain similar, in principle.

Remark 7. If the system is controllable (Assumption 1), then the state trajectory ($x_{ref}(\cdot)$ corresponding to the input that achieves optimal PPT yielding state trajectory) can be stabilized through standard techniques such as state feedback of the form $K[x(t) - x_{ref}(t)]$ to handle external perturbations and modeling errors.

4. Example

An example system with elastic dynamics is studied in this section to illustrate the proposed method. Simulation results are presented to show that substantial reduction of transition costs can be achieved with the use of the proposed optimal PPT method when compared to the use of the standard PPT approach.

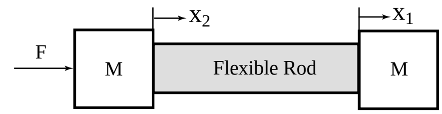

Fig. 3. Example: Two masses connected through a flexible rod. Input is force F applied to the mass at the left-hand side of the rod (with displacement x_2) and the output is the displacement of the mass to the right side of the rod, $y = x_1$.

4.1. Modeling

Example system. The example system is two masses connected by a flexible rod (see Fig. 3)—this is a benchmark problem used to study output-transition problems (e.g., Bhat & Miu, 1990). In this example system, the input and output are non-collocated; input is force F applied to the mass at the left side of the rod (with displacement x_2) and the output is the displacement of the mass to the right side of the rod, $y = x_1$ as shown in Fig. 3. Such non-collocation of the input and output points on a flexible structure results in non-minimum-phase dynamics. Therefore, rather than model the flexible rod as a spring and damper (which will result in a minimum-phase model), we model the flexible rod using the finite element method (FEM) to capture the non-minimum-phase dynamics of the system.

Modeling of the example system. The system dynamics is modeled as a simplified two-node axial rod (using FEM see, e.g., Bathe (1982)). The dynamics of the example system can be described by

$$[[M^l] + [M^r]]\ddot{U} + [C^r]\dot{U} + [K^r]U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} F, \tag{47}$$

where

$$U := [x_1 \quad x_2]^T, \quad [M^l] = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

is the mass-matrix term associated with the two lumped masses located at the ends of the rod, and the non-diagonal mass-matrix term

$$[M^r] = \frac{\rho_r A_r l_r}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

captures the distributed mass of the rod. Additionally, the stiffness matrix is

$$[K^r] = \frac{A_r E_r}{l_r} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and the structural damping matrix is $[C^r] = \alpha[K^r]$, where α is a scaling factor. Furthermore, the density of the rod, its cross-sectional area, length, and elastic modulus are represented by ρ_r, A_r, l_r, and E_r respectively. In the following simulations, the system parameters were chosen as (in appropriate units) $M = 10$, $m_2 := \frac{\rho_r A_r l_r}{6} = 1$, $k := \frac{A_r E_r}{l_r} = 1$, and $\alpha = 0.1$.

State-space equations. Let $x_3 := \dot{x}_1$ and $x_4 := \dot{x}_2$. Then, the state space representation of the system (as in Eq. (1)) can be obtained from Eq. (47) with $x := [x_1\ x_2\ x_3\ x_4]^T$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0909 & 0.0909 & -0.0091 & 0.0091 \\ 0.0909 & -0.0909 & 0.0091 & -0.0091 \end{bmatrix} \tag{48}$$

$$B = [0 \quad 0 \quad -0.0070 \quad 0.0839]^T$$

and

$$C = [1 \quad 0 \quad 0 \quad 0].$$

4.2. Solution to PPT problem

PPT problem. The goal of the PPT problem is to change the output from $y=0$ to 1 during the specified transition time $T_{tran}=t_f-t_i=10$ s. The weight on the input is chosen as $R=1$. The delimiting states ($\underline{x}$, and $\bar{x}$) are chosen as the following rigid body configurations of the system $\underline{x}=[0\ 0\ 0\ 0]^T$, $\underline{y}=0$, $\bar{x}=[1\ 1\ 0\ 0]^T$, $\bar{y}=1$.

4.2.1. Approach 1: standard PPT

If no pre- or post-actuation is used then the energy needed for the output transition can be minimized using standard PPT (see Definition 2). The input for standard PPT was found from Eq. (8) with

$$\hat{d}(t_i, t_f) = [15.8965\ 121.8660 - 534.7220 - 167.8326].$$

4.2.2. Approach 2: optimal PPT

Output-tracking form: The system equations were rewritten in the output-tracking form (with decoupled internal dynamics) using the following coordinate transformation (as in Eq. (12)):

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta_s \\ \eta_u \end{bmatrix}^T = \Phi^{-1}x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -0.7245 & -0.6892 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.6892 & -0.7245 \end{bmatrix}^{-1} x$$

and input law $u(t) = u_{ff}(t)$ with

$$u_{ff}(t) = B_s\eta_s(t) + B_u\eta_u(t) - [13 \quad 1.3 \quad 143]\mathbb{Y}(t),$$

where $B_s = -8.5231, B_u = -9.9018$ (as in Eq. (15)) and the output and its time-derivatives upto order $\rho=2$ is represented as $\mathbb{Y} = [y\ \dot{y}\ \ddot{y}]^T$ (See Eq. (16)). The internal dynamics for the example model has stable and unstable components, i.e., η_s and η_u, and it does not have components on the imaginary-axis of the complex plane, i.e., $\eta_c = \varnothing$, therefore, $B_c = \varnothing$.

System equations in output-tracking form: The system dynamics in the output-tracking coordinates is given by (as in Eq. (19))

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \ddot{y}_d, \\ \begin{bmatrix} \dot{\eta}_s \\ \dot{\eta}_u \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} \eta_s \\ \eta_u \end{bmatrix} + \begin{bmatrix} B_{int,s} \\ B_{int,u} \end{bmatrix} \mathbb{Y}_d, \end{aligned} \tag{49}$$

where $A_s = -0.9512$, $A_u = 1.0512$, $B_{int,s} = [-0.6892\ -0.0689\ -8.2707]$, and $B_{int,u} = [0.7245\ 0.0725\ 8.6946]$.

Cost for optimal PPT. Since $-A_u$ is Hurwitz, W_{pre} can be found by solving the Lyapunov Eq. (32) to obtain $W_{pre} = 46.6333$. Similarly, since A_s is Hurwitz, W_{post} can be found by solving the Lyapunov Eq. (38) to obtain $W_{post} = 38.1833$. As stated in Lemma 7, for a given choice of Ψ, the optimal cost for PPT is the following quadratic expression (see Eq. (44)):

$$J_{ppt}(\Psi) = \Psi^T \Lambda \Psi - 2\Psi^T b + c, \tag{50}$$

where

$$\Lambda = \begin{bmatrix} 400.1308 & -239.3198 \\ -239.3198 & 387.3621 \end{bmatrix}, \quad b = \begin{bmatrix} -85.9416 \\ -23.4111 \end{bmatrix}$$

and $c = 52.2438$.

Optimal PPT. The PPT cost (Eq. (50)) is minimized by the following unique choice of Ψ because Λ is invertible

$$\Psi^* = \begin{bmatrix} \eta_s^* \\ \eta_u^* \end{bmatrix} = \begin{bmatrix} \eta_s(t_f) \\ \eta_u(t_i) \end{bmatrix} = \Lambda^{-1}b = \begin{bmatrix} -0.3980 \\ -0.3063 \end{bmatrix}.$$

The corresponding optimal input $u_{ff} = u_{ppt}^*$ was obtained as described in Theorem 1 as follows.

$$u_{ppt}^*(t) = 3.0329e^{1.0512(t-t_i)} \quad \forall t < t_i,$$

$$u_{ppt}^*(t) = \begin{bmatrix} 0 \\ 0 \\ -0.0070 \\ 0.0839 \end{bmatrix}^T \exp\{A^T(t_f - t)\} \begin{bmatrix} 0.3936 \\ 10.4739 \\ -47.2569 \\ -7.0802 \end{bmatrix}, \quad \forall t_i \leqslant t \leqslant t_f$$

$$u_{ppt}^*(t) = -2.7828e^{-0.9512(t-t_f)} \quad \forall t > t_f.$$

4.3. Results

The standard and optimal inputs were calculated and applied as feedforward to the system. Feedback was not used in the following simulations because we did not want the feedback choice to affect the simulation results—this enables us to directly compare the solutions to the standard and optimal PPT problems. However, feedback can be used to further improve the performance, for example, to handle modeling errors and external perturbations.

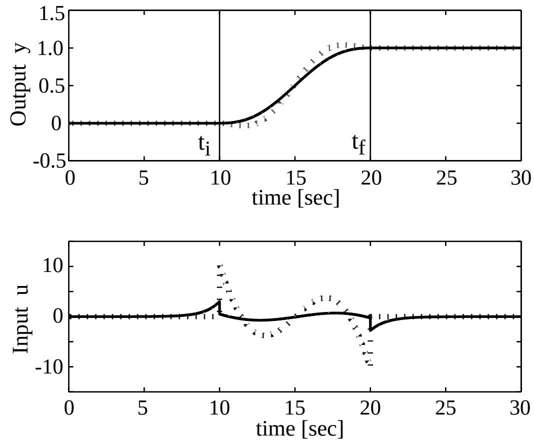

Fig. 4. Comparison of output and input time-trajectories for standard PPT (dotted line) and optimal PPT (solid line)—the optimal PPT input has pre- and post-actuation, however, the time for output transition is the same (10 s) for both methods. The output transition is initiated at time $t = t_i = 10$ s and completed at time $t = t_f = 20$ s.

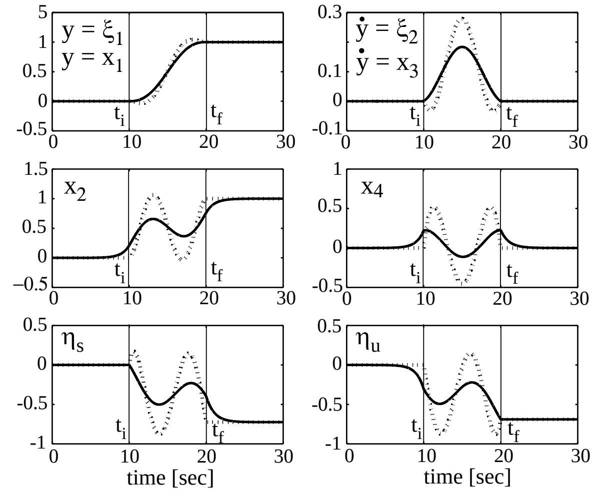

Fig. 5. Comparison of state time-trajectories for standard PPT (dotted line) and optimal PPT (solid line) with input weight $R = 1$. The stable and unstable components of the internal state (η_s, and η_u) are shown in the bottom plots.

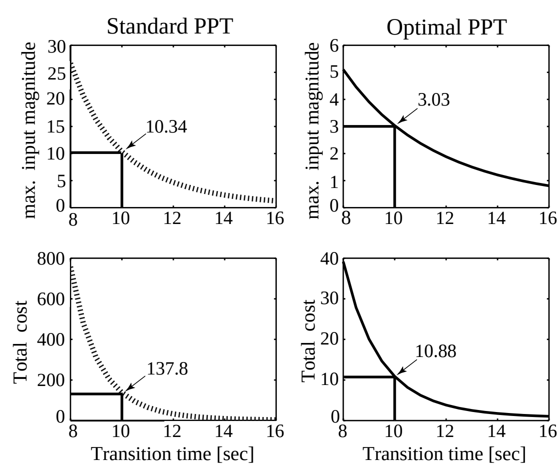

Fig. 6. Comparison of costs and maximum input magnitudes for standard PPT (dotted line) and optimal PPT (solid line) when the transition time $T_{tran} = t_f - t_i$ is varied. It is noted that the cost for optimal PPT is always less than the cost for standard PPT.

It is noted that the feedforward inputs are pre-computed using the simulation model; therefore, the actual system states are not needed in this implementation. Furthermore, the optimal PPT input (found through the inversion method) is applied before the initiation of the output transition; thus, the optimal input is non-causal in this sense. However, the inversion-based pre-actuation input can be computed and applied online if preview information of the output transition is available (see, Zou & Devasia, 1999 for preview-based computation of inverse inputs). In the simulations, the inputs were computed offline and pre- and post-actuation times were chosen as 10 s.

Remark 8. From Theorem 1, it can be seen that the pre-actuation input decays exponentially to zero as time $t \to -\infty$. The decay rate is proportional to the minimum distance D_{rhp} of the right half-plane zeros of the system (i.e., poles of A_u) from the imaginary axis of the complex plane. As this distance, increases, a smaller preactuation-time can be chosen to achieve the desired accuracy in output-tracking—typically, a preactuation time of $10/D_{rhp}$ is sufficient. Similarly, the post-actuation time needed depends on the minimum distance D_{lhp} of the left half-plane zeros of the system (i.e., poles of A_s) from the imaginary axis of the complex plane—typically, a post-actuation time of $10/D_{lhp}$ is sufficient.

Simulation results: Simulation results, when inputs for the standard PPT and the optimal PPT were applied to the example system, are presented in Figs. 4–6. Fig. 4 compares the inputs used in the standard and optimal PPT, and Fig. 5 compares the evolution of the states. Note that the input is only applied during the output-transition period, i.e., pre- and post-actuation are not used for the standard PPT; however, both pre- and post-actuation are used in the optimal PPT. Fig. 6 shows the optimal cost and maximum magnitudes of inputs found for standard and optimal PPT for various transition times.

Comparison of standard and optimal PPT: The simulation results in Fig. 6 show that the cost for optimal PPT input is always lower than the cost for standard PPT input (i.e., without pre- or post-actuation) for different transition times (as expected, see Corollary 1). For this example, with an output-transition time of 10 s and $R = 1$, the cost for the standard PPT (defined in Eq. (7)) is $J^*_{sst}(t_i, t_f) = 137.8$. The cost of optimal PPT input is 10.88 (for the same 10 s output-transition time), which is 12.7 times smaller than the cost of standard PPT input. Fig. 6 also shows that the magnitudes of the inputs used in optimal PPT are also substantially lower than the magnitudes of inputs needed for the standard PPT, especially as the transition time become smaller. For the 10 s output-transition time, the maximum magnitude of the standard PPT input is 10.34. The maximum amplitude of the input with optimal PPT is 3.03, which is 3.4 times smaller than the maximum amplitude for the input needed for the standard PPT (for the same 10 s output-transition time). Note that both methods result in zero residual vibrations in the output, however the optimal PPT achieves it with lower input cost by exploiting the use of pre- and post-actuation.

5. Conclusions

A technique to achieve optimal point-to-point output transition was presented. Freedom in the choice of the internal dynamics was exploited using pre- and post-actuation inputs (using inversion schemes) to optimize the input energy needed to achieve this output transition. The method was applied to a simplified flexible-structure model and simulation results were presented to illustrate the advantages of the technique over standard state-to-state optimal control techniques.

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