Statistics/EE 530, Spring Quarter 2018

Problem Set 7

Problem 23 (to be turned in – 5 points). Let $\{\varepsilon_t\}$ be a white noise process with zero mean and variance σ_ε^2 (by definition, such a process has an ACVS given by Equation (268d)). Note that Equation (267b) can be used to argue that the SDF for a white noise process is given by $S_\varepsilon(f) = \sigma_\varepsilon^2$ for all f (assuming for convenience that Δt is unity). Derive an explicit expression for the wavelet variance $\nu_\varepsilon^2(\tau_j)$ of $\{\varepsilon_t\}$. Using this expression, verify that Equation (296d) holds for a white noise process (i.e., that the sum of the wavelet variance over all possible dyadic scales is equal to the process variance σ_ε^2). (This is Exercise [8.1], p. 337.)

Problem 24 (to be turned in – 10 points). Show that, if $\{X_t\}$ is a stationary process, the expected difference between the unbiased and biased MODWT estimators of the wavelet variance (Equations (306b) and (306c), respectively) goes to 0 as $N \to \infty$. Does the same result hold if $\{X_t\}$ is a nonstationary process with stationary first order backward differences? Explain your answer. (This is Exercise [8.5], p. 337.)

Problem 25 (to be turned in – 5 points). Suppose that $\{X_t\}$ is a first order stationary autoregressive process with zero mean; i.e., we can write $X_t = \phi X_{t-1} + \varepsilon_t$, where $|\phi| < 1$ and $\{\varepsilon_t\}$ is a zero mean white noise process. The autocovariance sequence $\{s_\tau\}$ for $\{X_t\}$ is given by $s_{X,\tau} = \phi^{|\tau|} s_{X,0}$, where $s_{X,0}$ is the variance of the process. As defined by Equations (296a) and (306a) and assuming use of the Haar wavelet, let $\{\overline{W}_{1,t}\}$ be the unit scale MODWT wavelet coefficient process, and let $\{\widetilde{W}_{1,t}\}$ be the unit scale MODWT wavelet coefficients based upon the time series $X_0, \ldots, X_{N-1}$. For what range of t do we have $\widetilde{W}_{1,t} = \overline{W}_{1,t}$? Determine $E\{\widetilde{W}_{1,t}^2\}$ for all t. Under what circumstances do we have $E\{\widetilde{W}_{1,0}^2\} = \nu_X^2(\tau_1)$? (This is Exercise [8.6], pp. 337–8.)

Problem 26 (to be turned in – 10 points). Based upon Equation (313c) (with η estimated by $\hat{\eta}_1$ of Equation (313d)) and upon the D(4) MODWT unbiased estimator of the D(4) wavelet variance $\nu_X^2(\tau_1)$, compute a 95% confidence interval for $\nu_X^2(\tau_1)$ for the 16 point time series $\{X_{1,t}\}$ listed in the caption to Figure 42. Compute a second 95% confidence interval for $\nu_X^2(\tau_1)$, but this time using the EDOF η_3 of Equation (314c). Which measure of the EDOF would you regard as more appropriate to use here? Explain your answer. (Note that, if need be, Equation (264) can be used to compute the percentage points for the χ^2 distribution and that Table 263 gives certain percentage points that might be of use here.) (This is Exercise [8.11], pp. 338–9.)

Self-Graded Problem XVII (*not* to be turned in). Show that (a) the process $\{X_t\}$ defined by Equation (288a) is a nonstationary process; (b) the first order backward difference of $\{X_t\}$ is a stationary process with mean zero; and (c) at low frequencies the SDF for $\{X_t\}$ is approximately a power law process; i.e., $S_X(f) \propto |f|^\alpha$ approximately for f close to zero, where the power law exponent α is to be determined as part of the exercise. (This is Exercise [288a], p. 288.)

Self-Graded Problem XVIII (*not* to be turned in). Suppose that $\{X_t\}$ is a stochastic process whose dth order backward difference is a stationary process, say, $\{Y_t\}$, where $E\{Y_t\} = \mu_Y$. Let

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t = \ldots, -1, 0, 1, \ldots,$$

where the MODWT wavelet filter $\{\tilde{h}_{j,l}\}$ is based on a Daubechies wavelet filter of length L. Show that, on the one hand, if $L > 2d$ or if $L = 2d$ and $\mu_Y = 0$, then $E\{\overline{W}_{j,t}\} = 0$; and, on the other hand, if $\mu_Y \neq 0$ and $L = 2d$, then $E\{\overline{W}_{j,t}\} \neq 0$. (This is Exercise [305], p. 305.)

Self-Graded Problem XIX (*not* to be turned in). Under the assumption that $\hat{\nu}_X^2(\tau_j)$ is distributed as $a\chi^2_\eta$, determine a and η (the EDOF) by solving the two equations $E\{\hat{\nu}_X^2(\tau_j)\} = E\{a\chi^2_\eta\}$ and $\mathrm{var}\,\{\hat{\nu}_X^2(\tau_j)\} = \mathrm{var}\,\{a\chi^2_\eta\}$. In particular, show that

$$\eta = \frac{2\left(E\{\hat{\nu}_X^2(\tau_j)\}\right)^2}{\mathrm{var}\,\{\hat{\nu}_X^2(\tau_j)\}}.$$

Use this approach with the large sample approximations to the mean and variance of $\hat{\nu}_X^2(\tau_j)$ to show that the EDOF is given approximately by

$$\eta_1 \equiv \frac{M_j \nu_X^4(\tau_j)}{A_j}.$$

(This is Exercise [313a], p. 313.)

Self-Graded Problem XX (*not* to be turned in). Under the assumption that Equation (313a) holds, use an argument similar to the one leading up to Equation (311) to show that an approximate $100(1-2p)\%$ confidence interval for $\nu_X^2(\tau_j)$ is given by

$$\left[\frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(1-p)}, \frac{\eta\hat{\nu}_X^2(\tau_j)}{Q_\eta(p)}\right],$$

where $Q_\eta(p)$ is the $p \times 100\%$ percentage point for the χ^2_η distribution, i.e., $\mathbf{P}[\chi^2_\eta \leq Q_\eta(p)] = p$. (This is Exercise [313b], p. 313.)

Solutions to Problems 23 to 26 are due Wednesday, May 16, at the beginning of the class.