Statistics/EE 530, Spring Quarter 2018

Problem Set 3

Problem 8 (to be turned in – 10 points). Verify that use of the pyramid algorithm with the Haar wavelet and scaling filters yields a transform that agrees with the Haar DWT matrix $\mathcal{W}$ described in Section 4.1 for $N = 16$. In particular, describe the contents of $\mathcal{B}_j$ and $\mathcal{A}_j$ for $j = 1, 2, 3$ and 4, and show that $\mathcal{B}_2\mathcal{A}_1$, $\mathcal{B}_3\mathcal{A}_2\mathcal{A}_1$, $\mathcal{B}_4\mathcal{A}_3\mathcal{A}_2\mathcal{A}_1$ and $\mathcal{A}_4\mathcal{A}_3\mathcal{A}_2\mathcal{A}_1$ yield the bottom 8 rows of $\mathcal{W}$ as shown in the right-hand column of Figure 57. (This is Exercise [4.13], p. 157 of the course textbook, but with a typo corrected.)

Problem 9 (to be turned in – 5 points). Show that $\mathcal{W}_1^T\mathcal{W}_1 + \mathcal{V}_1^T\mathcal{V}_1 = I_N$, an equation that can be referred to as a level $J_0 = 1$ resolution of the Nth order identity matrix. State and prove a corresponding resolution of the Nth order identity matrix for $J_0 > 1$.

Problem 10 (to be turned in – 5 points for each of the 3 parts). Given a wavelet filter $\{h_l : l = 0, \ldots, L-1\}$ with $h_l \equiv 0$ for $l < 0$ and $l \geq L$, we have defined the corresponding scaling filter by $g_l = (-1)^{l+1}h_{L-1-l}$. This filter has a transfer function given by

$$G(f) = \sum_{l=-\infty}^{\infty} g_l e^{-i2\pi fl} = \sum_{l=0}^{L-1} g_l e^{-i2\pi fl} = e^{-i2\pi f(L-1)}H(\tfrac{1}{2} - f) \qquad (*)$$

(see Exercise [76a] – here $H(\cdot)$ is the transfer function for the wavelet filter). As noted on page 79 of the course textbook (and on lecture overhead IV–40), a second commonly used definition for the scaling filter is $\bar{g}_l = (-1)^{l-1}h_{1-l}$. Denote its transfer function by $\overline{G}(\cdot)$ and its squared gain function by $\overline{\mathcal{G}}(\cdot)$.

a. Find an expression for $\overline{G}(\cdot)$ in terms of $H(\cdot)$, i.e., an analog of $(*)$ above. Verify that $\overline{\mathcal{G}}(\cdot)$ is the same as $\mathcal{G}(\cdot)$ (the squared gain function associated with $G(\cdot)$).

b. A key step in Exercise [78] that establishes the orthonormality of the DWT based upon the scaling filter $\{g_l\}$ is to show that

$$G^*(\tfrac{k}{N})H(\tfrac{k}{N}) + G^*(\tfrac{k}{N} + \tfrac{1}{2})H(\tfrac{k}{N} + \tfrac{1}{2}) = 0.$$

Verify that the above still holds when we use the scaling filter $\{\bar{g}_l\}$; i.e., show that

$$\overline{G}^*(\tfrac{k}{N})H(\tfrac{k}{N}) + \overline{G}^*(\tfrac{k}{N} + \tfrac{1}{2})H(\tfrac{k}{N} + \tfrac{1}{2}) = 0.$$

c. In forming both $\{g_l\}$ and $\{\bar{g}_l\}$, we utilize the filter coefficients for the wavelet filter $\{h_l\}$ in reverse order. Explore the consequences on a level $J_0 = 1$ DWT of eliminating this reversal, i.e., of defining the scaling filter to be $\breve{g}_l = (-1)^{l+1}h_l$. (It might help to look over Exercise [77] and the discussion leading up to it.)

Self-Graded Problem V (*not* to be turned in). Suppose that $\{h_l\}$ is a wavelet filter, and let $H(\cdot)$ be its transfer function. As defined in Equation (75a), namely, $g_l \equiv (-1)^{l+1} h_{L-1-l}$, let $\{g_l\}$ be the scaling filter corresponding to $\{h_l\}$. Show that the transfer function $G(\cdot)$ for $\{g_l\}$ is given by

$$G(f) \equiv \sum_{l=-\infty}^{\infty} g_l e^{-i2\pi fl} = \sum_{l=0}^{L-1} g_l e^{-i2\pi fl} = e^{-i2\pi f(L-1)} H(\tfrac{1}{2} - f)$$

and hence

$$\mathcal{G}(f) = \mathcal{H}(\tfrac{1}{2} - f),$$

where $\mathcal{G}(f) \equiv |G(f)|^2$ is the squared gain function. (This is Exercise [76a], p. 76.)

Self-Graded Problem VI (*not* to be turned in). Show that we must have either

$$\sum_{l=0}^{L-1} g_l = \sqrt{2} \text{ or } \sum_{l=0}^{L-1} g_l = -\sqrt{2}.$$

Show further that

$$\sum_{l=0}^{L-1} g_l^2 = 1 \text{ and } \sum_{l=0}^{L-1} g_l g_{l+2n} = \sum_{l=-\infty}^{\infty} g_l g_{l+2n} = 0$$

for all nonzero integers n. (This is Exercise [76b], p. 76.)

Self-Graded Problem VII (*not* to be turned in). Use the inverse DFT relationship to argue that

$$g^\circ \star h^\circ_{2n} = \frac{1}{N} \sum_{k=0}^{\frac{N}{2}-1} \left[G^*(\tfrac{k}{N}) H(\tfrac{k}{N}) + G^*(\tfrac{k}{N} + \tfrac{1}{2}) H(\tfrac{k}{N} + \tfrac{1}{2}) \right] e^{i4\pi nk/N},$$

and then show that, for all k,

$$G^*(\tfrac{k}{N}) H(\tfrac{k}{N}) + G^*(\tfrac{k}{N} + \tfrac{1}{2}) H(\tfrac{k}{N} + \tfrac{1}{2}) = 0,$$

which immediately establishes Equation (77d). (This is Exercise [78], p. 78.)

Self-Graded Problem VIII (*not* to be turned in). Suppose that $\{h_l : l = 0, \ldots, L-1\}$ is a filter with transfer function $H(\cdot)$. Define a new filter by inserting m zeros between each of the elements of $\{h_l\}$:

$$h_0, \underbrace{0, \ldots, 0}_{m \text{ zeros}}, h_1, \ldots, h_{L-2}, \underbrace{0, \ldots, 0}_{m \text{ zeros}}, h_{L-1}.$$

Show that this filter has a transfer function given by $H([m+1]f)$. (This is Exercise [91], p. 91.)

Solutions to Problems 8, 9 and 10 are due Wednesday, April 18, at the beginning of the class.